Midterm 1 Solutions

1. $y' - y = 2t \quad y(0) = 5$

1) $y' = y \quad \int \frac{dy}{y} = \int dt \quad \ln|y| = t + C \quad y = ke^{t}$

2) Var. of parameters: $y(t) = k(t)e^{t}$

$$y' - y = 2t \to (k'(t)e^{t} + \cancel{k(t)e^{t}}) - (\cancel{k(t)e^{t}}) = 2t \to k'(t) = 2te^{-t}$$

$$k(t) = \int k'(t)dt = \int \underbrace{2e^{-t}}_{v'} \cdot \underbrace{t}_{u}\, dt = \underbrace{(-2e^{-t})}_{v}\underbrace{(t)}_{u} - \int \underbrace{(-2e^{-t})}_{v} \cdot \underbrace{(1)}_{u'}\, dt = -2te^{-t} - 2e^{-t} + C$$

$$y(t) = k(t)e^{t} = (-2te^{-t} - 2e^{-t} + C)e^{t} = -2t - 2 + Ce^{t}$$

$$5 = y(0) = \cancel{-2(0)} - 2 + Ce^{0} = -2 + C \to C = 7. \quad \boxed{y(t) = -2t - 2 + 7e^{t}}$$

Alternative #1: $y' + y = t. \quad y(0) = 5.$ Solution: $\boxed{y(t) = t - 1 + 6e^{-t}}$

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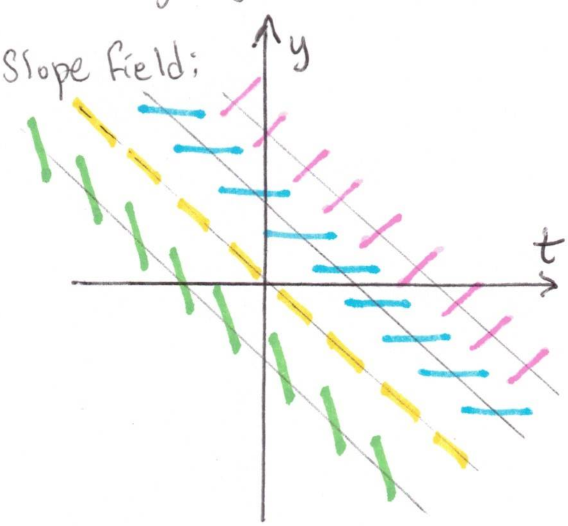
2. Consider $y' = y + t - 1$.

a) Slope field:



isoclines

$y' = -2$	$y = -t - 1$
$y' = -1$	$y = -t$
$y' = 0$	$y = -t + 1$
$y' = 1$	$y = -t + 2$

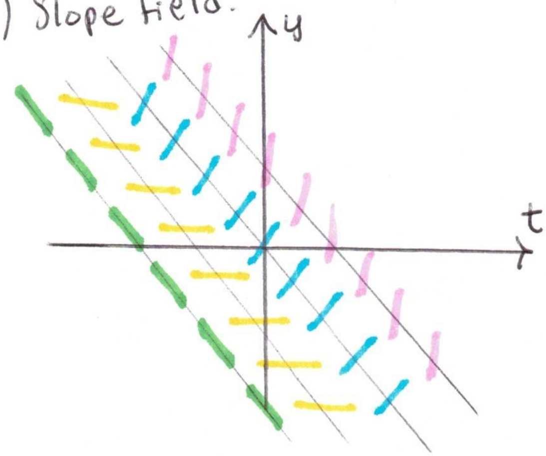
b) $y = -t \quad y' = -1$

thus $\underbrace{y'}_{-1} = -1 = \underbrace{y}_{-t} + t - 1.$ ✓

c) The solution curves never intersect because $f(y,t) = y + t - 1$ and $\frac{\partial f}{\partial y} = 1$ are continuous, so the existence & uniqueness theorem holds. (Uniqueness forbids intersections)

Alternative #2: Consider $y' = y + t + 1$.

a) Slope field:



Isoclines

$y' = -1$	$y = -t - 2$
$y' = 0$	$y = -t - 1$
$y' = 1$	$y = -t$
$y' = 2$	$y = -t + 1$

b) $y = -t - 2$

$y' = -1$

thus $\underbrace{y'}_{-1} = -1 = \underbrace{y}_{-t-2} + t + 1$. ✓

c) essentially same as before

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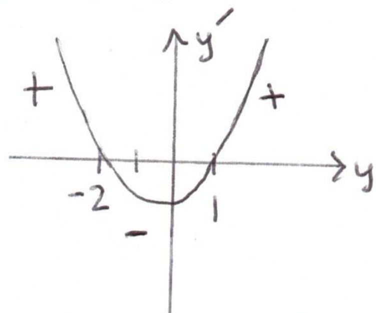
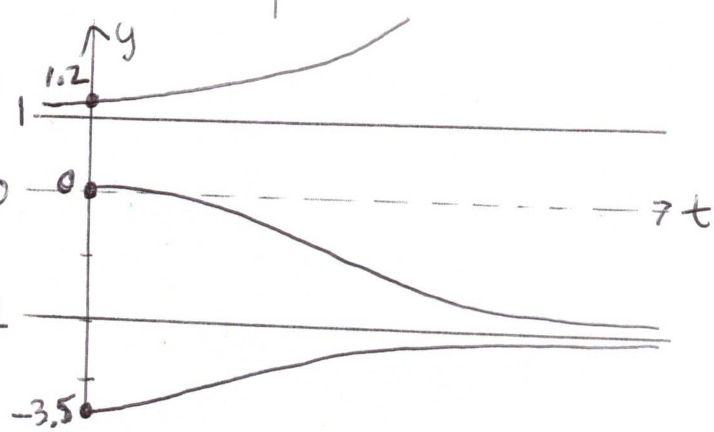
3. a) $y' = y^2 + y - 2 = (y+2)(y-1)$ equilibria: $y = -2$, $y = -1$



Phase diagram:

b)



c) equilibria of $y' = \cos(y)$ are y s.t. $\cos(y) = 0$

$$\boxed{y = \frac{k\pi}{2}, \text{ k odd integer.}}$$

Alternative #3. a) $y' = y^2 - 3y + 2 = (y-1)(y-2)$. equilibria: $y = 1$, $y = 2$.

Phase diagram:

b) the same picture as b) above (with different numbers)

c) equilibria of $y' = \sin(y)$ are numbers y such that $\sin(y) = 0$

$$\boxed{y = k\pi, \text{ k integer}}$$

4. $y' = 1 + yt$ $\quad y(1) = 1$ $\quad \Delta t = 1$ $\quad$ estimate $y(4)$

$t_0 = 1 \quad y_0 = 1$

$t_1 = 2 \quad y_1 = y_0 + \Delta t \cdot (1 + y_0 t_0) = 1 + (1)\cdot(1 + 1\cdot 1) = 1 + 2 = 3$

$t_2 = 3 \quad y_2 = y_1 + \Delta t \cdot (1 + y_1 t_1) = 3 + (1)\cdot(1 + 3\cdot 2) = 3 + 7 = 10$

$t_3 = 4 \quad y_3 = y_2 + \Delta t \cdot (1 + y_2 \cdot t_2) = 10 + (1)\cdot(1 + 10\cdot 3) = 10 + 31 = \boxed{41} \approx y(4)$

Alternative #4.

$y' = yt - 1$ $\quad y(1) = 2$ $\quad \Delta t = 1$ $\quad$ estimate $y(4)$

$t_0 = 1 \quad y_0 = 2$

$t_1 = 2 \quad y_1 = y_0 + \Delta t\,(y_0 t_0 - 1) = 2 + (1)(2\cdot 1 - 1) = 2 + 1 = 3$

$t_2 = 3 \quad y_2 = y_1 + \Delta t\,(y_1 t_1 - 1) = 3 + (1)(3\cdot 2 - 1) = 3 + 5 = 8$

$t_3 = 4 \quad y_3 = y_2 + \Delta t\,(y_2 t_2 - 1) = 8 + (1)(8\cdot 3 - 1) = 8 + 23 = \boxed{31} \approx y(4).$

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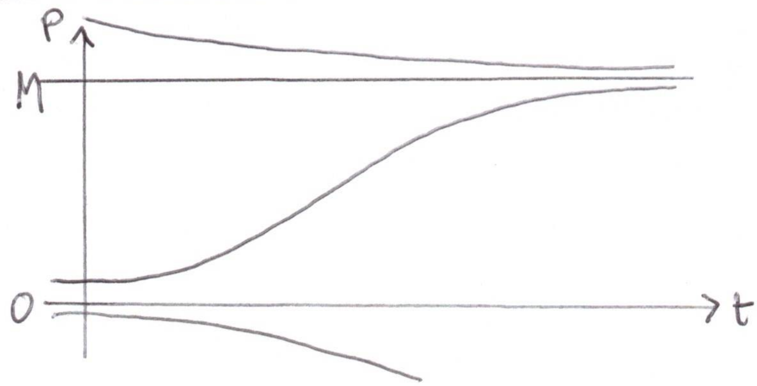
5. a) equilibria: $P = 0$, $P = M$

Phase diagram:



b) $P' = 100P - 4P^2$
$= 4P(25 - P)$
$\Rightarrow \boxed{M = 25}$

Alternative #5 b) $P' = 100P - 5P^2$
$= 5P(20 - P)$
$\Rightarrow \boxed{M = 20}$

c) $P' = -kP^2$ separable

$$\int \frac{dP}{P^2} = \int -k\,dt$$

$$-\frac{1}{P} = -kt + C$$

$\rightarrow P = \dfrac{1}{kt - C}$

$k = 3$ (given)

$200 = P(0) = \dfrac{1}{-C}$

$-C = -\dfrac{1}{200}$

$\rightarrow P(t) = \boxed{\dfrac{1}{3t + \frac{1}{200}}}$

Alternative #5c)
instead
$k = 2$
$P(0) = 3000$

$\rightarrow P(t) = \boxed{\dfrac{1}{2t + \frac{1}{3000}}}$

6. $ty' - y + t\ln(t) = 0$

$$y' - \frac{1}{t}y = -\ln(t)$$

1) $y' - \frac{1}{t}y = 0$ separable $\quad \int \frac{dy}{y} = \int \frac{dt}{t} \quad \ln|y| = \ln|t| + C \quad y = kt$, k nonzero constant

2) var. of parameters: $y(t) = k(t)\,t$

$$y' - \frac{1}{t}y = -t\ln(t) \rightarrow (k'(t)t + \cancel{k(t)\cdot 1}) - \cancel{\frac{1}{t}(k(t)t)} = -\ln(t)$$

$$k'(t)t = -\ln(t)$$

$$k'(t) = -\frac{1}{t}\ln(t)$$

$$k(t) = \int k'(t)dt = \int -\frac{1}{t}\ln(t)\,dt = -\int u\,du = -\frac{u^2}{2} + C = -\frac{\ln(t)^2}{2} + C$$

$u = \ln(t)$

$du = \frac{1}{t}dt$

$$y(t) = k(t)\cdot t = \left(-\frac{\ln(t)^2}{2} + C\right)t = \boxed{-t\cdot\frac{\ln(t)^2}{2} + Ct}$$

Alternative #6.

$$ty' - y = t\ln(t)$$

Solution steps are very similar

Answer: $y(t) = \boxed{\frac{t\ln(t)^2}{2} + Ct}$

The End.