Practice Problems for Exam 1

MTH 211 9/26/23

1. $\vec{v} = 2\vec{i} + 2\vec{k}$ $\quad \vec{w} = 2\vec{i} + \vec{j} + \vec{k}$

(a) $|\vec{v}| = \sqrt{2^2+2^2} = \sqrt{4+4} = 2\sqrt{2}.$ $\quad$ (b) $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\sqrt{2}}{2}\vec{i} + \dfrac{\sqrt{2}}{2}\vec{k}.$

(c) $\vec{v}\cdot\vec{w} = (2)(2) + (0)(1) + (2)(1) = 6$

(d) $\cos\theta = \vec{v}\cdot\vec{w} \,/\, |\vec{v}||\vec{w}|$

(e) $|\vec{w}| = \sqrt{2^2+1^2+1^2} = \sqrt{6}.$ $\quad \theta = \arccos(\sqrt{3}/2) = \pi/6.$

(f) $\vec{v}+\vec{w} = 4\vec{i} + \vec{j} + 3\vec{k}$

(g) $\dfrac{\vec{v}\cdot\vec{w}}{|\vec{v}|} = \dfrac{6}{2\sqrt{2}} = 3/\sqrt{2}.$ $\quad$ (use: $\mathrm{proj}_{\vec{v}}(\vec{w}) = \left(\dfrac{\vec{w}\cdot\vec{v}}{|\vec{v}|}\right)\dfrac{\vec{v}}{|\vec{v}|}.$)

(h) $\vec{w} = \mathrm{proj}_{\vec{v}}(\vec{w}) + (\vec{w} - \mathrm{proj}_{\vec{v}}(\vec{w}))$

$= \langle 3/2, 0, 3/2\rangle + \langle 2,1,1\rangle - \langle 3/2, 0, 3/2\rangle$

$= \underbrace{\langle 3/2, 0, 3/2\rangle}_{\text{parallel to } \vec{v}} + \underbrace{\langle \tfrac{1}{2}, 1, -\tfrac{1}{2}\rangle}_{\text{perpendicular to } \vec{v}}.$

(i) $\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 0\vec{i} + 4\vec{j} + 2\vec{k} - 0\vec{k} - 2\vec{i} - 2\vec{j}$

$= \langle -2, 2, 2\rangle.$

2. (a) F (b) F (c) F

3. (a) $r_1(t) = \langle 0,2,-3\rangle + t\langle 1,0,-1\rangle$
$r_2(t) = \langle -2,3,-6\rangle + s\langle 3,-1,2\rangle$
} two lines

Intersect @ point $P = (1,2,-4)$:

$$\left.\begin{aligned} 0 + t &= 1 \\ 2 + 0t &= 2 \\ -3 - t &= -4 \end{aligned}\right\} \quad t=1 \text{ works, so } P \text{ on line 1. } \checkmark$$

$$\left.\begin{aligned} -2 + 3s &= 1 \\ 3 - s &= 2 \\ -6 + 2s &= -4 \end{aligned}\right\} \quad s=1 \text{ works, so } P \text{ on line 2. } \checkmark$$

(b) dir. vectors: $\vec{a} = \langle 1,0,-1\rangle$, $\vec{b} = \langle 3,-1,2\rangle$.

$\vec{a}\cdot\vec{b} = 3 + 0 - 2 = 1$.

$|\vec{a}| = \sqrt{2}$ $\qquad |\vec{b}| = \sqrt{9+1+4} = \sqrt{14}$

so $\theta = \arccos\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \arccos\left(\dfrac{1}{2\sqrt{7}}\right)$.

4. $\vec{u} = \vec{i} - \vec{j} + \vec{k}$ $\qquad \vec{v} = 2\vec{j} - 3\vec{k}$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 0 & 2 & -3 \end{vmatrix} = 3\vec{i} + 2\vec{k} - 2\vec{i} + 3\vec{j} = \langle 1, 3, 2\rangle.$$

(b) $|\vec{u}\times\vec{v}| = \sqrt{1^2+3^2+2^2} = \sqrt{1+9+4} = \sqrt{14}.$

5. Plane passing through $(0,1,-2)$ and the line

$$\vec{r}(t) = \langle 0,2,-3\rangle + t\langle 1,0,-1\rangle.$$

$\vec{a} = \langle 1,0,-1\rangle$ $\qquad$ $\vec{b} = \langle 0,2,-3\rangle - \langle 0,1,-2\rangle = \langle 0,1,-1\rangle$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{k} + \vec{i} + \vec{j} = \overset{a\ \ b\ \ c}{\langle 1, 1, 1\rangle}.$$

plane is $\quad a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$$x + y - 1 + z + 2 = 0$$

$$x + y + z + 1 = 0.$$

6. line: $\vec{r}(t) = \langle -2, 3, -6\rangle + t\langle 3, -1, -2\rangle$

plane: $2x - y + 3z = 5$

(a) intersection: $\quad x = -2+3t \qquad$ plug into eq of plane:

$y = 3 - t \qquad 2(-2+3t) - (3-t)$

$z = -6 - 2t \qquad + 3(-6-2t) = 5$

$$\rightarrow t = 30.$$

Thus intersect @ $(88, -27, -66)$.

(b) $\vec{a} = \langle 3,-1,-2 \rangle$ $\vec{b} = \langle 2,-1,3 \rangle$

$\vec{a}\cdot\vec{b} = 6 + 1 - 6 = 1$

$|\vec{a}| = \sqrt{9+1+4} = \sqrt{14}$ $|\vec{b}| = \sqrt{4+1+9} = \sqrt{14}$

$\theta = \arccos\left(\frac{1}{14}\right).$

7. Are $P(3,1,2)$, $Q(6,-1,6)$, $R(1,4,7)$, $S(2,1,5)$ coplanar?

$\vec{a} = \overrightarrow{PQ} = \langle 3,-2,4 \rangle$

$\vec{b} = \overrightarrow{PR} = \langle -2,3,5 \rangle$

$\vec{c} = \overrightarrow{PS} = \langle -1,0,3 \rangle$

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} 3 & -2 & 4 \\ -2 & 3 & 5 \\ -1 & 0 & 3 \end{vmatrix} = 27 + 10 + 0 + 12 - 0 - 12 \neq 0$$

so they are not.

8. Sym. eqs for line which is intersection of

$x + y - z = 2$ and $3x - 4y + 5z = 6$.

Dir. vector: $\langle 1,1,-1 \rangle \times \langle 3,-4,5 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 3 & -4 & 5 \end{vmatrix}$

$= \langle 5-4, -3-5, -4-3 \rangle = \langle 1,-8,-7 \rangle.$

Point: $(2, 0, 0)$. line is $\langle 2, 0, 0\rangle + t\langle 1, -8, -7\rangle$.

Sym. eq: $x - 2 = y/_{-8} = z/_{-7}$.

9. $\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 3\vec{k}$.

(a) $\vec{r}(t) = \langle \cos t, \sin t, 3t\rangle + \overset{\text{consts.}}{\langle c_1, c_2, c_3\rangle}$

$\vec{r}(0) = \langle 1, 0, 3\rangle$ given. So we get

$\vec{r}(t) = \langle \cos t, \sin t, 3t + 3\rangle$.

(b) unit ~~tangent~~ vector @ $t = \pi$:

$\vec{r}\,'(\pi) = \langle 0, -1, 3\rangle$. $|\vec{r}\,'(\pi)| = \sqrt{1+9} = \sqrt{10}$.

So $\dfrac{\vec{r}\,'(\pi)}{|\vec{r}\,'(\pi)|} = \langle 0, -1/\sqrt{10}, 3/\sqrt{10}\rangle$.

(c) acceleration @ $t = \pi$:

$\vec{r}\,''(t) = \langle -\cos t, -\sin t, 0\rangle$. $\vec{r}\,''(\pi) = \langle +1, 0, 0\rangle$.

(d) dist from $t=0$ to $t = \pi$: $\displaystyle\int_0^{\pi} |\vec{r}\,'(t)|\,dt = \int_0^{\pi} \sqrt{(\sin t)^2 + \cos^2 t + 3^2}\,dt = \int_0^{\pi} \sqrt{10}\,dt$

$= \sqrt{10}\,\pi$.

10. $z + \cos(xy) = x^2 + y^2$

this is the graph of $g(x,y) = x^2 + y^2 - \cos(xy)$

it's level surface of $f(x,y,z) = x^2 + y^2 - z - \cos(xy)$.

11. (a) $f(x,y,z) = \sin(xy + z)$

$f_x = y\cos(xy+z)$ $\quad f_y = x\cos(xy+z)$ $\quad f_z = \cos(xy+z)$

$f_{xx} = -y^2 \sin(xy+z)$ $\quad f_{xy} = \cos(xy+z) - yx\sin(xy+z)$

$f_{xz} = -y\sin(xy+z)$ $\quad f_{yy} = -x^2\sin(xy+z)$

$f_{yz} = -x\sin(xy+z)$ $\quad f_{zz} = -\sin(xy+z)$.

(b) $$\frac{\partial^{100}}{\partial x^{95}\,\partial y^2\,\partial x^3}\left(ye^x x + \cos(x)\right) = 0.$$

12. tangent plane to graph $z = \underbrace{-x^2 + 4y^2 + 1}_{f(x,y)}$ @ $(2,1,1)$.

$f_x = -2x$ $\quad f_y = 8y$

$f_x(2,1) = -4$ $\quad f_y(2,1) = 8$.

$z = 1 + (-4)(x-2) + (8)(y-1)$

$z = -4x + 8y + 1$.

13. $f(x,y)$ satisfies $f(1,1)=3$, $f_x(1,1)=2$, $f_y(1,1)=-1$.

estimate $f(1.1, 0.9)$.

$$f(1.1, 0.9) \approx 3 + 2(1.1-1) + (-1)(0.9-1)$$

$$= 3 + 0.2 + 0.1 = 3.3.$$

14.

Volume $V(r,h) = \frac{1}{3}\pi r^2 h$.

$$dV = V_r\,dr + V_h\,dh = \frac{2}{3}\pi r h\,dr + \frac{1}{3}\pi r^2\,dh$$

$$dV(120,140) = \frac{2}{3}\pi(120)(140)\,dr + \frac{1}{3}\pi(120)^2\,dh$$

$$\le \pi(80)\cdot(140)\cdot(1.8) + \pi(40)(120)\cdot(2.5)$$

$$= 32160\pi.$$