# Homework #4 solutions

1. (a) $y' = -8ty$, $y(0)=2$ $\quad \Delta t = \frac{1}{4}$

$t_0 = 0 \qquad y_0 = 2$

$t_1 = \frac{1}{4} \qquad y_1 = y_0 + \Delta t \cdot (-8 t_0 \cdot y_0) = 2 + \cancel{(\tfrac{1}{4})(-8 \cdot 0 \cdot 2)} = 2$

$t_2 = \frac{1}{2} \qquad y_2 = y_1 + \Delta t \cdot (-8 t_1 \cdot y_1) = 2 + (\tfrac{1}{4})(-8 \cdot \tfrac{1}{4} \cdot 2) = 2 - 1 = 1$

$t_3 = \frac{3}{4} \qquad y_3 = y_2 + \Delta t \cdot (-8 t_2 \cdot y_2) = 1 + (\tfrac{1}{4})(-8 \cdot \tfrac{1}{2} \cdot 1) = 1 - 1 = 0$

$t_4 = 1 \qquad y_4 = y_3 + \Delta t \cdot (-8 t_3 \cdot y_3) = 0 + \cancel{(\tfrac{1}{4})(-8 \cdot \tfrac{3}{4} \cdot 0)} = 0$

$y_4 = \boxed{0 \approx y(1).}$

(b) separable eq. $y' = -8ty \qquad \int \frac{dy}{y} = \int -8t\,dt$

$$\ln|y| = -4t^2 + C$$

$$y = ke^{-4t^2}$$

$$y(0) = 2 \Rightarrow 2 = ke^{-0} = k$$

$$y(t) = 2e^{-4t^2}.$$

$$\boxed{y(1) = 2e^{-4} \approx 0.0366}$$

2. $\frac{dP}{dt} = aP - bP^2$ $\quad$ $B = aP$ birth rate $\quad$ ($a, b$ constants)
$D = bP^2$ death rate

General logistic:
$P' = kP(M-P)$, $M$ = carrying capacity, or limiting population.

$$aP - bP^2 = kP(M-P) = -kP^2 + kMP \Rightarrow k = b,\ kM = a$$

Then $M = \frac{kM}{k} = \boxed{\frac{a}{b}}$

3. At $t=0$, birth rate $= 6$ /month $\Rightarrow$ $6 = B(0) = aP_0$
death rate $= 2$ /month $\Rightarrow$ $2 = D(0) = bP_0^2$

Initial population is $P_0 = 30$ ibis. Thus $a = \frac{6}{30} = \frac{1}{5}$

$b = \frac{2}{30^2} = \frac{1}{450}$

Then $M = \frac{a}{b} = \frac{1/5}{1/450} = 90.$

General solution (from class or text):

$$P(t) = \frac{P_0 M}{(M-P_0)e^{-Mkt} + P_0}$$

Plugging all constants in we obtain

$$P(t) = \frac{90 \cdot 30}{(90-30)e^{-90 \cdot \frac{1}{450} \cdot t} + 30} = \frac{90}{2e^{-t/5} + 1}.$$

$\frac{2}{3}$ of limiting population $= \frac{2}{3} \cdot 90 = 60.$

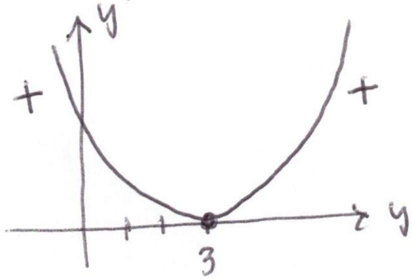
Find t: $\quad 60 = \dfrac{90}{2e^{-t/5}+1} \qquad \dfrac{2}{3} = \dfrac{1}{2e^{-t/5}+1}$

$2e^{-t/5}+1 = 3/2 \qquad 2e^{-t/5} = \frac{1}{2} \qquad e^{-t/5} = \frac{1}{4}$

$-t/5 = \ln\left(\frac{1}{4}\right) \qquad t = -5\ln\left(\frac{1}{4}\right) = \boxed{5\ln(4) \approx 6.931 \text{ months}}$

4. $y' = (y-3)^2$

(a) equilibria: $y(t) = 3$

(b) 

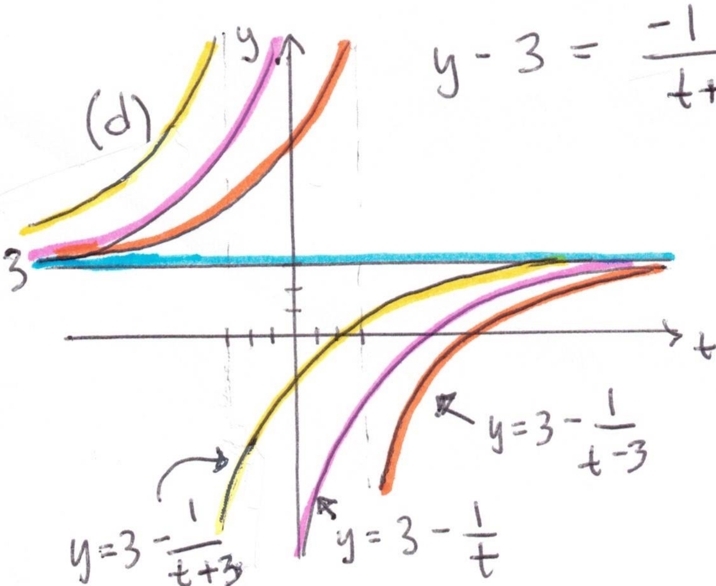

Phase diagram 


$\boxed{\text{neither stable / unstable}}$

(c) separable $\displaystyle\int \frac{dy}{(y-3)^2} = \int dt \qquad -\frac{1}{y-3} = t + C$

$y - 3 = \dfrac{-1}{t+C} \qquad \boxed{y(t) = 3 - \dfrac{1}{t+C}}$

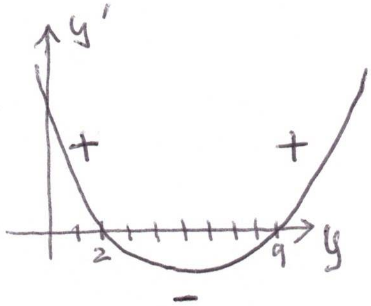
$\boxed{\text{and the equilibrium } y(t) = 3.}$

(d) 

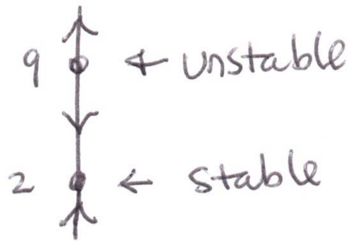
5. $y' = -18 + 11y - y^2 = -(y-2)(y-9)$

(a) equilibria: $\boxed{y(t) = 2,\ y(t) = 9}$

(b)




Phase diagram




(c)

$$-dt = \frac{dy}{(y-2)(y-9)} = -\frac{1}{7}dy\left(\frac{1}{y-2} - \frac{1}{y-9}\right)$$

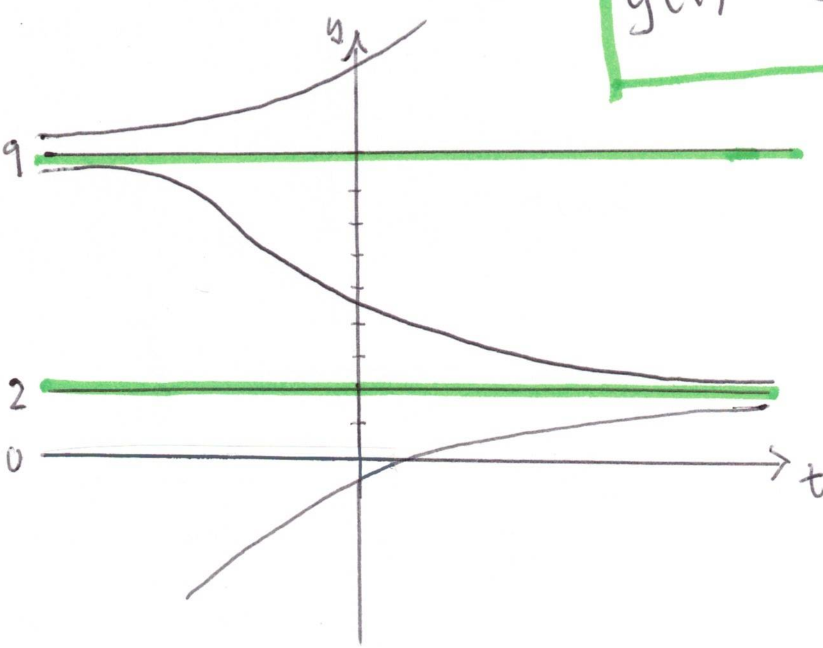
$$7t + C = 7\int dt = \int \frac{dy}{y-2} - \int \frac{dy}{y-9} = \ln|y-2| - \ln|y-9| = \ln\left|\frac{y-2}{y-9}\right|$$

$$\frac{y-2}{y-9} = ke^{7t} \qquad y - 2 = (y-9)ke^{7t} = yke^{7t} - 9ke^{7t}$$

$$y(1-ke^{7t}) = 2 - 9ke^{7t}$$

$$\boxed{y(t) = \frac{2-9ke^{7t}}{1-ke^{7t}}}$$

(d)




(Sketch is fine here.)

6. $\frac{dP}{dt} = \frac{1}{2}P(3-P) - H = -\frac{1}{2}P^2 + \frac{3}{2}P - H$ (assume $H \geq 0$)

equilibria: $\frac{-\frac{3}{2} \pm \sqrt{\frac{9}{4} - 4(-\frac{1}{2})(-H)}}{2(-\frac{1}{2})} = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2H}$

Let $\left.\begin{aligned} A &= \frac{3}{2} - \sqrt{\frac{9}{4} - 2H} \\ B &= \frac{3}{2} + \sqrt{\frac{9}{4} - 2H} \end{aligned}\right\}$ distinct when $0 \leq H < \frac{9}{8}$

$\frac{3}{2} = A = B$ when $H = 9/8$. No (real) equilibria when $H > \frac{9}{8}$.