HOMEWORK 5 SOLUTIONS

1. $y_1(t) = \sqrt{t}$ $\quad y_1'(t) = \frac{1}{2}\frac{1}{\sqrt{t}}$ $\quad y_1''(t) = -\frac{1}{4}t^{-3/2}$

Therefore $y_1 \cdot y_1'' + (y_1')^2 = (t^{1/2})\left(-\frac{1}{4}t^{-3/2}\right) + \left(\frac{1}{2}t^{-1/2}\right)^2 = -\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = 0.$ ✓

$y_2(t) = 1.$ $\quad y_2' = 0,\ y_2'' = 0.$ $\quad y_2 \cdot y_2'' + (y_2')^2 = (1)\cdot(0) + (0)^2 = 0.$ ✓

We have shown y_1, y_2 are solutions to $yy'' + (y')^2 = 0.$

However, $(y_1+y_2)(y_1''+y_2'') + (y_1'+y_2')^2 = (\sqrt{t}+1)\left(-\frac{1}{4}t^{-3/2}\right) + \left(\frac{1}{2}t^{-1/2}\right)^2$

$$= \cancel{-\frac{1}{4}t^{-1}} - \frac{1}{4}t^{-3/2} + \cancel{\frac{1}{4}t^{-1/2}} = -\frac{1}{4}t^{-3/2} \neq 0$$

thus the sum $y_1 + y_2$ is not a solution.

2. (a) $y_1 = e^t\sin(t),\ y_2 = e^t\cos(t)$

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = e^t\sin(t)\cdot\left(e^t\cos(t) - e^t\sin(t)\right) - \left(e^t\sin(t) + e^t\cos(t)\right)\cdot e^t\cos(t)$$

$$= -e^{2t}\left(\sin^2(t) + \cos^2(t)\right) = -e^{2t} \neq 0$$

$\rightarrow$ y_1, y_2 are linearly independent.

(b) $y_1 = \sin^2(t),\ y_2 = 1 - 2\cos(2t)$

Suppose there are constants c_1, c_2 such that $c_1y_1 + c_2y_2 = 0.$

Then at $t = 0$: $\quad c_1\cancel{\sin^2(0)} + c_2(1 - 2\cos(2\cdot 0)) = 0$

$$-c_2 = 0, \text{ so } c_2 = 0.$$

At $t = \frac{\pi}{2}$: $\quad c_1\sin^2\left(\frac{\pi}{2}\right) + c_2(1 - 2\cos(2\pi/2)) = 0$

$$c_1 + 3c_2 = 0, \text{ but } c_2 = 0 \text{ so } c_1 = 0.$$

We conclude c_1, c_2 must be zero. Thus y_1, y_2 are linearly independent.

(c) $y_1 = t^3,\ y_2 = t^2|t|.$ Suppose again there are constants c_1, c_2 so that $c_1y_1 + c_2y_2 = 0.$ At $t = 1$: $c_1(1)^3 + c_2(1)^2|1| = 0 \rightarrow c_1 + c_2 = 0$

$\rightarrow c_1 = -c_2.$

At $t=-1$: $c_1(-1)^3 + c_2(-1)^2|-1| = 0$

$-c_1 + c_2 = 0 \rightarrow c_1 = c_2$. Together with $c_1 = -c_2$ this implies $c_1 = c_2 = 0$.

Thus y_1, y_2 are linearly independent.

3. (a) $y'' - 3y' + 2y = 0$

Characteristic polynomial $P(r) = r^2 - 3r + 2 = (r-1)(r-2)$ two distinct real roots: 1, 2

$\rightarrow \boxed{y(t) = c_1 e^t + c_2 e^{2t}}$

(b) $y'' - 10y' = 0$

$P(r) = r^2 - 10r = r(r-10)$, two distinct real roots: 0, 10

$\rightarrow \boxed{y(t) = c_1 + c_2 e^{10t}}$

(c) $y'' + y' - y = 0$

$P(r) = r^2 + r - 1$ roots: $\dfrac{-1 \pm \sqrt{1-(4)(-1)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ two distinct real roots

$\rightarrow \boxed{y(t) = c_1 \exp\left(\left(-\tfrac{1}{2} + \tfrac{\sqrt{5}}{2}\right)t\right) + c_2 \exp\left(\left(-\tfrac{1}{2} - \tfrac{\sqrt{5}}{2}\right)t\right).}$

(d) $y'' + 2y' + y = 0$

$P(r) = r^2 + 2r + 1 = (r+1)^2$ root is -1, multiplicity 2

$\rightarrow \boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t}}$

4. $y'' + 5y' + 6y = 0$, $y(0) = 0$, $y'(0) = 1$.

$P(r) = r^2 + 5r + 6 = (r+2)(r+3) \rightarrow y(t) = c_1 e^{-2t} + c_2 e^{-3t}$

$0 = y(0) = c_1 + c_2 \rightarrow c_1 = -c_2$

$y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$

$1 = y'(0) = -2c_1 - 3c_2 = -2c_1 + 3c_1 = c_1 \rightarrow c_1 = 1$, $c_2 = -1$

Thus $\boxed{y(t) = e^{-2t} - e^{-3t}}$

5. (a) $3y^{(4)} + 4y^{(3)} = 0$

$P(r) = 3r^4 + 4r^3 = r^3(3r+4)$ roots: 0 (mult. 3), $-\frac{4}{3}$ (mult. 1)

$\rightarrow \boxed{y(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^{-4t/3}}$

(b) $y^{(4)} - 3y^{(3)} + 3y'' - y' = 0$

$P(r) = r^4 - 3r^3 + 3r^2 - r = r(r^3 - 3r^2 + 3r - 1) = r(r-1)^3$

roots: 0 (mult. 1), 1 (mult. 3)

$\rightarrow \boxed{y(t) = c_1 + c_2 e^t + c_2 t e^t + c_3 t^2 e^t.}$

6. (a) $y'' - 2y' + 2y = 0$

$P(r) = r^2 - 2r + 2$ roots: $\dfrac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm i$

$\rightarrow \boxed{y(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t)}$

(b) $y'' + y' + y = 0$ $P(r) = r^2 + r + 1$ roots: $\dfrac{-1 \pm \sqrt{1^2 - 4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\rightarrow \boxed{y(t) = c_1 e^{-t/2} \cos(\sqrt{3}t/2) + c_2 e^{-t/2} \sin(\sqrt{3}t/2).}$

7. (a) $y(t) = c_1 e^{-5t} + c_2 t e^{-5t}$

comes from root -5 of mult. 2. So $P(r) = (r+5)^2 = r^2 + 10r + 25$.

Thus y solves $\boxed{y'' + 10y' + 25y = 0.}$

(b) $y(t) = e^t(c_1 e^{t\sqrt{3}} + c_2 e^{-t\sqrt{3}})$ comes from roots $1 \pm \sqrt{3}$

So $P(r) = (r - 1 - \sqrt{3})(r - 1 + \sqrt{3}) = r^2 - 2r - 2 \rightarrow \boxed{y'' - 2y' - 2y = 0.}$

(c) $y(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$

Comes from roots $1 \pm 2i$ so $P(r) = (r - 1 - 2i)(r - 1 + 2i)$

$= r^2 - 2r + 5$

$\rightarrow \boxed{y'' - 2y' + 5y = 0.}$