Practice Problems for Midterm 2 — Solutions.

1. Which of the following are subspaces?

(a) vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ with $a_1 a_2 = 0$

No. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are two such vectors,
but their sum $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has $a_1 a_2 = 1 \cdot 1 \neq 0$.

(b) vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ with $a_1 + 3a_2 = 0$.

Yes. Answer 1: we just check such vectors are preserved under addition & scaling:
If $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ has $a_1 + 3a_2 = 0$, $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has $b_1 + 3b_2 = 0$
then the sum $\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix}$ satisfies $(a_1+b_1) + 3(a_1+b_1) = 0$.
Similarly, if $c \in \mathbb{R}$ then $c\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix}$ has $(ca_1) + 3(ca_2) = 0$.
(Also need that $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ satisfies $0 + 3 0 = 0$.)
Answer 2: $a_1 + 3a_2 = 0$ is a line in $\mathbb{R}^2$ passing through the origin, which is a subspace.

(c) vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ with $a_1^2 + a_2 = 0$.

No. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ satisfy $a_1^2 + a_2 = 0$.
But the sum $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ has $a_1^2 + a_2 = 0^2 + (-2)^2 = 4 \neq 0$.

(d) vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ with $a_1 < a_2$.

No. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ satisfies $a_1 = 0 < 1 = a_2$
But the scaled vector $(-1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
has $a_1 = 0 > -1 = a_2$. (Alternative: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ does not satisfy $0 < 0$.)

(e) vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ with $a_1 + a_2 = 1$.

No. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ satisfy $a_1 + a_2 = 1$.

No. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ satisfy $a_1+a_2=1$.
But the sum $\begin{bmatrix}1\\1\end{bmatrix}$ has $a_1+a_2 = 1+1 = 2 \neq 1$.

(Alternative: $\begin{bmatrix}0\\0\end{bmatrix}$ does not satisfy $0+0=1$!)

(f) all linear combinations of $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}$.

Yes. This is $\operatorname{span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}\right)$.

(g) vectors $\begin{bmatrix}a_1\\a_2\end{bmatrix}$ with a_1, a_2 both integers.

No. $\begin{bmatrix}1\\0\end{bmatrix}$ is such a vector, but the scaled vector $\left(\frac{1}{2}\right)\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1/2\\0\end{bmatrix}$ is not.

2. Consider the vector space of 3×3 matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Which is a subspace?

(a) Matrices with $a_{11}+a_{22}+a_{33}=0$.

Yes. (similar to 1(b).)

(b) Matrices A such that $A\vec{x}=\vec{0}$ has one solution.

No. If $A = 0$-matrix, every $\vec{x}$ in $\mathbb{R}^3$ solves $A\vec{x}=\vec{0}$.

So the "zero vector" is not in the set.

(c) Matrices such that $A = A^T$.

Yes. Straightforward to check $A^T + B^T = (A+B)^T$ and $(cA)^T = cA^T$

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Then: if $A = A^T$ and $B = B^T$ we show the sum also satisfies the property: $A+B = A^T + B^T = (A+B)^T$.

If $A = A^T$ and $c \in \mathbb{R}$ then $cA = cA^T = (cA)^T$.

Note also the zero 3×3 matrix satisfies $A = A^T$.

3. (a) $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\frac{1}{2}(\vec{v}_1 + \vec{v}_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \frac{1}{2}(\vec{v}_1 + \vec{v}_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \frac{1}{2}(\vec{v}_2 + \vec{v}_3) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

We recovered the standard basis of $\mathbb{R}^3$.

Thus $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^3$.

The vectors are independant and their span has dimension 3.

(b)

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

These are independent (if dependent, $a_1\vec{v}_1 + a_2\vec{v}_2 = 0$ where a_1, a_2 not both zero. Say $a_1 \neq 0$. Then $\vec{v}_1 = \left(-\frac{a_2}{a_1}\right)\vec{v}_2$. But $\vec{v}_1$ is not a multiple of $\vec{v}_2$! Similar argument if $a_2 \neq 0$.)

They form a basis for the subspace they span, which is of dimension 2.

(c) $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 6 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 3 & 2 & 6 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{-1} & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \rightarrow \begin{array}{c} \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{-1} & 3 & -2 \\ 0 & 0 & \boxed{3} & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \qquad \\ \text{pivot columns} \end{array}$$

They are not independent.
A basis is $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Dimension of $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ is 3.

4. (a) False.
(b) False.
(c) True.
(d) True.

5. (a) $N(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$ is plane in $\mathbb{R}^{3 = n}$, so has dimension 2.

Rank-Nullity Theorem: $\dim(C(A)) + \underbrace{\dim(N(A))}_{=2} = \underbrace{n}_{=3}$

$\rightarrow \dim(C(A)) = 3 - 2 = 1$.

(b) A is $110 \times \overset{n}{54}$. $\dim(C(A)) = 33$.

RN Thm. $\underbrace{\dim(C(A))}_{=33} + \dim(N(A)) = \underbrace{n}_{=54}$

$\rightarrow \dim(N(A)) = 54 - 33 = 21$.

6. (a) $\mathcal{U} + W$ is a subspace:
By definition, $\mathcal{U} + W$ consists of vectors of the form $\vec{u} + \vec{w}$ where $\vec{u}$ is in $\mathcal{U}$ and $\vec{w}$ is in W.

First note $\vec{0} = \vec{0} + \vec{0}$ is in $\mathcal{U} + W$.

First note $\vec{0} = \underbrace{\vec{0}}_{\text{in } \mathcal{U}} + \underbrace{\vec{0}}_{\text{in } W}$ is in $\mathcal{U}+W$.

Next, suppose $\underbrace{\vec{u}}_{\text{in } \mathcal{U}} + \underbrace{\vec{w}}_{\text{in } W}$ and $\underbrace{\vec{u}'}_{\text{in } \mathcal{U}} + \underbrace{\vec{w}'}_{\text{in } W}$ are in $\mathcal{U}+W$.

Then

$$(\vec{u}+\vec{w}) + (\vec{u}'+\vec{w}') = \underbrace{(\vec{u}+\vec{u}')}_{\substack{\text{in } \mathcal{U} \\ \text{since } \mathcal{U} \text{ is} \\ \text{a subspace}}} + \underbrace{(\vec{w}+\vec{w}')}_{\substack{\text{in } W \\ \text{since} \\ W \text{ is a subspace}}}, \text{ so the sum is in } \mathcal{U}+W.$$

If $\vec{u}+\vec{w}$ is in $\mathcal{U}+W$ and $c \in \mathbb{R}$ then

$$c(\vec{u}+\vec{w}) = \underbrace{c\vec{u}}_{\substack{\text{in } \mathcal{U} \\ \text{b/c } \mathcal{U} \text{ is} \\ \text{a subspace}}} + \underbrace{c\vec{w}}_{\substack{\text{in } W \\ \text{b/c } W \\ \text{a subspace}}}, \text{ so this is in } \mathcal{U}+W.$$

(b) We will use: for two subspaces $\mathcal{U}, W \subset$ some vectorspace

$$\dim \mathcal{U} + \dim W = \dim(\mathcal{U}+W) + \dim(\mathcal{U}\cap W).$$

Suppose $\mathcal{U}, W$ are planes through $\vec{0}$ in $\mathbb{R}^4$ and $\mathcal{U}+W = \mathbb{R}^4$.

Then $\dim \mathcal{U} = 2$, $\dim W = 2$, $\dim(\mathcal{U}+W) = 4$. So

$$2 + 2 = 4 + \dim(\mathcal{U}\cap W) \longrightarrow \dim(\mathcal{U}\cap W) = 0.$$

This means $\mathcal{U}\cap W = \{\vec{0}\}$.

(c) $\mathcal{U}, W \subset \mathbb{R}^n$ subspaces of $\dim = 3$, $\mathcal{U}\cap W =$ line.

Then

$$\underbrace{\dim \mathcal{U}}_{=3} + \underbrace{\dim W}_{=3} = \dim(\mathcal{U}+W) + \underbrace{\dim(\mathcal{U}\cap W)}_{=1}$$

So $\dim(\mathcal{U}+W) = 5$. Since $\mathcal{U}+W \subset \mathbb{R}^n$,

$\dim(\mathcal{U}+W) \leq \dim(\mathbb{R}^n) = n$. So $n \geq 5$.

7. $V_n = \left\{ \begin{matrix} \text{polynomials of} \\ \text{degree } \leq n \end{matrix} \right\}$

$T: V_n \to V_{n-1}$, $T(p(x)) = \frac{d}{dx} p(x)$.

(a) T is a linear transformation.

(i) let $p(x), g(x)$ be in V_n. Then

$$T(p(x)+g(x)) = \frac{d}{dx}\left(p(x)+g(x)\right) = \frac{d}{dx}p(x) + \frac{d}{dx}g(x) = T(p(x)) + T(g(x)).$$

(ii) let $p(x)$ be in V_n, and $c \in \mathbb{R}$. Then

$$T(c\,p(x)) = \frac{d}{dx}(c\,p(x)) = c\frac{d}{dx}p(x) = c\,T(p(x)).$$

(b) $N(T) = \{ p(x) \text{ of } \deg \leq n : T(p(x)) = 0 \}$

$T(p(x)) = 0$ means $\frac{d}{dx}p(x) = 0$.

Derivative zero $\to$ $p(x)$ must be constant.

$N(T) = \left\{ \begin{matrix} \text{constant} \\ \text{polynomials} \end{matrix} \right\} = \left\{ \begin{matrix} \text{degree } 0 \\ \text{polynomials} \end{matrix} \right\} = V_0$

(c) $$\dim\left(\left\{ \begin{matrix} \text{outputs} \\ \text{of } T \end{matrix} \right\}\right) + \underbrace{\dim N(T)}_{= \dim V_0 = 1} = \underbrace{\dim V_n}_{= n+1}$$ (see HW 7)

$\to \dim\left(\left\{ \begin{matrix} \text{outputs} \\ \text{of } T \end{matrix} \right\}\right) = n.$

(c)

We have $\underbrace{\left\{ \begin{matrix} \text{outputs of} \\ T \end{matrix} \right\}}_{\dim = n} \subset \underbrace{V_{n-1}}_{\dim = n}$

we have $\underbrace{\{\quad T \quad\}}_{\text{dim} = n} \subseteq \underbrace{V_{n-1}}_{\text{dim} = n}$

dims equal $\longrightarrow$ so in fact $\left\{\begin{matrix}\text{outputs} \\ \text{of } T\end{matrix}\right\} = V_{n-1}$.

Thus every polynomial of $\deg \leq n-1$ is some $T(p(x))$ where $p(x)$ has $\deg \leq n$. But $T(p(x)) = \frac{d}{dx} p(x)$.

n is arbitrary here.

So every polynomial is a derivative.