Final Exam Practice (Solutions)

1(a) $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$ eigenvalues: $\det(A - \lambda I) = \det\begin{pmatrix} 5-\lambda & 6 \\ -2 & -2-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2$

$= (\lambda-1)(\lambda-2)$

$\lambda_1 = 1,\ \lambda_2 = 2$

eigenvectors: $\lambda_1 = 1$: $(A - \lambda_1 I)v_1 = 0$

Let $v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $\begin{pmatrix} 4 & 6 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $4a + 6b = 0 \Rightarrow v_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$\lambda_2 = 2$: $\begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{v_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $3a + 6b = 0 \Rightarrow v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

General solution: $\boxed{\vec{x}(t) = c_1 e^{t}\begin{pmatrix} 3 \\ -2 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.}$

1(b) $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ eigenvalues: $\det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^2$; one eigenvalue $\lambda_1 = 3$ (mult. 2)

eigenvectors: $\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{(A-\lambda_1 I)}\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{v_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $b = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Only one eigenvector up to scaling.

Need a generalized eigenvector w_1: $(A - \lambda_1)w_1 = v_1$. Let $w_1 = \begin{pmatrix} a \\ b \end{pmatrix}$.

$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $b = 1 \Rightarrow w_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

General solution: $\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\left(t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)e^{3t} = \boxed{\begin{pmatrix} c_1 e^{3t} + c_2 t e^{3t} \\ c_2 e^{3t} \end{pmatrix}}$

1(c) $A = \begin{pmatrix} 0 & 1 \\ \frac{3}{4} & -1 \end{pmatrix}$ eigenvalues: $\det\begin{pmatrix} -\lambda & 1 \\ 3/4 & -1-\lambda \end{pmatrix} = \lambda^2 + \lambda - \frac{3}{4} = (\lambda + \frac{3}{2})(\lambda - \frac{1}{2})$

$\lambda_1 = -\frac{3}{2},\ \lambda_2 = \frac{1}{2}$

eigenvectors: $\lambda_1 = -\frac{3}{2}$: $\begin{pmatrix} 3/2 & 1 \\ 3/4 & 1/2 \end{pmatrix}\overbrace{\begin{pmatrix} a \\ b \end{pmatrix}}^{v_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\frac{3}{2}a + b = 0 \Rightarrow v_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$\lambda_2 = \frac{1}{2}$: $\begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{4} & -\frac{3}{2} \end{pmatrix}\overbrace{\begin{pmatrix} a \\ b \end{pmatrix}}^{v_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $-\frac{a}{2} + b = 0 \Rightarrow v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

General solution: $\vec{x}(t) = c_1 e^{-3/2 t} \begin{pmatrix} 2 \\ -3 \end{pmatrix} + c_2 e^{t/2} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

1(d) $A = \begin{pmatrix} -1 & -5 \\ 1 & 1 \end{pmatrix}$ eigenvalues: $\det \begin{pmatrix} -1-\lambda & -5 \\ 1 & 1-\lambda \end{pmatrix} = \lambda^2 + 4$, roots $\pm 2i$. Let $\lambda_1 = 2$

eigenvector for $\lambda_1 = 2i$: $\begin{pmatrix} -1-2i & -5 \\ 1 & -1-2i \end{pmatrix} \overset{v_1}{\begin{pmatrix} a \\ b \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (-1-2i)a - 5b = 0$

$\Rightarrow v_1 = \begin{pmatrix} -1+2i \\ 1 \end{pmatrix}$.

General soln: $\vec{x}(t) = c_1 \operatorname{Re}(e^{\lambda_1 t} \cdot v_1) + c_2 \operatorname{Im}(e^{\lambda_1 t} \cdot v_1)$

where $e^{\lambda_1 t} \cdot v_1 = e^{2it} \begin{pmatrix} -1+2i \\ 1 \end{pmatrix} = (\cos(2t) + i\sin(2t)) \begin{pmatrix} -1+2i \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + i \begin{pmatrix} 2\cos(2t) - \sin(2t) \\ \sin(2t) \end{pmatrix}$

Thus $\vec{x}(t) = \boxed{c_1 \begin{pmatrix} -\cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 \begin{pmatrix} 2\cos(2t) - \sin(2t) \\ \sin(2t) \end{pmatrix}}$

1(e) $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ eigenvalues: $\det \begin{pmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 5$

roots: $\dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$, Let $\lambda_1 = 2 + i$.

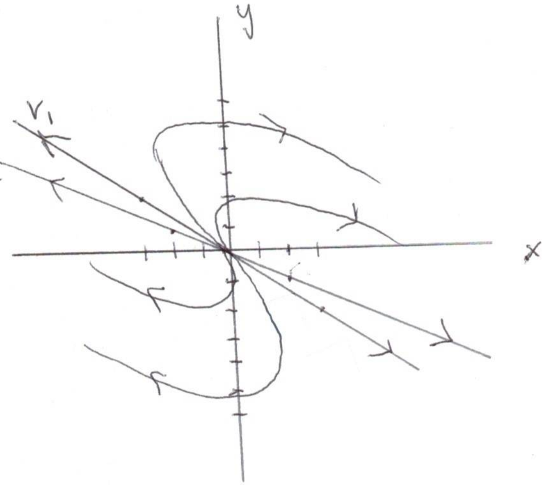
Eigenvector: $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \underset{v_1}{\begin{pmatrix} a \\ b \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $-ia - b = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Gen. solution: $\vec{x}(t) = c_1 \operatorname{Re}(e^{\lambda_1 t} \cdot v_1) + c_2 \operatorname{Im}(e^{\lambda_1 t} \cdot v_1)$ where

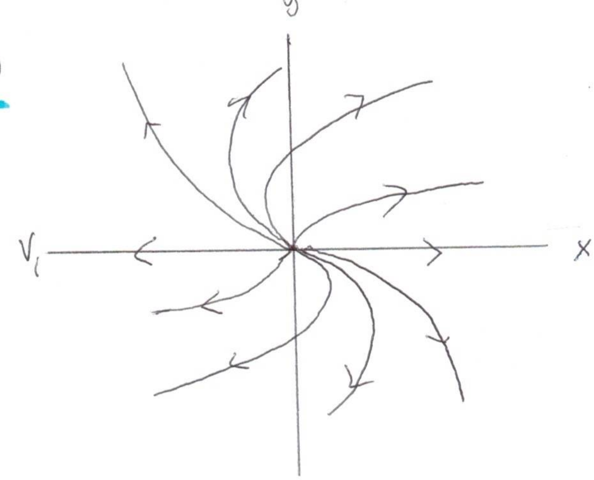
$e^{\lambda t} \cdot v_1 = e^{(2+i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{2t} (\cos(t) + i\sin(t)) \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{2t} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + i e^{2t} \begin{pmatrix} \sin \\ -c \end{pmatrix}$

Thus $\vec{x}(t) = \boxed{c_1 e^{2t} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin(t) \\ -\cos(t) \end{pmatrix}}$

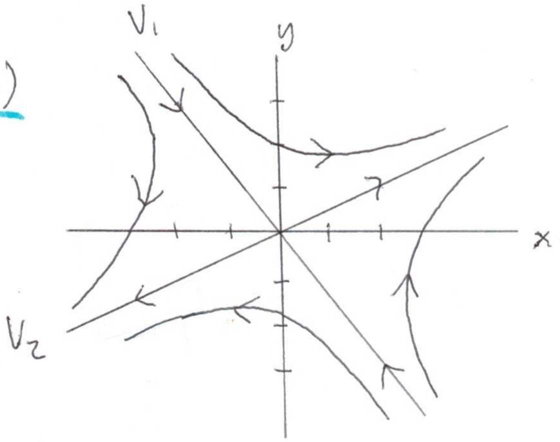
2(a)



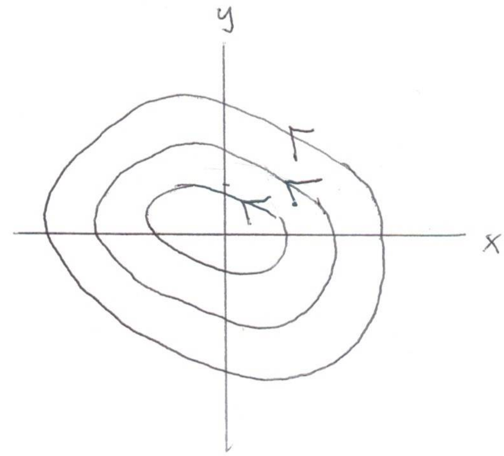
2(b)



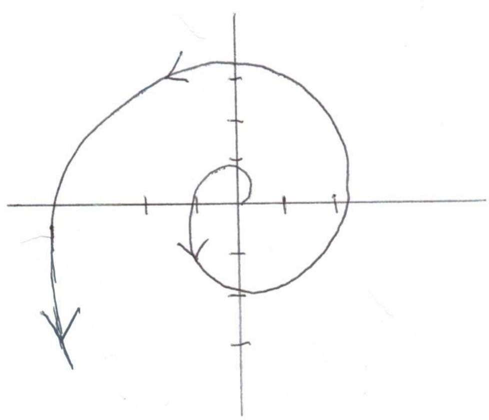
2(c)



2(d)



2(e)



3. $\begin{cases} x' = -4x + 4xy \\ y' = 2y - y^2 - xy \end{cases}$

(a) Find equilibria: $\begin{cases} 0 = -4x + 4xy = 4x(y-1) \\ 0 = 2y - y^2 - xy = y(2-y-x) \end{cases}$

Either $x=0$ or $y=1$. If $x=0$, either $y=0$ or $2-y=0 \Rightarrow y=2$.
If $y=1$ then $2-1-x=0 \Rightarrow x=1$.

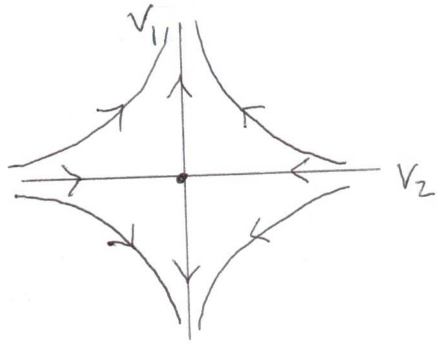
Thus the equilibria are: $\boxed{(0,0),\ (0,2),\ (1,1).}$

(b) Linearize near each equilibrium. $A(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -4+4y & 4x \\ -y & 2-2y-x \end{pmatrix}$

$(0,0)$: $A(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}$. eigenvalues: $\det\begin{pmatrix} -4-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = (-4-\lambda)(2-\lambda)$, $\lambda_1 = 2$, $\lambda_2 = -4$

eigenvectors: $\lambda_1 = 2$: $\begin{pmatrix} -6 & 0 \\ 0 & 0 \end{pmatrix}\underset{v_1}{\begin{pmatrix} a \\ b \end{pmatrix}} = 0$, $-6a = 0 \Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2 = -4$: $\begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}\underset{v_2}{\begin{pmatrix} a \\ b \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $6b = 0 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

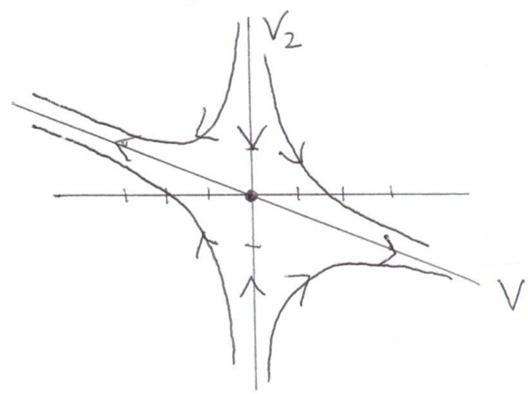


$(0,2)$: $A(0,2) = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}$. eigenvalues: $\det\begin{pmatrix} 4-\lambda & 0 \\ -2 & -2-\lambda \end{pmatrix} = (4-\lambda)(-2-\lambda)$

$\lambda_1 = 4$, $\lambda_2 = -2$

eigenvectors: $\lambda_1 = 4$: $\begin{pmatrix} 0 & 0 \\ -2 & -6 \end{pmatrix}\overset{v_1}{\begin{pmatrix} a \\ b \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $-2a-6b = 0 \Rightarrow v_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\lambda_2 = -2$: $\begin{pmatrix} 6 & 0 \\ -2 & 0 \end{pmatrix}\underset{v_2}{\begin{pmatrix} a \\ b \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $6a = 0 \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

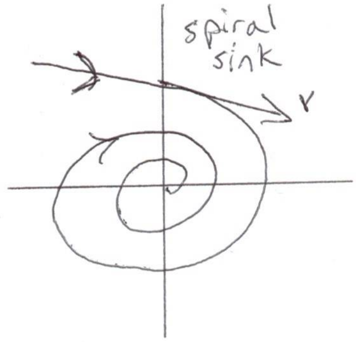


$(1,1)$: $A(1,1) = \begin{pmatrix} 0 & 4 \\ -1 & -1 \end{pmatrix}$

eigenvalues: $\det\begin{pmatrix} -\lambda & 4 \\ -1 & -1-\lambda \end{pmatrix} = \lambda^2 + \lambda + 4$

roots: $\dfrac{-1 \pm \sqrt{1-16}}{2} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{15}}{2}$

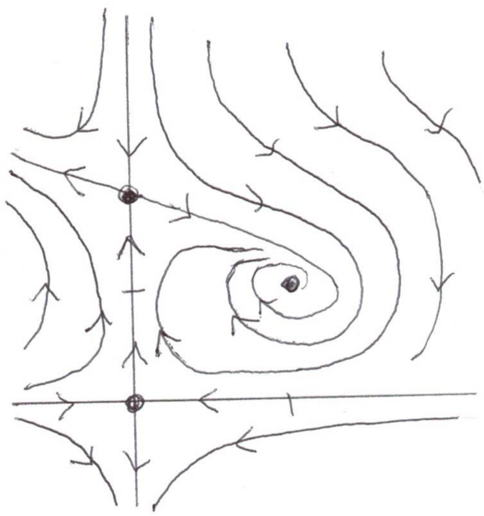
complex eigenvalues, negative real part (Hartman-Grobman applies)



tangent vector v at $\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$v = A(1,1)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

(c)



4. $\begin{cases} x' = 2x - 2x^2 + 5xy \\ y' = y - 2y^2 + 2xy \end{cases}$

(a) Find equilibria. $\begin{cases} 0 = 2x - 2x^2 + 5xy = x(2 - 2x + 5y) \\ 0 = y - 2y^2 + 2xy = y(1 - 2y + 2x) \end{cases}$

$y \neq 0,\quad x = 0 \Rightarrow 1 - 2y = 0 \Rightarrow y = \frac{1}{2} \Rightarrow (0, \frac{1}{2})$

$x \neq 0,\quad y = 0 \Rightarrow 2 - 2x = 0 \Rightarrow x = 1 \Rightarrow (1, 0)$

$x \neq 0, y \neq 0 \Rightarrow \begin{cases} 2 - 2x + 5y = 0 \\ 1 - 2y + 2x = 0 \end{cases}$ solve $\Rightarrow (-\frac{3}{2}, -1)$

$x = 0, y = 0 \Rightarrow (0, 0)$. In total: $\boxed{(0,0), (1,0), (0,\frac{1}{2}), (-\frac{3}{2}, -1)}$

(b) Linearize near each equilibrium.

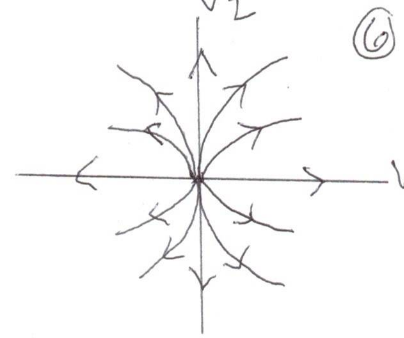
$$A(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 2 - 4x + 5y & 5x \\ 2y & 1 - 4y + 2x \end{pmatrix}$$

$(0,0)$: $A(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ eigenvalues: $\det\begin{pmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda)$

$\lambda_1 = 2,\ \lambda_2 = 1.$

eigenvectors: $\lambda_1 = 2$: $\underset{v_1}{\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$

$\lambda_2 = 1$: $\underset{v_2}{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_2 = \begin{pmatrix}0\\ 1\end{pmatrix}$

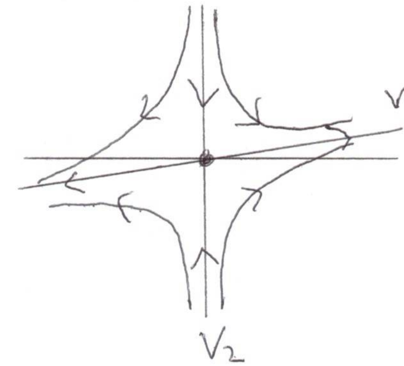


$(0, \frac{1}{2})$: $A(0, \frac{1}{2}) = \begin{pmatrix} 9/2 & 0\\ 1 & -1\end{pmatrix}$ eigenvalues: $\det\begin{pmatrix} 9/2 - \lambda & 0\\ 1 & -1-\lambda\end{pmatrix} = (\frac{9}{2} - \lambda)(-1-\lambda)$

$\lambda_1 = 9/2$, $\lambda_2 = -1$.

eigenvectors: $\lambda_1 = \frac{9}{2}$: $\begin{pmatrix}0 & 0\\ 1 & -\frac{11}{2}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_1 = \begin{pmatrix}11\\ 2\end{pmatrix}$

$\lambda_2 = -1$: $\begin{pmatrix}11/2 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_2 = \begin{pmatrix}0\\ 1\end{pmatrix}$

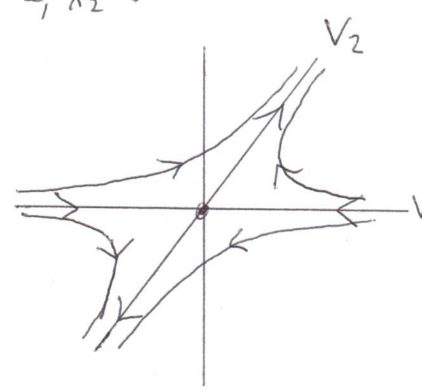


$(1, 0)$: $A(1,0) = \begin{pmatrix}-2 & 5\\ 0 & 3\end{pmatrix}$ eigenvalues: $\det\begin{pmatrix}-2-\lambda & 5\\ 0 & 3-\lambda\end{pmatrix}$

$= (-2-\lambda)(3-\lambda)$, $\lambda_1 = -2$, $\lambda_2 = 3$.

eigenvectors: $\lambda_1 = -2$: $\begin{pmatrix}0 & 5\\ 0 & 5\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$

$\lambda_2 = 3$: $\begin{pmatrix}-5 & 5\\ 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_2 = \begin{pmatrix}1\\ 1\end{pmatrix}$



$(-\frac{3}{2}, -1)$: $A(-\frac{3}{2}, -1) = \begin{pmatrix}3 & -15/2\\ -2 & 2\end{pmatrix}$

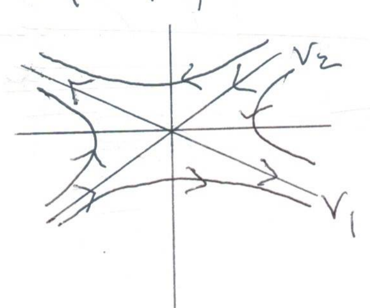
eigenvalues: $\det\begin{pmatrix}3-\lambda & -15/2\\ -2 & 2-\lambda\end{pmatrix} = \lambda^2 - 5\lambda - 9$ roots: $\dfrac{5 \pm \sqrt{25+36}}{2} = \dfrac{5}{2} \pm \dfrac{\sqrt{61}}{2}$

eigenvectors: $\lambda_1 = \frac{5}{2} + \frac{\sqrt{61}}{2}$: $\begin{pmatrix}\frac{1}{2} - \frac{\sqrt{61}}{2} & -\frac{15}{2}\\ -2 & -\frac{1}{2} - \frac{\sqrt{61}}{2}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_1 = \begin{pmatrix}1+\sqrt{61}\\ -4\end{pmatrix}$

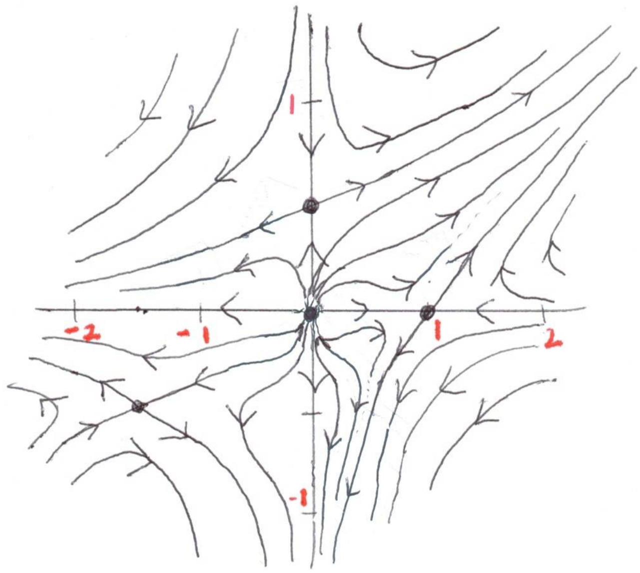
(note $\lambda_1 > 0$)

$\lambda_2 = \frac{5}{2} - \frac{\sqrt{61}}{2}$ similar, get (for example) $v_2 = \begin{pmatrix}1-\sqrt{61}\\ -4\end{pmatrix}$

(note $\lambda_2 < 0$)



(c)



5(a) $y'' - 2y' - 3y = 0,\ y(0) = 2,\ y'(0) = 0$. Set $Y = \mathcal{L}\{y\}$.

$\mathcal{L}\{y'\} = s\cdot Y - y(0) = s\cdot Y - 2.$

$\mathcal{L}\{y''\} = s^2\cdot Y - s\cdot y(0) - y'(0) = s^2\cdot Y - 2s$

Thus transformed eq. is:

$$(s^2\cdot Y - 2s) - 2\cdot(s\cdot Y - 2) - 3\cdot Y = 0$$

$$(s^2 - 2s - 3)\cdot Y + (-2s + 4) = 0$$

$$Y = \frac{2s-4}{s^2-2s-3} = \frac{2s-4}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$2s - 4 = (s+1)A + (s-3)B.$ $\quad s = -1:\ -6 = -4B \Rightarrow B = \frac{3}{2}$

$s = 3:\ 2 = 4A \Rightarrow A = \frac{1}{2}$

$Y = \dfrac{1/2}{s-3} + \dfrac{3/2}{s+1}$. Now use $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ to obtain

$$y(t) = \mathcal{L}^{-1}\{Y\} = \boxed{\tfrac{1}{2}e^{3t} + \tfrac{3}{2}e^{-t}}$$

5(b) $y''+y=2\sin(t)$, $y(0)=y'(0)=0$. Set $Y=\mathcal{L}\{y\}$.

$\mathcal{L}\{y''\}=s^2\cdot Y-sy(0)-y'(0)=s^2 Y$. $\mathcal{L}\{\sin(t)\}=\frac{1}{s^2+1}$

The transformed eq. is then

$$s^2\cdot Y+Y=\frac{2}{s^2+1}\Rightarrow Y=\frac{2}{(s^2+1)^2}\overset{\text{use table}}{\Rightarrow} y(t)=\boxed{\sin(t)-t\cos(t)}$$

5(c) $y''+5y'+4y=1-u_2(t)$, $y(0)=y'(0)=0$. Set $Y=\mathcal{L}\{y\}$.

$\mathcal{L}\{y''\}=s^2\cdot Y$. $\mathcal{L}\{y'\}=s\cdot Y$. $\mathcal{L}\{1\}=\frac{1}{s}$. $\mathcal{L}\{u_2(t)\}=e^{-2s}/s$.

The transformed eq. is:

$$s^2\cdot Y+5s\cdot Y+4\cdot Y=\frac{1}{s}-\frac{e^{-2s}}{s}$$

$$(s^2+5s+4)\,Y=\frac{1}{s}-\frac{e^{-2s}}{s} \qquad (s+1)(s+4)\,Y=\frac{1}{s}-\frac{e^{-2s}}{s}$$

$$Y=\frac{1}{s(s+1)(s+4)}-\frac{e^{-2s}}{s(s+1)(s+4)}$$

$$\frac{1}{s(s+1)(s+4)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+4}\Rightarrow A(s+1)(s+4)+Bs(s+4)+Cs(s+1)=1$$

$s=-1$: $-3B=1\Rightarrow B=-\frac{1}{3}$

$s=-4$: $12C=1\Rightarrow C=\frac{1}{12}$

$s=0$: $4A=1\Rightarrow A=\frac{1}{4}$

Thus $Y=(1-e^{-2s})\cdot\left(\frac{1}{4}\cdot\frac{1}{s}-\frac{1}{3}\cdot\frac{1}{s+1}+\frac{1}{12}\cdot\frac{1}{s+4}\right)$.

Use $\mathcal{L}\{u_a(t)\cdot f(t-a)\}=e^{-as}\cdot\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{e^{at}\}=\frac{1}{s-a}$ to get:

$$y(t)=\boxed{\left(\tfrac{1}{4}\right)\cdot(1-u_2(t))+\left(-\tfrac{1}{3}\right)\left(e^{-t}-u_2(t)e^{-t+2}\right)+\left(\tfrac{1}{12}\right)\left(e^{-4t}-u_2(t)e^{-4t+8}\right)}$$

5(d) $y'' + 9y = \delta(t-3\pi) + \cos(3t)$, $y(0) = y'(0) = 0$

Set $Y = \mathcal{L}\{y\}$ as usual.

$\mathcal{L}\{y''\} = s^2 Y$. $\mathcal{L}\{\delta(t-3\pi)\} = e^{-3\pi s}$. $\mathcal{L}\{\cos(3t)\} = \dfrac{s}{s^2+9}$

The transformed eq. is then

$$s^2 \cdot Y + 9 \cdot Y = e^{-3\pi s} + \frac{s}{s^2+9}$$

$$Y = \frac{e^{-3\pi s}}{s^2+9} + \frac{s}{(s^2+9)^2}$$

Use $\mathcal{L}\{u_a(t) f(t-a)\} = e^{-as} \cdot \mathcal{L}\{f(t)\}$ to get $\mathcal{L}^{-1}\left\{\dfrac{e^{-3\pi s}}{s^2+9}\right\} = \frac{1}{3} u_{3\pi}(t) \sin(3(t-3\pi))$

$$= -\frac{1}{3} u_{3\pi}(t) \sin(3t)$$

Use table (for example) to get $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+9)^2}\right\} = \dfrac{t}{6}\sin(3t)$.

Thus $y(t) = \mathcal{L}^{-1}\{Y\} = -\frac{1}{3} u_{3\pi}(t) \sin(3t) + \frac{t}{6} \sin(3t)$.

$$= \boxed{\left(\frac{t}{6} - \frac{u_{3\pi}(t)}{3}\right) \sin(3t)}$$