Homework 7 Solutions

1. (a) $y''+16y = 3\sec(4t)$

Homogeneous eq. $y''+16y=0 \to P(r)=r^2+16=(r+4i)(r-4i)$

$y_1=\cos(4t),\quad y_2=\sin(4t)$

$y_p(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$ where, setting $f(t)=3\sec(4t)$,

$u_1=-\int\frac{y_2 f}{W}$, $u_2=\int\frac{y_1 f}{W}$. Here $W=W(y_1,y_2)=y_1y_2'-y_1'y_2$

$$= \cos(4t)\cdot 4\cos(4t)-(-4\sin(4t))\cdot\sin(4t)$$
$$=4\left(\cos^2(4t)+\sin^2(4t)\right)=4.$$

Thus $u_1=-\int\frac{\sin(4t)\cdot 3\sec(4t)}{4}dt=-\frac{3}{4}\int\frac{\sin(4t)}{\cos(4t)}dt \underset{v=\cos(4t)}{=} -\frac{3}{4}\left(-\frac{1}{4}\right)\int\frac{dv}{v}=\frac{3}{16}\ln|v|$

$$=\frac{3}{16}\ln|\cos(4t)|.$$

Next, $u_2=\int\frac{\cos(4t)\cdot 3\sec(4t)}{4}dt=\frac{3}{4}\int dt=\frac{3}{4}t$

$y_p=u_1y_1+u_2y_2=\frac{3}{16}\ln|\cos(4t)|\cdot\cos(4t)+\frac{3}{4}t\cdot\sin(4t).$

The general solution is

$$y(t)=c_1y_1+c_2y_2+y_p=\boxed{c_1\cos(4t)+c_2\sin(4t)+\frac{3}{16}\ln|\cos(4t)|\cdot\cos(4t)+\frac{3}{4}t\sin(4t)}$$

(b) $y''-5y=te^t$

Homogeneous eq.: $P(r)=r^2-5=(r-\sqrt5)(r+\sqrt5)\to y_1=e^{\sqrt5 t},\ y_2=e^{-\sqrt5 t}$

$y_p=u_1y_1+u_2y_2$ $\quad W(y_1,y_2)=y_1y_2'-y_1'y_2=(e^{\sqrt5 t})(-\sqrt5 e^{-\sqrt5 t})-(\sqrt5 e^{\sqrt5 t})(e^{-\sqrt5 t})$

$$=-2\sqrt5.$$

$$u_1 = -\int \frac{y_2 f}{W} = -\int \frac{e^{-\sqrt{5}t} \cdot te^t}{-2\sqrt{5}}\,dt = \frac{1}{2\sqrt{5}} \int \underbrace{t}_{u} \underbrace{e^{(1-\sqrt{5})t}}_{v'}\,dt$$

$$= \frac{1}{2\sqrt{5}} \cdot \underbrace{t}_{u} \cdot \underbrace{\left(\frac{1}{1-\sqrt{5}}\right) e^{(1-\sqrt{5})t}}_{v} - \frac{1}{2\sqrt{5}} \int \underbrace{(1)}_{u'} \cdot \underbrace{\left(\frac{1}{1-\sqrt{5}}\right) e^{(1-\sqrt{5})t}}_{v}\,dt$$

$$= \frac{1}{2\sqrt{5}} \cdot \frac{1}{(1-\sqrt{5})}\, te^{(1-\sqrt{5})t} - \frac{1}{2\sqrt{5}} \cdot \frac{1}{(1-\sqrt{5})^2}\, e^{(1-\sqrt{5})t}$$

Similar computation gives $u_2 = -\frac{1}{2\sqrt{5}} \cdot \frac{1}{(1+\sqrt{5})} \cdot te^{(1+\sqrt{5})t} + \frac{1}{2\sqrt{5}} \frac{1}{(1+\sqrt{5})^2} e^{(1+\sqrt{5})t}$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{1}{2\sqrt{5}} \cdot \frac{1}{(1-\sqrt{5})}\, te^t - \frac{1}{2\sqrt{5}} \frac{1}{(1-\sqrt{5})^2} e^t - \frac{1}{2\sqrt{5}} \cdot \frac{1}{(1+\sqrt{5})}\, te^t + \frac{1}{2\sqrt{5}} \cdot \frac{1}{(1+\sqrt{5})^2} e^t$$

$$= \frac{1}{2\sqrt{5}}\left(\frac{1}{1-\sqrt{5}} - \frac{1}{1+\sqrt{5}}\right) te^t + \frac{1}{2\sqrt{5}}\left(\frac{1}{(1+\sqrt{5})^2} - \frac{1}{(1-\sqrt{5})^2}\right)e^t$$

$$= \frac{1}{2\sqrt{5}}\left(\frac{1+\sqrt{5} - (1-\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})}\right) te^t + \frac{1}{2\sqrt{5}}\left(\frac{(1-\sqrt{5})^2 - (1+\sqrt{5})^2}{(1+\sqrt{5})^2(1-\sqrt{5})^2}\right)e^t$$

$$= \frac{1}{2\sqrt{5}} \cdot \left(\frac{2\sqrt{5}}{-4}\right) te^t + \frac{1}{2\sqrt{5}}\left(\frac{-4\sqrt{5}}{16}\right)e^t$$

$$= -\frac{te^t}{4} - \frac{e^t}{8}$$

General solution:

$$y(t) = c_1 y_1 + c_2 y_2 + y_p = \boxed{c_1 e^{\sqrt{5}t} + c_2 e^{-\sqrt{5}t} - \frac{te^t}{4} - \frac{e^t}{8}}$$

2. $LI'' + RI' + \frac{1}{C} I = E'$ $\quad R = 16,\ L = 2,\ C = \frac{1}{100},\ E = 100 \to E' = 0$

$\to 2I'' + 16I' + 100I = 0$, $I'' + 8I' + 50I = 0$. $P(r) = r^2 + 8r + 50$ has roots

$$\frac{-8 \pm \sqrt{64 - 4 \cdot 50}}{2} = -4 \pm i\sqrt{34} \longrightarrow \boxed{y(t) = c_1 e^{-4t} \cos(\sqrt{34}t) + c_2 e^{-4t} \sin(\sqrt{34}\,t).}$$