

1. Prove or disprove that the given map is a homomorphism. If it is a homomorphism, determine whether it is an isomorphism.
 - (a) The map $\phi : (\mathbb{Z}_{20}, +) \rightarrow (\mathbb{Z}_{20}, +)$ given by $\phi(k \pmod{20}) = 8k \pmod{20}$.
 - (b) The map $\phi : (\mathbb{Z}_{10}, +) \rightarrow (\mathbb{Z}_{10}, +)$ given by $\phi(k \pmod{10}) = k(k+4) \pmod{10}$.
 - (c) The map $\phi : (\mathbb{C}^\times, \times) \rightarrow (\mathbb{C}^\times, \times)$ given by $\phi(z) = z^3$.

2. If H is a non-trivial normal subgroup of G , is it possible that $G/H \cong G$? Consider the problem when G is finite and the case when G is possibly infinite.

3. Define the following rotation and reflection matrices in $\text{GL}_2(\mathbb{R})$, where $\theta \in \mathbb{R}$:

$$A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The matrices $G = \{A_\theta : \theta \in \mathbb{R}\} \cup \{BA_\theta : \theta \in \mathbb{R}\}$ form a group (no need to check this).

- (a) Prove or disprove that $H = \{A_\theta : \theta \in \mathbb{R}\}$ is a normal subgroup of G .
 - (b) Is $\langle B \rangle$, the subgroup generated by B , a normal subgroup of G ?
4. Prove or disprove whether the two groups listed are isomorphic.
 - (a) S_3 and $\mathbb{Z}_2 \times \mathbb{Z}_6$
 - (b) $(\mathbb{Z}_3, +)$ and the complex numbers $\{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\}$ with multiplication
 - (c) Q_8 =quaternion group and G =symmetries of a square
 - (d) \mathbb{Z} and $\mathbb{Z} \times \mathbb{Z}$
 - (e) \mathbb{R}^\times and \mathbb{C}^\times
 - (f) $A_4 \times \mathbb{Z}_2$ and $S_3 \times \mathbb{Z}_4$ ¹

5. Use the method of Cayley's Theorem to produce an explicit injective homomorphism from the group \mathbb{Z}_8^\times into a symmetric group.

¹In a product of two groups $G_1 \times G_2$, the order of $(a, b) \in G_1 \times G_2$ is equal to $\text{lcm}(\text{ord}(a), \text{ord}(b))$.