Eigenvalues & Eigenvectors (cont.)

Finding eigenvalues / eigenvectors for $\overset{n\times n}{A}$:

1. Compute $\det(A-\lambda I)$. This is a polynomial of degree n in the variable λ.

2. Find the roots of this polynomial. These are the eigenvalues. Label them $\lambda_1, \lambda_2, \lambda_3 \ldots$

3. For each eigenvalue λ_i solve $(A-\lambda_i I)\vec{x}=\vec{0}$. Nonzero solutions are eigenvectors associated to λ_i.

Ex: $A=\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad \det(A-\lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 2 & -\lambda & 2 \\ 0 & 1 & -\lambda \end{bmatrix}\right)$

$$= -\lambda^3 - (1)(2)(-\lambda) - (-\lambda)(2)(1)$$

$$= -\lambda^3 + 4\lambda \quad = -\lambda(\lambda-2)(\lambda+2)$$

Roots are $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = -2$.

Now find the eigenvectors.

$\lambda_1 = 0$: Solve $(A-\lambda_1 I)\vec{x}=\vec{0} \xrightarrow{\lambda_1=0} A\vec{x}=\vec{0}$

So solve $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

One soln. is $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. In fact any scalar multiple is a solution. $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\lambda_2 = 2$: Solve $(A - \lambda_2 I)\vec{x} = \vec{0}$ $\quad A - \lambda_2 I = \begin{bmatrix} -2 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & -2 \end{bmatrix}$

$$\begin{bmatrix} -2 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus RREF of $(A - \lambda_2 I)\vec{x} = \vec{0}$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

↑ free: $z = t$

$$\longrightarrow \begin{matrix} x - z = 0 \\ y - 2z = 0 \end{matrix} \longrightarrow \begin{matrix} x = t \\ y = 2t \end{matrix}$$

Solutions are $\vec{x} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. So $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector for λ_2.

$\lambda_3 = -2$:

$$A - \lambda_3 I = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

get solns $\vec{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ so $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector for λ_3.

Ex. Suppose $\lambda = 0$ is an eigenvalue of A.
Then $\det(A - \lambda I) = 0$ which is just $\det A = 0$.
So A is not invertible.

$\vec{x}$ is an eigenvector associated to $\lambda = 0$ $\Longleftrightarrow$ $A\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$ $\Longleftrightarrow$ $\vec{x}$ is in $N(A)$ (nullspace) and $\vec{x} \neq \vec{0}$

Ex.

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

diagonal matrix

$$\det(\Lambda - \lambda I) = \det\left(\begin{bmatrix} \lambda_1 - \lambda & & & \\ & \lambda_2 - \lambda & & \\ & & \ddots & \\ & & & \lambda_n - \lambda \end{bmatrix}\right)$$

$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

The roots (the eigenvalues of Λ) are $\lambda_1, \lambda_2, \dots, \lambda_n$

Ex. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 3$, $\lambda_3 = 1$

eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for λ_1 $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for λ_2 $\quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for λ_3

Ex.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}\right) = -(1-\lambda)^2 \lambda$$

eigenvalues: $\lambda_1 = 0, \quad \lambda_2 = 1$

Let's find eigenvectors,

$\lambda_1 = 0$: Solve $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ solutions are $t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

an eigenvector is $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 0$: $A - \lambda_2 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ solutions: $t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

an eigenvector is $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Can only find 2 independent eigenvectors!

Diagonalizing an $n \times n$ matrix A:

Suppose we can find n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ for A where $\vec{v}_i$ has associated eigenvalue λ_i. Set:

$$X = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Then we can diagonalize A as follows:

$$A = X \Lambda X^{-1}$$

Ex:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \qquad \lambda_1 = 0 \qquad \lambda_2 = 2 \qquad \lambda_3 = -2$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ -1 & 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$X^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$ Then $A = X \Lambda X^{-1}$!

Remember what this is telling us:

in the coordinate system determined by $\vec{v}_1, \vec{v}_2, \vec{v}_3$

the matrix A is diagonal;

it sends $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ to $2c_2\vec{v}_2 - 2c_3\vec{v}_3$.

Sometimes eigenvalues are complex #'s.

Ex. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1$

roots are $\sqrt{-1}$, $-\sqrt{-1}$

so these are the eigenvalues

Can proceed to diagonalize A if we allow our matrices to have complex # entries.

$\lambda_1 = \sqrt{-1}$: $\begin{bmatrix} -\sqrt{-1} & 1 \\ -1 & -\sqrt{-1} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Solns. are $\vec{x} = t\begin{bmatrix} \sqrt{-1} \\ -1 \end{bmatrix}$

So an eigenvector is $\vec{v}_1 = \begin{bmatrix} \sqrt{-1} \\ -1 \end{bmatrix}$

$\lambda_2 = -\sqrt{-1}$: $\begin{bmatrix} \sqrt{-1} & 1 \\ -1 & \sqrt{-1} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Solns. are $\vec{x} = \begin{bmatrix} \sqrt{-1} \\ 1 \end{bmatrix}$

So an eigenvector is $\vec{v}_2 = \begin{bmatrix} \sqrt{-1} \\ 1 \end{bmatrix}$

Then

$$\underset{A}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \underset{X}{\begin{bmatrix} \sqrt{-1} & \sqrt{-1} \\ -1 & 1 \end{bmatrix}}\underset{\Lambda}{\begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}}\underset{X^{-1}}{\begin{bmatrix} -\frac{\sqrt{-1}}{2} & -\frac{1}{2} \\ -\sqrt{-1}/2 & \frac{1}{2} \end{bmatrix}}$$

Relation to determinants?

Suppose A is diagonalizable, i.e. $A = X \Lambda X^{-1}$.

Then
$$\det(A) = \det(X \Lambda X^{-1})$$

$$= \det(X)\det(\Lambda)\det(X^{-1})$$

$$= \cancel{\det(X)}\ \det(\Lambda)\ \frac{1}{\cancel{\det(X)}}$$

$$= \det(\Lambda) = \det\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right)$$

$$= \lambda_1 \lambda_2 \cdots \lambda_n .$$

Thus det(A) is the product of the eigenvalues of A (with eigenvalues repeating according to multiplicity).

This holds for all $n \times n$ A! (not just diagonalizable)

Indeed,
$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

$$\Big\downarrow \lambda = 0$$

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n .$$