Last time we discussed for two subspaces $\mathcal{U}, W$ in a vector space V the constructions

intersection $\mathcal{U} \cap W = \{$ vectors $\vec{v}$ in V that are in $\mathcal{U}$ and $W\}$

and the sum $\mathcal{U} + W = \{$ vectors $\vec{u}+\vec{w}$ where $\vec{u}$ in $\mathcal{U}$, $\vec{w}$ in $W\}$.

These are subspaces of V.

We also gave evidence for a formula:

$$\boxed{\dim(\mathcal{U}) + \dim(W) = \dim(\mathcal{U}+W) + \dim(\mathcal{U}\cap W).} \qquad (*)$$

We'll briefly discuss now how this follows from the Rank-Nullity Thm.

First, given any two vector spaces $\mathcal{U}, W$ (possibly unrelated) we can define the product vector space

$$\mathcal{U} \times W = \{ (\vec{u}, \vec{w}) \mid \vec{u} \text{ is in } \mathcal{U}, \ \vec{w} \text{ is in } W \}$$

i.e. ordered pairs of vectors

Addition & scalar mult. are straightforward to define on $\mathcal{U}\times W$, and one can check the vector space axioms.

Ex. If $\mathcal{U} = \mathbb{R}^n$, $W = \mathbb{R}^m$ then $\mathcal{U}\times W = \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$.

You can also show $\boxed{\dim(\mathcal{U}\times W) = \dim \mathcal{U} + \dim W.}$

Now to explain (*), let $\mathcal{U}, W$ be subspaces of V.

Define a linear transformation

$$T : \mathcal{U} \times W \longrightarrow V$$

by $T((\vec{u}, \vec{w})) = \vec{u} + \vec{w}$.

Note $\operatorname{im}(T) = \{\text{outputs of } T\} = \{\vec{u} + \vec{w} \mid \vec{u} \text{ in } \mathcal{U}, \vec{w} \text{ in } W\} = \mathcal{U} + W$.

On the other hand,

$$N(T) = \{(\vec{u}, \vec{w}) \mid \vec{u} \text{ in } \mathcal{U}, \ \vec{w} \text{ in } W, \ \vec{u} + \vec{w} = \vec{0}\}$$

Note $\vec{u} + \vec{w} = \vec{0}$ implies $\vec{u} = -\vec{w}$. So $\vec{u}$ determines $\vec{w}$.

Also, $\vec{u}$ is in $\mathcal{U}$ & $\vec{u} = -\vec{w}$ is in W

$\rightarrow \vec{u}$ is in $\mathcal{U} \cap W$.

In this way, $N(T)$ is naturally in correspondence with $\mathcal{U} \cap W$.

So Rank - Nullity Theorem for T gives

$$\dim(\operatorname{im}(T)) + \dim(N(T)) = \dim(\mathcal{U} \times W)$$

$$\rightarrow \dim(\mathcal{U} + W) + \dim(\mathcal{U} \cap W) = \dim \mathcal{U} + \dim W,$$

which is (*).

Example Suppose a plane $\mathcal{U}$ through $\vec{0}$ satisfies

$$\mathcal{U} + W = \mathbb{R}^5$$

where $W = \{(x, y, z, 0, 0) \mid x, y, z \in \mathbb{R}\} \subset \mathbb{R}^5$.

What is $\mathcal{U} \cap W$?

To solve this, use formula (*):

$$\underbrace{\dim(\mathcal{U})}_{=2} + \underbrace{\dim(W)}_{=3} = \underbrace{\dim(\mathcal{U}+W)}_{5} + \dim(\mathcal{U} \cap W)$$

$\rightarrow \dim(\mathcal{U} \cap W) = 0$. This means $\mathcal{U} \cap W = \{\vec{0}\}$.

Example Suppose two subspaces $\mathcal{U}, W \subset \mathbb{R}^n$ each $\dim = 2$ intersect in a point.

What can n be?

Again use (*):

$$\underbrace{\dim(\mathcal{U})}_{=2} + \underbrace{\dim(W)}_{=2} = \dim(\mathcal{U}+W) + \underbrace{\dim(\mathcal{U} \cap W)}_{=0}$$

$\rightarrow \dim(\mathcal{U}+W) = 4$.

Since $\mathcal{U}+W \subset \mathbb{R}^n$ we get $\dim(\mathcal{U}+W) = 4 \leq n$.

So n can be any integer ≥ 4.

More practice w/ Rank-Nullity Thm.

Example Suppose the space of solutions to $A\vec{x} = \vec{0}$ is a line in $\mathbb{R}^4$. What is the dimension of the column space of A?

We're given $N(A) = \{ \vec{x} \mid A\vec{x} = \vec{0} \}$ is a line in $\mathbb{R}^4$.

$$\dim C(A) + \underbrace{\dim N(A)}_{=1} = \underbrace{n}_{4}$$

$\rightarrow$ $\dim C(A) = 3.$

Example Suppose a 99×81 matrix A has a column space of dimension 50.

Compute the dimension of the space of solns. to $A\vec{x} = \vec{0}$.

$$\underbrace{\dim C(A)}_{50} + \dim N(A) = \underbrace{n}_{81}$$

$\rightarrow$ $\dim N(A) = \underline{31}.$

This is the dimension we're after.