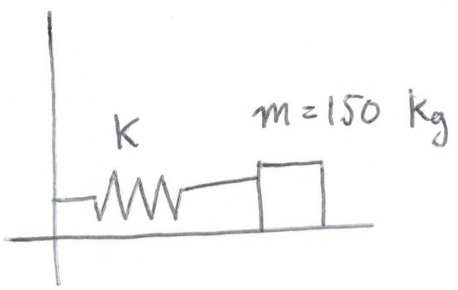
Homework 6 solutions.

1.



Spring stretches $\overbrace{10\text{ cm}}^{0.1\text{ m}}$ by force of 5 N
$\to 5 = k(0.1)$, $k = 50$.

$mx'' + kx = 0 \to 150x'' + 50x = 0$
$\to 3x'' + x = 0$

(a) Characteristic polynomial of $3x'' + x = 0$:
$P(r) = 3r^2 + 1 = 3(r^2 + \frac{1}{3}) = 3(r + \frac{i}{\sqrt{3}})(r - \frac{i}{\sqrt{3}})$ complex conjugate roots $\pm i/\sqrt{3}$

$\to x(t) = c_1 \cos(t/\sqrt{3}) + c_2 \sin(t/\sqrt{3})$ $\omega_0 = 1/\sqrt{3}$
$= C\cos(t/\sqrt{3} - \alpha)$

where $C = \sqrt{c_1^2 + c_2^2}$, $0 \le \alpha < 2\pi$ satisfies $\cos(\alpha) = \frac{c_1}{C}$, $\sin(\alpha) = \frac{c_2}{C}$.

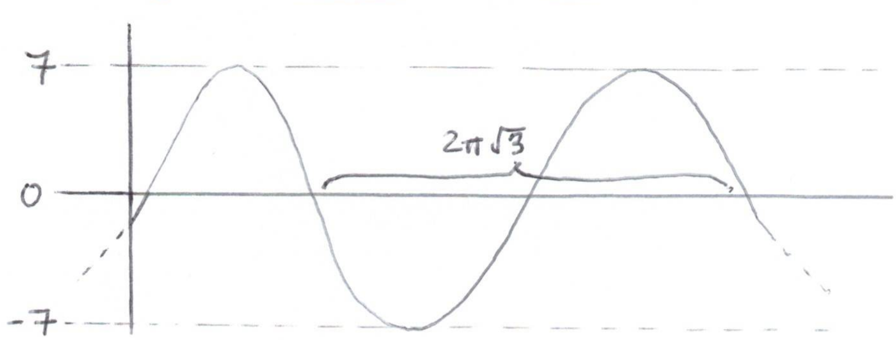
Initial conditions: $x(0) = -1$ m, $x'(0) = 4$ m/s.

$-1 = x(0) = c_1$ $\quad 4 = x'(0) = c_2/\sqrt{3}$, $c_2 = 4\sqrt{3}$

$C = \sqrt{(-1)^2 + (4\sqrt{3})^2} = \sqrt{1+48} = \sqrt{49} = 7$

$\alpha = \arccos(\frac{c_1}{C}) = \arccos(\frac{-1}{7}) \approx 1.714$ (check $\sin\alpha \approx c_2/C$.)

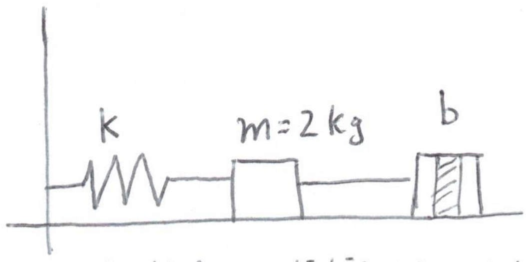
Thus $\boxed{x(t) \approx 7\cos(t/\sqrt{3} - 1.714)}$



(b) amplitude $= \boxed{7 \text{ m}}$

period $= \frac{2\pi}{\omega_0} = \frac{2\pi}{1/\sqrt{3}} = 2\pi\sqrt{3} \approx \boxed{10.88 \text{ s}}$

2.



Spring stretches 2 m by force of 4 N

$\rightarrow 4 = k\cdot(2),\ k = 2$

$mx'' + bx' + kx = 0 \rightarrow 2x'' + bx' + 2x = 0$

initial conditions: $x(0) = 2$ m, $x'(0) = 1$ m/s.

(a) $b = 6$. $2x'' + 6x' + 2x = 0 \rightarrow x'' + 3x' + x = 0$

Characteristic polynomial $P(r) = r^2 + 3r + 1$

roots: $\dfrac{-3 \pm \sqrt{3^2 - 4}}{2} = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ (two distinct real roots)

$$x(t) = c_1 \exp\left(\left(-\tfrac{3}{2} + \tfrac{\sqrt{5}}{2}\right)t\right) + c_2 \exp\left(\left(-\tfrac{3}{2} - \tfrac{\sqrt{5}}{2}\right)t\right)$$

$2 = x(0) = c_1 + c_2$

$1 = x'(0) = c_1\left(-\frac{3}{2} + \frac{\sqrt{5}}{2}\right) + c_2\left(-\frac{3}{2} - \frac{\sqrt{5}}{2}\right) = \sqrt{5}c_1 - 3 - \sqrt{5}$

$\rightarrow c_1 = \frac{4}{\sqrt{5}} + 1,\ c_2 = -\frac{4}{\sqrt{5}} + 1$

$$\boxed{x(t) = \left(\tfrac{4}{\sqrt{5}} + 1\right)\exp\left(\left(-\tfrac{3}{2} + \tfrac{\sqrt{5}}{2}\right)t\right) + \left(-\tfrac{4}{\sqrt{5}} + 1\right)\exp\left(\left(-\tfrac{3}{2} - \tfrac{\sqrt{5}}{2}\right)t\right)}$$

(b) $b = 4$. $2x'' + 4x' + 2x = 0 \rightarrow x'' + 2x' + x = 0$

Char. poly. $P(r) = r^2 + 2r + 1 = (r+1)^2$ root: -1, multiplicity 2

$x(t) = c_1 e^{-t} + c_2 t e^{-t}$

$2 = x(0) = c_1 \quad 1 = x'(0) = -c_1 + c_2 \rightarrow c_2 = 3$

$$\boxed{x(t) = 2e^{-t} + 3te^{-t}}$$

(c) $b = 2$. $2x'' + 2x' + 2x = 0 \rightarrow x'' + x' + x = 0$ complex conjugate pair

Char. Poly. $P(r) = r^2 + r + 1$ roots: $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$x(t) = c_1 e^{-t/2}\cos(t\sqrt{3}/2) + c_2 e^{-t/2}\sin(t\sqrt{3}/2)$

$2 = x(0) = c_1 \quad 1 = x'(0) = -\frac{c_1}{2} + \frac{c_2\sqrt{3}}{2} = -1 + \frac{c_2\sqrt{3}}{2} \rightarrow c_2 = 4/\sqrt{3}$

$$\boxed{x(t) = 2e^{-t/2}\cos(t\sqrt{3}/2) + \tfrac{4}{\sqrt{3}}e^{-t/2}\sin(t\sqrt{3}/2)}$$

3. (a) $y'' + 4y = e^{5t}$

$f(t) = e^{5t}$ root $= 5$ $P_f(r) = r - 5$

$y'' + 4y = 0$ has $P(r) = r^2 + 4r = (r+2i)(r-2i)$

$\Big\}$ $P(r)$, $P_f(r)$ have no common roots

$\longrightarrow y_p(t) = ce^{5t}$

$y_p'' + 4y_p = e^{5t} \longrightarrow 25ce^{5t} + 4ce^{5t} = e^{5t} \to 29c = 1,\ c = \frac{1}{29}$

Thus $\boxed{y_p(t) = \frac{1}{29}e^{5t}}$

(b) $4y'' + 4y' + y = 3te^t$

$f(t) = 3te^t$ ← comes from root $= 1$ of multiplicity 2; $P_f(r) = (r-1)^2$

$4y'' + 4y' + y = 0$ has char. poly. $P(r) = 4r^2 + 4r + 1 = 4(r + \frac{1}{2})^2$

root $= -\frac{1}{2}$, multiplicity 2

$P_f(r)$, $P(r)$ have no common roots $\longrightarrow y_p(t) = c_1e^t + c_2te^t$

$4y_p'' + 4y_p' + y_p = 3te^t \to 4((c_1 + 2c_2)e^t + c_2te^t) + 4((c_1 + c_2)e^t + c_2te^t)$
$+ (c_1e^t + c_2te^t) = 3te^t$

$\to \begin{cases} 4(c_1 + 2c_2) + 4(c_1 + c_2) + c_1 = 0 \\ 4c_2 + 4c_2 + c_2 = 3 \to c_2 = 1/3 \end{cases}$

$\to c_1 = -4/9.$

$$\boxed{y_p(t) = -\frac{4}{9}e^t + \frac{1}{3}te^t}$$

(c) $y'' + 4y' + 2y = t^2$

$f(t) = t^2$ ← comes from root $= 0$, mult. 3 $\to P_f(r) = r^3$

$y'' + 4y' + 2y = 0 \to P(r) = r^2 + 4r + 2$ roots: $-2 \pm \sqrt{2}$

$\Big\}$ no common roots

$\rightarrow y_p(t) = c_1 + c_2 t + c_3 t^2$

$y_p'' + 4y_p' + 2y_p = t^2 \rightarrow 2c_3 + 4(c_2 + 2c_3 t) + 2(c_1 + c_2 t + c_3 t^2) = t^2$

$(2c_3 + 4c_2 + 2c_1) + (8c_3 + 2c_2)t + (2c_3)t^2 = t^2$

$$\rightarrow \begin{cases} 2c_3 + 4c_2 + 2c_1 = 0 \\ 8c_3 + 2c_2 = 0 \\ 2c_3 = 1 \rightarrow c_3 = 1/2 \end{cases}$$

2^{nd} eq: $8c_3 + 2c_2 = 8(\frac{1}{2}) + 2c_2 = 4 + 2c_2 = 0 \rightarrow c_2 = -2$

1^{st} eq: $2c_3 + 4c_2 + 2c_1 = 2(\frac{1}{2}) + 4(-2) + 2c_1$

$= 1 - 8 + 2c_1 = -7 + 2c_1 = 0$

$\rightarrow c_1 = 7/2$

$$\boxed{y_p(t) = \frac{7}{2} - 2t + \frac{1}{2}t^2}$$

(d) $y'' + 9y = \cos(3t) + 4\sin(3t)$

$f(t) = \cos(3t) + 4\sin(3t)$ ← comes from roots $\pm i3$

$P_f(r) = (r - i3)(r + i3) = r^2 + 9$

$y'' + 9y = 0$ has char. poly $P(r) = r^2 + 9$

General solution associated to $P_f(r)P(r) = (r^2+9)^2 = (r-i3)^2(r+i3)^2$

involves:

$\cos(3t)$, $\sin(3t)$, $t\cos(3t)$, $t\sin(3t)$

However, $\cos(3t), \sin(3t)$ are solutions of $y'' + 9y = 0$ so forget them

$y_p(t) = c_1 t\cos(3t) + c_2 t\sin(3t)$

$y_p'(t) = c_1\cos(3t) - 3c_1 t\sin(3t) + c_2\sin(3t) + 3c_2 t\cos(3t)$

$y_p''(t) = -3c_1\sin(3t) - 3c_1\sin(3t) - 9c_1 t\cos(3t) + 3c_2\cos(3t)$

$+ 3c_2\cos(3t) - 9c_2 t\sin(3t)$

$$y_p'' + 9y_p = \cos(3t) + 4\sin(3t)$$

$$\to -6c_1\sin(3t) + 6c_2\cos(3t) - 9c_1\cancel{t\cos(3t)} - 9\cancel{c_2 t\sin(3t)}$$
$$+ 9\left(\cancel{c_1 t\cos(3t)} + c_2 t\cancel{\sin(3t)}\right) = \cos(3t) + 4\sin(3t)$$

$$\to -6c_1 = 4, \quad 6c_2 = 1 \quad \to \quad c_1 = -2/3, \quad c_2 = 1/6$$

$$\boxed{y_p(t) = -\frac{2}{3}t\cos(3t) + \frac{1}{6}t\sin(3t)}$$

$\boxed{4.}$ (a) $y'' - 2y' + 2y = e^t\sin(t)$

$f(t) = e^t\sin(t)$ ← comes from root $1 \pm i$, $P_f(r) = (r-1+i)(r-1-i)$
$$= r^2 - 2r + 2$$

$y'' - 2y' + 2y = 0$ has char. poly $P(r) = r^2 - 2r + 2$

General solution associated to $P(r)P_f(r) = (r^2-2r+2)^2$ involves

$$e^t\cos(t), \; e^t\sin(t), \; te^t\cos(t), \; te^t\sin(t)$$

But the 1st two solve $y'' - 2y' + 2y = 0$, so we set

$$y_p(t) = c_1 te^t\cos(t) + c_2 te^t\sin(t)$$

Then
$$y_p'(t) = c_1 e^t\cos(t) + c_1 te^t\cos(t) - c_1 te^t\sin(t)$$
$$+ c_2 e^t\sin(t) + c_2 te^t\sin(t) + c_2 te^t\cos(t)$$
$$= (c_1 e^t + c_1 te^t + c_2 te^t)\cos(t) + (c_2 e^t + c_2 te^t - c_1 te^t)\sin(t)$$

$$y_p''(t) = (2c_1 e^t + \cancel{c_1 te^t} + c_2 e^t + c_2 te^t)\cos(t)$$
$$- (c_1 e^t + c_1 te^t + \cancel{c_2 te^t})\sin(t)$$
$$+ (2c_2 e^t + \cancel{c_2 te^t} - c_1 e^t - c_1 te^t)\sin(t)$$
$$+ (c_2 e^t + c_2 te^t - \cancel{c_1 te^t})\cos(t)$$

$y_p'' - 2y_p' + 2y_p = e^t \sin(t)$ yields

$$\left(2(\cancel{c_1}+c_2)e^t + \cancel{2c_2te^t}\right)\cos(t) + \left(2(\cancel{c_2}-c_1)e^t - \cancel{2c_1te^t}\right)\sin(t) \quad \} \, y_p''$$

$$-2\left(\left(\cancel{c_1e^t} + \cancel{(c_1}+\cancel{c_2)te^t}\right)\cos(t) + \left(\cancel{c_2e^t} + \cancel{(c_2-c_1)te^t}\right)\sin(t)\right) \quad \} \, -2y_p'$$

$$+2\left(\cancel{c_1te^t}\cos(t) + \cancel{c_2te^t}\sin(t)\right) \quad \} \, 2y_p$$

$$= e^t \sin(t).$$

$$\rightarrow \quad 2c_2e^t\cos(t) - 2c_1e^t\sin(t) = e^t\sin(t)$$

$$\rightarrow \quad c_2 = 0, \; c_1 = -\tfrac{1}{2} \quad \rightarrow \quad y_p(t) = -\frac{t}{2}e^t\cos(t).$$

General solution: $\boxed{y(t) = -\frac{t}{2}e^t\cos(t) + c_1e^t\cos(t) + c_2e^t\sin(t)}$

(b) $y^{(4)} - 4y'' + 4y = e^t - te^{2t}$

$f(t) = e^t - te^{2t}$ comes from roots 1 (mult. 1) and 2 (mult. 2)

$$P_f(r) = (r-1)(r-2)^2$$

$$y^{(4)} - 4y'' + 4y = 0 \rightarrow P(r) = r^4 - 4r^2 + 4 = (r^2-2)^2 = (r-\sqrt{2})^2(r+\sqrt{2})^2$$

$P(r)$, $P_f(r)$ have no common roots. $\rightarrow$ $y_p(t) = c_1e^t + c_2e^{2t} + c_3te^{2t}$

$$y_p' = c_1e^t + 2c_2e^{2t} + c_3e^{2t} + 2c_3te^{2t}$$

$$y_p'' = c_1e^t + 2(2c_2+c_3)e^{2t} + 2c_3e^{2t} + 4c_3te^{2t}$$

$$y_p''' = c_1e^t + 2(4c_2+4c_3)e^{2t} + 4c_3e^{2t} + 8c_3te^{2t}$$

$$y_p^{(4)} = c_1e^t + 2(8c_2+12c_3)e^{2t} + 8c_3e^{2t} + 16c_3te^{2t}$$

$$y^{(4)} - 4y'' + 4y = c_1 e^{t} + (16c_2 + 32c_3)e^{2t} + 16c_3 te^{2t}$$
$$-4\left(c_1e^{t} + 4(c_2+c_3)e^{2t} + 4c_3 te^{2t}\right)$$
$$+4\left(c_1e^{t} + c_2e^{2t} + c_3te^{2t}\right) = e^{t} - te^{2t}$$

Equating coefficients,

e^t terms: $c_1 - 4c_1 + 4c_1 = 1 \to c_1 = 1$

e^{2t} terms: $\cancel{16c_2} + 32c_3 - \cancel{16c_2} - 16c_3 + 4c_2 = 0 \to c_2 = -4c_3$

te^{2t} terms: $\cancel{16c_3} - \cancel{16c_3} + 4c_3 = -1 \to c_3 = -1/4 \to c_2 = 1$

$y_p(t) = e^{t} + e^{2t} - \frac{1}{4}te^{2t}$

General solution: $\boxed{y(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + c_3 te^{\sqrt{2}t} + c_4 te^{-\sqrt{2}t} + e^{t} + e^{2t} - \frac{t}{4}e^{2t}}$

5. $mx'' + kx = \cos(t)$, $m = 1$, $2\,N = k\cdot(2m) \to k = 1$

$\to x'' + x = \cos(t)$. Initial conditions $x(0) = -1$, $x'(0) = 1$ m/s

$f(t) = \cos(t) \leftarrow$ comes from root $\pm i$, so $P_f(r) = (r-i)(r+i) = r^2+1$.

$x'' + x = 0$ has char. poly. $P(r) = r^2+1$. General solution for

$P(r)P_f(r) = (r+1)^2$ includes $\cancel{\cos(t)}$, $\cancel{\sin(t)}$, $t\cos(t)$, $t\sin(t)$.

$\to x_p(t) = c_1 t\cos(t) + c_2 t\sin(t)$. $x_p' = c_1\cos(t) - c_1 t\sin(t) + c_2\sin(t) + c_2 t\cos(t)$

$$x_p'' = -c_1\sin(t) - c_1\sin(t) - c_1 t\cos(t) + c_2\cos(t) + c_2\cos(t) - c_2 t\sin(t)$$

$$x_p'' + x_p = \cos(t) \to -2c_1\sin(t) - \cancel{c_1 t\cos(t)} + 2c_2\cos(t) - \cancel{c_2 t\sin(t)}$$
$$+ \cancel{c_1 t\cos(t)} + \cancel{c_2 t\sin(t)} = \cos(t)$$

$\to c_1 = 0$, $c_2 = \frac{1}{2}$ $\to x_p(t) = \frac{1}{2}t\sin(t) \nearrow$ $x(t) = c_1\cos t + c_2\sin t + \frac{t}{2}\sin t$ use IC's $\to$ $\boxed{-\cos t + \sin t + \frac{t}{2}\sin t}$