

## More isomorphism theorems

Last lecture we learned about the 1st Isomorphism Theorem. As its name suggests, it has a few sequels, whose names are just as imaginative!

For subsets  $H$  and  $N$  of a group  $G$  we define the subset  $HN \subset G$  to be:

$$HN = \{ab : a \in H, b \in N\}$$

► **2nd Isomorphism Theorem:** Let  $H, N$  be subgroups a group  $G$  with  $N$  normal. Then  $HN$  is a subgroup of  $G$ ,  $H \cap N$  is a normal subgroup of  $H$ , and

$$\frac{H}{H \cap N} \cong \frac{HN}{N}$$

*Proof.* Let us first show  $HN$  is a subgroup of  $G$ . Clearly  $e \in HN$ , since  $e = ee$  and  $e \in H, e \in N$ . Let  $ab, a'b' \in HN$  where  $a, a' \in H, b, b' \in N$ . Then  $Na' = a'N$  since  $N$  is normal, and this implies  $ba' = a'b''$  for some  $b'' \in N$ . Thus

$$(ab)(a'b') = a(ba')b' = (aa')(b''b') \in HN.$$

Next, let  $ab \in HN$ . Then  $Na^{-1} = a^{-1}N$  implies  $b^{-1}a^{-1} = a^{-1}b'$  for some  $b' \in N$ . Thus  $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b' \in HN$ . Thus  $HN$  is a subgroup of  $G$ .

The argument that  $H \cap N$  is a normal subgroup of  $H$  is also straightforward, and is omitted.

To prove the isomorphism, we construct a homomorphism

$$\phi : H \longrightarrow HN/N$$

by setting, for each  $a \in H$ ,  $\phi(a) = aN$ . Any coset in  $HN/N$  is of the form  $abN = aN$  for  $ab \in HN$ , and  $\phi(a) = aN$ , so  $\phi$  is onto. The map  $\phi$  is a homomorphism:

$$\phi(aa') = aa'N = aNa'N = \phi(a)\phi(a')$$

Now let us compute  $\ker(\phi)$ . Suppose  $a \in H$  and  $\phi(a) = aN = N$ . This means  $a \in N$ . Thus  $\ker(\phi) = H \cap N$ . The 1st Isomorphism Theorem then gives

$$HN/N \cong H/\ker(\phi) = H/H \cap N \quad \square$$

► **Example:** Let  $m, n$  be positive integers. Let  $G = \mathbb{Z}$  and  $H = m\mathbb{Z}$ ,  $N = n\mathbb{Z}$ . Recall that  $\gcd(m, n)$  is characterized as being the smallest positive integer contained in the set

$$m\mathbb{Z} + n\mathbb{Z} = \{am + bn : a, b \in \mathbb{Z}\}$$

This subset of  $\mathbb{Z}$  is the subgroup “ $HN$ ” in this example. We see that  $m\mathbb{Z} + n\mathbb{Z} = \gcd(m, n)\mathbb{Z}$ . On the other hand, the subset corresponding to  $H \cap N$  is given by

$$m\mathbb{Z} \cap n\mathbb{Z} = \{k \in \mathbb{Z} : k = am = an \text{ for some } a, b \in \mathbb{Z}\}$$

Noting that the smallest positive integer in this subset is the lcm of  $m$  and  $n$ , we obtain  $m\mathbb{Z} \cap n\mathbb{Z} = \text{lcm}(m, n)\mathbb{Z}$ . The 2nd Isomorphism Theorem gives us

$$\frac{m\mathbb{Z}}{\text{lcm}(m, n)\mathbb{Z}} = \frac{H}{H \cap N} \cong \frac{HN}{N} = \frac{\text{gcd}(m, n)\mathbb{Z}}{n\mathbb{Z}}$$

If  $k$  divides  $l$ , then  $|k\mathbb{Z}/l\mathbb{Z}| = l/k$ . (This can be shown explicitly, but is also deduced as a consequence of the 3rd Isomorphism Theorem below.) By comparing orders of groups,

$$\text{lcm}(m, n)/m = n/\text{gcd}(m, n) \implies \text{lcm}(m, n)\text{gcd}(m, n) = mn.$$

► **Example:** Consider  $G = S_4$  and the normal subgroup  $N = \{e, (12)(34), (13)(24), (14)(23)\}$  of order 4 inside of  $S_4$ . Then we saw from last lecture that  $S_4/N \cong S_3$  by constructing an onto homomorphism  $\phi: S_4 \rightarrow S_3$ , recognizing its kernel to be  $N$ , and using the 1st Isomorphism Theorem. Let us give a different method for showing  $S_4/N \cong S_3$ .

Take  $H = S_3 \subset S_4$ . This is a (non-normal) subgroup. Then we first claim

$$HN = S_4.$$

To see this, we proceed as follows. First,  $S_3 = S_3e \in HN$ , so in particular  $(12), (23), (31) \in HN$ . Next, as  $(12), (23), (31) \in S_3$  and  $(12)(34), (23)(14), (31)(24) \in N$  we get the products

$$(12)((12)(34)) = (34), \quad (23)((23)(14)) = (14), \quad (31)((31)(24)) = (24)$$

also as elements of  $HN$ . Thus every transposition  $(12), (13), (14), (23), (24), (34)$  in  $S_4$  is in  $HN$ . Every permutation in  $S_4$  is a product of transpositions, and therefore  $S_4 = HN$ . Next note that  $H \cap N = \{e\}$  by direct inspection. Finally, the 2nd Isomorphism Theorem gives

$$S_4/N = HN/N \cong S_3/S_3 \cap N = S_3/\{e\} \cong S_3$$

► **3rd Isomorphism Theorem:** Let  $G$  be a group and suppose  $H, N \subset G$  are normal subgroups with  $N \subset H$ . Then we have an isomorphism

$$\frac{G}{H} \cong \frac{G/N}{H/N}$$

*Proof.* We define a map  $\phi: G/N \rightarrow G/H$  by setting  $\phi(aN) = aH$ . Note that if  $aN = bN$ , then  $ab^{-1} \in N \subset H$ , and thus  $aH = bH$ . Thus  $\phi(aN) = aH = bH = \phi(bN)$ , and  $\phi$  is well-defined. Furthermore,  $\phi$  is an onto homomorphism, by repeating earlier arguments.

The kernel of  $\phi$  is given by  $aN \in G/N$  such that  $\phi(aN) = aH = H$ , which is true if and only if  $a \in H$ . Thus  $\ker(\phi) = \{aN : a \in H\} = H/N$ . The 1st Isomorphism Theorem gives

$$G/H \cong \frac{G/N}{\ker(\phi)} = \frac{G/N}{H/N} \quad \square$$

► **Example:** Consider  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ . Note that  $mn\mathbb{Z} \subset n\mathbb{Z}$  for any integer  $m$ . With  $G = \mathbb{Z}$ ,  $H = n\mathbb{Z}$  and  $N = mn\mathbb{Z}$  we obtain from the 3rd Isomorphism Theorem:

$$\mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}/mn\mathbb{Z}}{n\mathbb{Z}/mn\mathbb{Z}}$$

Taking orders, we get  $n = |\mathbb{Z}_n| = |\mathbb{Z}/mn\mathbb{Z}|/|n\mathbb{Z}/mn\mathbb{Z}| = mn/|n\mathbb{Z}/mn\mathbb{Z}|$ . In particular,

$$|n\mathbb{Z}/mn\mathbb{Z}| = m.$$

Setting  $l = mn$  and  $k = n$  we deduce in general that if  $k$  divides  $l$  then  $|k\mathbb{Z}/l\mathbb{Z}| = l/k$ .