The Elimination Algorithm

MTH 260 2/9/23

Recall the Elementary Row Operations:

(1) multiply a row by a nonzero scalar

(2) add/subtract a multiple of one row from another

(3) row exchange

Suppose we're given an equation to solve:

$$A\vec{x} = \vec{b}$$

where A is an $m\times n$-matrix, $\vec{x}$ (unknown) in $\mathbb{R}^n$, $\vec{b}$ vector in $\mathbb{R}^m$.

Write the augmented matrix:

$$[A \mid \vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \\ a_{m1} & \cdots & & a_{mn} & b_m \end{array}\right]$$

(i) Start at the left-most nonzero column.

Swap rows (operation (3)) to get a nonzero entry in the top row.

This will be the 1st pivot.

can assume $a_{11} \neq 0$ by (i)

$$\left[\begin{array}{ccc|c} \boxed{a_{11}} & a_{12} \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right] \xrightarrow{(ii)} \left[\begin{array}{ccc|c} \boxed{a_{11}} & a_{12} \cdots & a_{1n} & b_1 \\ 0 & ? & ? & ? \\ \vdots & & & \vdots \\ 0 & ? & ? & ? \end{array}\right]$$

(ii) Eliminate all entries below the pivot using operation (2).
(see above picture)

$$\left[\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & ? & & ? & ? \\ \vdots & & & & \\ 0 & ? & & ? & ? \end{array}\right]$$

repeat!

(iii) Forget the first row and repeat above steps on remaining rows below.

End Result:

Echelon Form

$$\left[\begin{array}{cccccc|c} \boxed{a} & ? & ? & ? & & & ? \\ & \boxed{b} & ? & ? & & & \vdots \\ & & \boxed{c} & ? & & & \vdots \\ & \text{all zeros} & & & \ddots & & ? \end{array}\right]$$

Pivots are $a, b, c, \ldots$ (all nonzero)

Can do better: (iv) multiply each pivot row (operation (1)) to make the pivot $= 1$.

Reduced Echelon Form (REF)

$$\left[\begin{array}{cccccc|c} \boxed{1} & ? & ? & & ? & & ? \\ & \boxed{1} & ? & & ? & & \vdots \\ & & & \boxed{1} & ? & & ? \\ & \text{all zeros} & & & & \ddots & \end{array}\right]$$

(v) use operation (2) to eliminate all entries above each pivot.

Reduced Row Echelon Form (RREF)

$$\left[\begin{array}{cccccc|c} \boxed{1} & ? & 0 & ? & 0 & & ? \\ & & \boxed{1} & ? & 0 & & \vdots \\ & & & & \boxed{1} & ? & \vdots \\ & \text{all zeros} & & & & \ddots & ? \end{array}\right]$$

In this final form the algorithm is called Gauss-Jordan Elimination.

$$\left\{\begin{array}{l}\text{system of}\\ \text{linear equations}\end{array}\right\} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \{\text{RREF}\}$$

The system is easy to solve once it's in RREF.
(similar to upper triangular systems.)

Ex.

$$\begin{cases} x + 2y = 0 \\ x + y = -1 \\ -x + 2y = -1 \end{cases} \qquad \left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & \boxed{-1} & -1 \\ 0 & 4 & -1 \end{array}\right]$$

$$\downarrow$$

$$\underset{\text{RREF}}{\left[\begin{array}{cc|c} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & -5 \end{array}\right]} \xleftarrow[\text{from } 1^{st}]{\text{subtract twice the } 2^{nd} \text{ eq}} \underset{\text{REF}}{\left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & -5 \end{array}\right]} \xleftarrow[\text{by } -1]{\text{mult. } 2^{nd} \text{ row}} \underset{\text{Echelon Form}}{\left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & \boxed{-1} & -1 \\ 0 & 0 & -5 \end{array}\right]}$$

The last equation says $0y = -5 \to 0 = -5$, impossible.
So the system has no solutions.

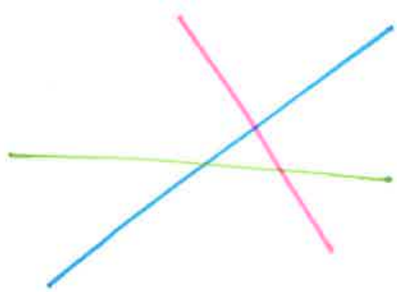
Whenever you see a row of the form

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$$

where $c \neq 0$, the system has no solutions.

Geometry:

3 lines in $\mathbb{R}^2$ usually don't intersect all at once



That's what happened here.

If we had considered

$$\begin{cases} x+2y=0 \\ x+y=-1 \\ -x+2y=4 \end{cases} \leftarrow \text{only change}$$

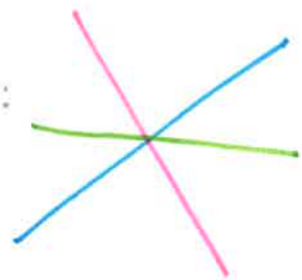
Elimination gives
$$\left[\begin{array}{cc|c} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Last eq. is $0 \cdot y = 0$, no contradiction.

When you see a row of the form $[0\ 0 \cdots 0 \mid 0]$ forget about it; it just says $0=0$.

The system can then be solved; there's one solution.

Geometry:



Ex.
$$\begin{cases} x_1 - 3x_2 + x_3 + x_4 = 0 \\ 2x_1 - 5x_2 - x_3 - x_4 = 1 \end{cases}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & -3 & 1 & 1 & 0 \\ 2 & -5 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & -3 & 1 & 1 & 0 \\ 0 & \boxed{1} & -3 & -3 & 1 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{c} \left[\begin{array}{cccc|c} \boxed{1} & 0 & -8 & -8 & 3 \\ 0 & \boxed{1} & -3 & -3 & 1 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array} \quad \text{RREF}$$

Some new terminology:

Pivot variables — variables at corners of staircase (pivots)

Free variables — other remaining variables

Here: Pivot variables are x_1, x_2; Free variables are x_3, x_4

Usually give free variables new names:

$$x_3 = s, \qquad x_4 = t$$

Then we solve the system in terms of the free vars.

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & -8 & -8 & 3 \\ 0 & \boxed{1} & -3 & -3 & 1 \end{array}\right]$$

2nd eq: $x_2 - 3x_3 - 3x_4 = 1$

$x_2 - 3s - 3t = 1 \longrightarrow x_2 = 1 + 3s + 3t$

1st eq: $x_1 - 8x_3 - 8x_4 = 3$

$x_1 - 8s - 8t = 3 \longrightarrow x_1 = 3 + 8s + 8t$

So solutions are:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3+8s+8t \\ 1+3s+3t \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}} + s\underbrace{\begin{bmatrix} 8 \\ 3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + t\underbrace{\begin{bmatrix} 8 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

where s,t range over all reals

Geometry:

the two original equations give 3D hyperplanes in $\mathbb{R}^4$

they intersect in the 2D plane

$$P = \{\vec{p} + s\vec{u} + t\vec{v} \mid s,t \in \mathbb{R}\}$$

living inside $\mathbb{R}^4$.

Ex.

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -1 & -3 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 2 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -1 & -3 \\ 0 & \boxed{1} & -2 & 1 & 6 \\ 0 & 2 & -4 & 3 & 11 \end{array}\right]$$

$$\downarrow$$

$$\text{RREF} \quad \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & 0 & -4 \\ 0 & \boxed{1} & -2 & 0 & 7 \\ 0 & 0 & 0 & \boxed{1} & -1 \end{array}\right] \end{array} \longleftarrow \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -1 & -3 \\ 0 & \boxed{1} & -2 & 1 & 6 \\ 0 & 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

Pivot variables: x_1, x_2, x_4

Free variables: $x_3 = t$

3rd eq: $x_4 = -1$

2nd eq: $x_2 - 2x_3 = 7 \rightarrow x_2 - 2t = 7 \rightarrow x_2 = 7 + 2t$

1st eq: $x_1 + 3x_3 = -4 \rightarrow x_1 + 3t = -4 \rightarrow x_1 = -4 - 3t$

Solutions:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4-3t \\ 7+2t \\ t \\ -1 \end{bmatrix} = \underbrace{\begin{bmatrix} -4 \\ 7 \\ 0 \\ -1 \end{bmatrix}}_{\vec{p}} + t \underbrace{\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}}$$

Solutions form a line $\{\vec{p} + t\vec{v} \mid t \in \mathbb{R}\}$ inside $\mathbb{R}^4$.