Practice Problems for Exam 2

1. Determine whether the given map is a homomorphism. If it's a homomorphism also determine if it's an isom.

(a) $\phi: \mathbb{Z}_{20} \to \mathbb{Z}_{20}$, $\phi(k \pmod{20}) \equiv 8k \pmod{20}$.

$\phi(k+k') \equiv 8(k+k') \equiv 8k+8k' \equiv \phi(k)+\phi(k')$,

So it's a homomorphism.

$\phi(5) \equiv 8 \cdot 5 \equiv 40 \equiv 0 \Rightarrow \ker\phi \neq \{0\}$

$\Rightarrow$ ϕ not injective, So not isom.

(b) $\phi: \mathbb{Z}_{10} \to \mathbb{Z}_{10}$, $\phi(k \pmod{10}) \equiv k(k+4) \pmod{10}$

Not a homomorphism:

$\phi(1) + \phi(-1) \equiv 1(1+4) + (-1)(-1+4) \equiv 5 - 3 \equiv 2$

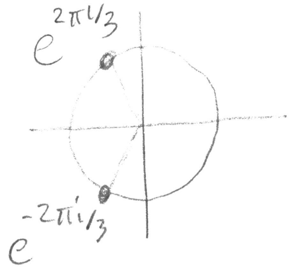
$\not\equiv 0 \equiv \phi(0) \equiv \phi(1-1)$

(c) $\phi: \mathbb{C}^{\times} \to \mathbb{C}^{\times}$, $\phi(z) = z^3$

$\phi(zw) = (zw)^3 = z^3 w^3 = \phi(z)\phi(w) \Rightarrow \phi$ homomorphism

$\phi(e^{\pm 2\pi i/3}) = 1 \Rightarrow \ker\phi \neq \{1\} \Rightarrow \phi$ not inject.

So not isom.



(Note ϕ is surjective, though.)

2. If H is a nontrivial normal subgp of G,

is it possible that $G/H \cong G$?

G finite: $|G/H| = |G|/|H| < |G|$, since H nontrivial $\Rightarrow |H| > 1$.

So $G/H \not\cong G$ if G is finite.

G infinite: it's possible. Example:

componentwise addition

$$G = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{Z}\} = \mathbb{Z} \times \mathbb{Z} \times \ldots$$

"Shift map" $\phi: G \to G$, $\phi(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$

is a homomorphism. It's onto.

1^{st} Isom. Thm $\Rightarrow G/\ker\phi \cong G$

Finally, note $\ker\phi = \{(a_1, 0, \ldots) \mid a_1 \in \mathbb{Z}\}$ is nontrivial. (2)

3. Define the following rotation & reflection matrices in $GL_2(\mathbb{R})$, where $\theta \in \mathbb{R}$:

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$G = \{A_\theta \mid \theta \in \mathbb{R}\} \cup \{BA_\theta \mid \theta \in \mathbb{R}\}$ is a subgp of $GL_2(\mathbb{R})$

(a) Prove or disprove $H = \{A_\theta \mid \theta \in \mathbb{R}\}$ is normal in G.

A_θ is rotation CC by angle θ, so $A_\theta A_\phi = A_{\theta+\phi}$,

$(A_\theta)^{-1} = A_{-\theta}$.

Let $A_\theta \in H$ and $g \in G$. If $g = A_\phi$ then

$$gA_\theta g^{-1} = A_\phi A_\theta A_\phi^{-1} = A_{\phi+\theta-\phi} = A_\theta \in H.$$

If $g = BA_\phi$ then

$$gA_\theta g^{-1} = BA_\phi A_\theta A_\phi^{-1} B^{-1} = BA_\theta B^{-1}$$

(note $B^{-1} = B$)

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A_{-\theta}$$

which is also in H.

Thus $gHg^{-1} \subset H$ for all $g \in G \Rightarrow H$ normal.

<u>Alt. proof</u>: H is the kernel of the homomorphism

$\det : G \to \mathbb{R}^{\times}$, and kernels are normal.

(b) Is the subgp in G generated by B normal?

We computed $BA_\theta B^{-1} = A_{-\theta} \Rightarrow BA_\theta = A_{-\theta}B$.

$$A_\theta B A_\theta^{-1} = A_\theta B A_{-\theta} = A_\theta A_\theta B = A_{2\theta} B$$

$A_{2\theta}B \notin \langle B \rangle = \{I, B\}$ if 2θ is not in $2\pi\mathbb{Z}$,

so $\langle B \rangle$ is not normal.

4. Prove or disprove whether the two gps listed are isomorphic.

(a) S_3, $\mathbb{Z}_2 \times \mathbb{Z}_6$. <u>Not isom.</u> S^3 nonabelian, $\mathbb{Z}_2 \times \mathbb{Z}_6$ abelian

(b) $\mathbb{Z}_3$, $\{1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i\} = G \subset \mathbb{C}^\times$

<u>Isomorphic.</u> $\phi : \mathbb{Z}_3 \to G$, $\phi(\pm 1 (\text{mod } 3)) = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\phi(0 (\text{mod } 3)) = 1$

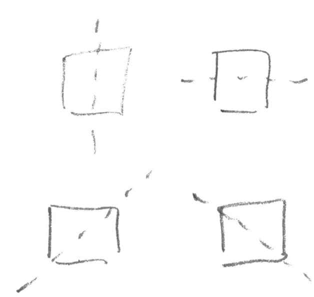
((Can also use what we proved in class: every group of prime order is cyclic, and $\mathbb{Z}_n$ is the only finite cyclic group up to isomorphism)

(c) Q_8 = quaternion gp, G = symmetries of a square

<u>Not isomorphic.</u> $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$

has only one element of order 2

G has more than one element of order 2:



(d) $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$

<u>Not Isomorphic.</u> $\mathbb{Z}$ is cyclic, while

$\mathbb{Z} \times \mathbb{Z}$ is not cyclic.

(e) $\mathbb{R}^\times$, $\mathbb{C}^\times$ Not Isomorphic

$\mathbb{C}^\times$ has elements of any given order ($e^{2\pi i/n}$ has ord. n)

while the only finite order elements in $\mathbb{R}^\times$ are ± 1

(f) $A_4 \times \mathbb{Z}_2$, $S_3 \times \mathbb{Z}_4$. Not Isomorphic:

Use: in product of two groups G_1, G_2,

$$\operatorname{ord}(g_1, g_2) = \operatorname{lcm}(\operatorname{ord}(g_1), \operatorname{ord}(g_2))$$

orders of elements in S_3: 1, 2, 3

——— " ——— $\mathbb{Z}_4$: 1, 2, 4

——— " ——— $S_3 \times \mathbb{Z}_4$: 1, 2, 3, 4, 6, 12

——— " ——— A_4: 1, 2, 3
e $(12)(34)$ (123)

——— " ——— $\mathbb{Z}_2$: 1, 2

——— " ——— $A_4 \times \mathbb{Z}_2$: 1, 2, 3, 6

$S^3 \times \mathbb{Z}_4$ has an element of order 12 (ex. $((123), 1 \pmod 4)$) while $A_4 \times \mathbb{Z}_2$ doesn't.

5. Use the method of Cayley's Thm to construct an injective homomorphism from $\mathbb{Z}_8^\times$ to a symmetric group.

$\mathbb{Z}_8^\times = \{1, 3, 5, 7\}$ (units mod 8)

$1 = a_1,\ 3 = a_2,\ 5 = a_3,\ 7 = a_4$

Define $\phi : \mathbb{Z}_8^\times \to S_4$ by $\phi(a) = \sigma_a$ where

$\sigma_a : \{1,2,3,4\} \to \{1,2,3,4\}$ is determined by

$\sigma_a(i) = j \iff a a_i = a_j$.

$a = 1$ gives $\sigma_a = e$ (identity), of course $\Rightarrow \phi(1) = e$

$a = 3$:

$$\left.\begin{array}{ll} a a_1 \equiv 3\cdot 1 \equiv 3 \equiv a_2 & \Rightarrow \sigma_a(1) = 2 \\ a a_2 \equiv 3\cdot 3 \equiv 9 \equiv 1 \equiv a_1 & \Rightarrow \sigma_a(2) = 1 \\ a a_3 \equiv 3\cdot 5 \equiv 15 \equiv 7 \equiv a_4 & \Rightarrow \sigma_a(3) = 4 \\ a a_4 \equiv 3\cdot 7 \equiv -3 \equiv 5 \equiv a_3 & \Rightarrow \sigma_a(4) = 3 \end{array}\right\} \Rightarrow \phi(3) = (12)(34)$$

Similar computations give

$\phi(5) = (13)(24), \quad \phi(7) = (14)(23)$

$\operatorname{im}\phi = \{e, (12)(34), (13)(24), (14)(23)\}$