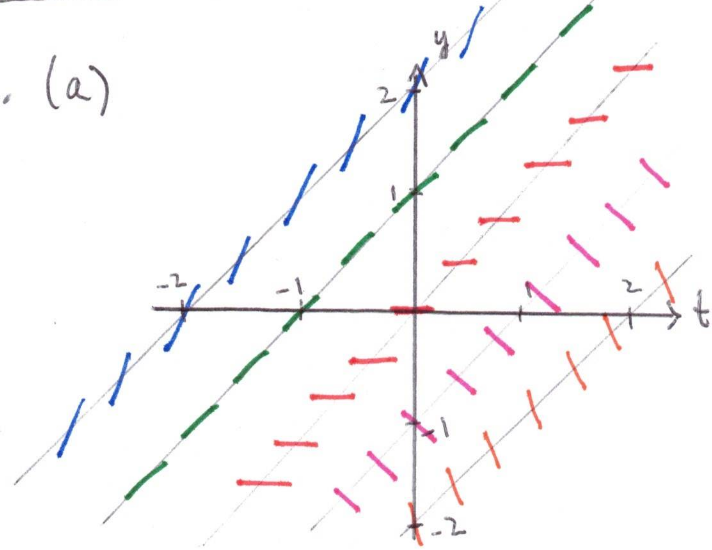
Homework #3 Solutions

1. (a)



slope field for $y' = y - t$

isoclines

$y' = 0$: $y = t$

$y' = 1$: $y = 1 + t$

$y' = 2$: $y = 2 + t$

$y' = -1$: $y = -1 + t$

$y' = -2$: $y = -2 + t$

(b) Solve $y' = y$ first: $\int \frac{dy}{y} = \int dt \quad \ln|y| = t + C \quad y = ke^t$

Variation of parameters: set $y(t) = k(t)e^t$

$$y' = y - t \longrightarrow k'(t)e^t + \cancel{k(t)e^t} = \cancel{k(t)e^t} - t$$

$$k'(t) = -te^{-t}$$

$$k(t) = \int k'(t)dt = \int \underset{u}{(t)}\underset{v'}{(-e^{-t})} = \underset{u}{(t)}\underset{v}{(e^{-t})} - \int \underset{u'}{(1)} \cdot \underset{v}{(e^{-t})}dt$$

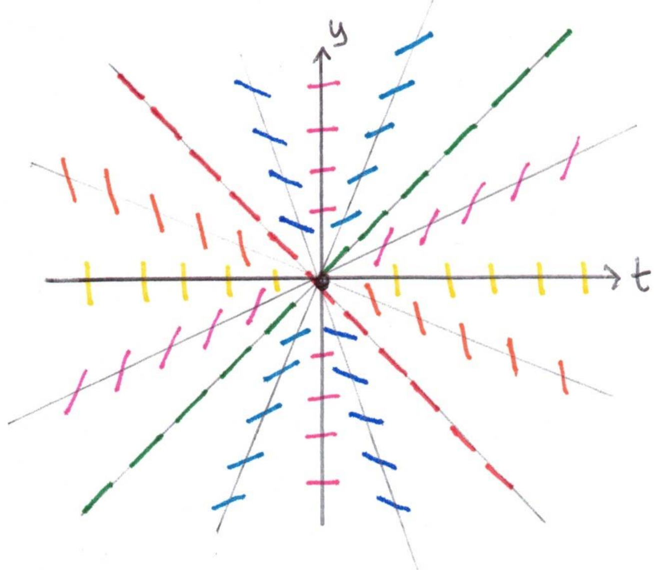
$$= te^{-t} + e^{-t} + C$$

$$y(t) = k(t)e^t = (te^{-t} + e^{-t} + C)e^t = \boxed{t + 1 + Ce^t}$$

(c) The solution which is a line is $y(t) = 1 + t$ (obtained from general solution by choosing $C = 0$)

It is the green isocline in (a).

2. (a)



Slope field for $y' = t/y$ omit $(0,0)$

isoclines

$y' = 1$: $y = t$

$y' = \frac{1}{2}$: $y = 2t$

$y' = -1$: $y = -t$

$y' = 2$: $y = \frac{1}{2}t$

$y' = -\frac{1}{2}$: $y = -2t$

$y' = -2$: $y = -\frac{1}{2}t$

$y \neq 0,\ t = 0 \Rightarrow y' = 0$

(can also draw vertical slope lines for:
$y = 0,\ t \neq 0 \Rightarrow y' = \pm\infty$)

(b) separable: $\int y\,dy = \int t\,dt$ $\quad \frac{1}{2}y^2 = \frac{1}{2}t^2 + C$

(no constant solns.) $\quad \boxed{y = \pm\sqrt{t^2 + C_1}}$

(c)

The two solutions which are lines are

$y(t) = t$ and $y(t) = -t$

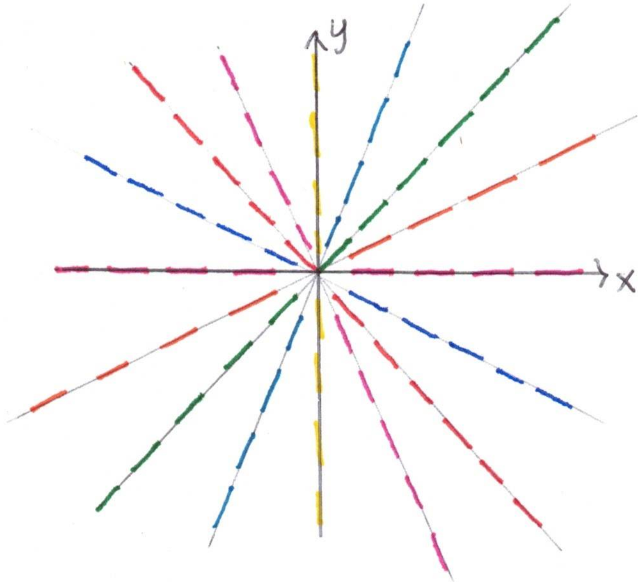
obtained from general solution by taking $\pm = +$ and $C_1 = 0$

This is the green isocline in (a).

obtained from general solution by taking $\pm = -$ and $C_1 = 0$

This is the red isocline in (a).

3. (a)



Slope field for $y' = y/x$ omit $(0,0)$

isoclines

$y' = 1$: $y = x$

$y' = 2$: $y = 2x$

$y' = \frac{1}{2}$: $y = \frac{1}{2}x$

$y' = -1$: $y = -x$

$y' = -2$: $y = -2x$

$y' = -\frac{1}{2}$: $y = -\frac{1}{2}x$

$x \neq 0,\ y = 0 \Rightarrow y' = 0$

(can also draw vertical slope lines for: $x = 0,\ y \neq 0 \Rightarrow y' = \pm\infty$)

(b) Separable: $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x}$ $\quad \ln|y| = \ln|x| + C$

$y = \pm e^{C} \cdot x = kx$ for $k \neq 0$ constant

Also have the constant solution $y(x) = 0$ which fits into above family if we allow $k = 0$.

So general solution is $\boxed{y(x) = kx}$ k any constant

(c)

All solutions pass through the origin.

Thus infinitely many solutions, $y(x) = kx$, as k ranges over all real numbers, pass through the origin!

4.

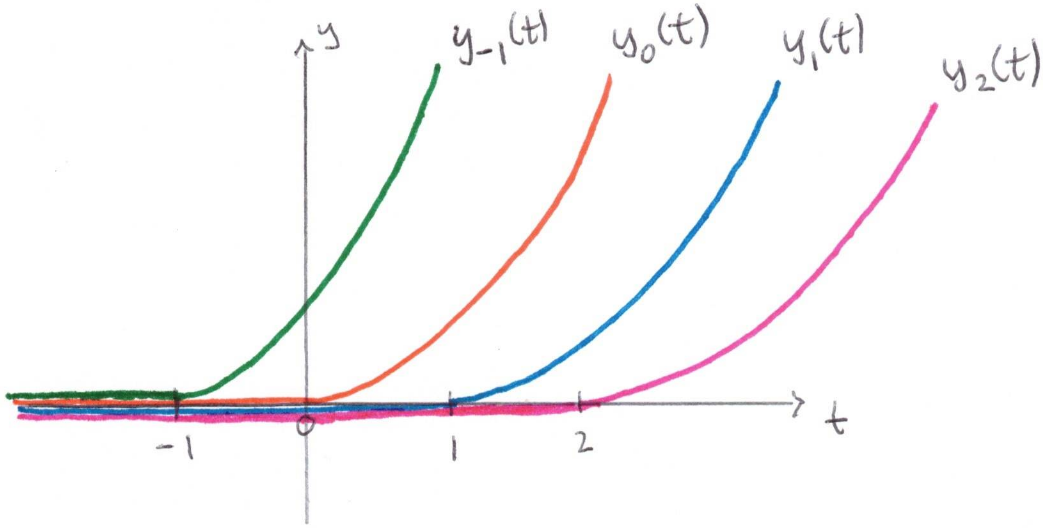
$$y_c(t) = \begin{cases} 0 & t \le c \\ (t-c)^2 & t > c \end{cases} \qquad (c \text{ any fixed real number})$$

(a)

$$y_c'(t) = \begin{cases} 0 & t \le c \\ 2(t-c) & t > c \end{cases}.$$ This is clearly equal to $2\sqrt{y_c(t)}$.

Therefore $y_c'(t) = 2\sqrt{y_c(t)}$ as desired.

(b)



(c)

There are infinitely many solutions to $y' = 2\sqrt{y}$ which satisfy $y(0)=0$: the functions $y_c(t)$ with $c \geq 0$ all satisfy $y(0)=0$.

In particular, $y_0(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$, $y_4(t)$, ... are all examples of solutions to $y' = 2\sqrt{y}$, $y(0)=0$.