System of m linear equations in n unknowns:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} m \text{ equations}$$

n unknowns: $x_1, \dots, x_n$

"Row Picture"

"Algebraic Interpretation" of Linear Algebra:
study of solving such systems

Each equation gives a hyperplane in $\mathbb{R}^n$

So: solutions to all m equations = intersections of the corresponding hyperplanes

Let's continue to build our intuition through low-dimensional examples.

Ex. ($m=n=2$, i.e. 2 equations in 2 unknowns)

$$\begin{cases} x+2y=1 & (1) \\ x+2y=0 & (2) \end{cases}$$

$\to$ 2nd equation gives $x=-2y$

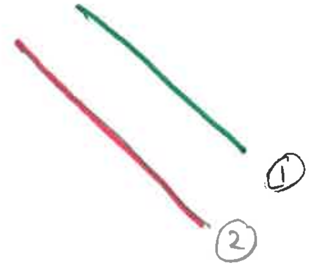
plug into 1st eq.: $(\cancel{-2y}) + \cancel{2y} = 1$

$\to$ $0=1$, contradiction!

Thus this system has no solutions.

Geometric meaning:

two parallel lines that don't intersect



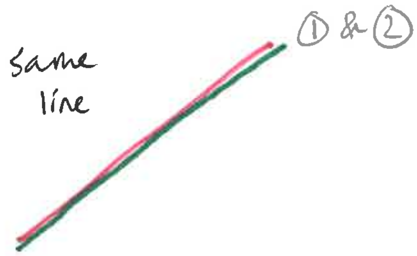
Ex $\begin{cases} x - y = 5 & (1) \\ 2x - 2y = 10 & (2) \end{cases}$

$2 \times (1) = (2)$
so equations define same line

∞ solutions

Geometric interpretation:

same line



In general, given

$$\begin{cases} a x + b y = e \\ c x + d y = f \end{cases}$$

there are 3 qualitative cases for solutions:

1 solution (unique)

0 solutions

∞ solutions

Given a system of linear equations we may also use the "column picture" viewpoint:

$$\boxed{\begin{matrix} a_{11} x_1 \\ a_{21} x_1 \\ \vdots \\ a_{m1} x_1 \end{matrix}} \begin{matrix} + \\ + \\ \\ + \end{matrix} \boxed{\begin{matrix} a_{12} x_2 \\ a_{22} x_2 \\ \vdots \\ a_{m2} x_2 \end{matrix}} \begin{matrix} + \cdots + \\ + \cdots + \\ \\ + \cdots + \end{matrix} \boxed{\begin{matrix} a_{1n} x_n \\ a_{2n} x_n \\ \vdots \\ a_{mn} x_n \end{matrix}} \begin{matrix} = \\ = \\ \\ = \end{matrix} \boxed{\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{matrix}}$$

Can rewrite as:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\underbrace{}_{\vec{a}_1} \qquad \underbrace{}_{\vec{a}_2} \qquad \underbrace{}_{\vec{a}_n} \qquad \underbrace{}_{\vec{b}}$$

which becomes the vector equation:

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + \cdots + x_n\vec{a}_n = \vec{b}$$

Here $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n, \vec{b}$ are given (constant) vectors

and $x_1, \ldots, x_n$ are unknown (variable) scalars

The question: "does the system of linear eqs. have a solution?" becomes: "is $\vec{b}$ a linear combination of $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$?"

Ex.

$$\begin{cases} x + 2y + 3z = 0 & (1) \\ x + y + z = 0 & (2) \end{cases}$$

"Row Picture" (geometry): Intersection of two planes in $\mathbb{R}^3$

"Column Picture" (vector equation): rewrite as $x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}2\\1\end{bmatrix} + z\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.

Solve by elimination: (1) − (2):

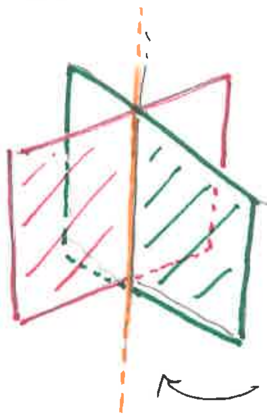
$$\begin{aligned} \cancel{x} + 2y + 3z &= 0 \\ -(\cancel{x} + y + z &= 0) \\ \hline y + 2z &= 0 \end{aligned} \quad \longrightarrow \quad y = -2z$$

Plug $y=-2z$ into ②: $x+(-2z)+z=0$

$x-z=0 \longrightarrow x=z$

So $\vec{x}=\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ Let $z=t$

Then solution set is $\left\{ t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$ ← this is a line in $\mathbb{R}^3$.



← planes, one for each eq ① & ②

← solutions give a line

Could not eliminate all 3 variables b/c started w 2 eqs!

Idea of "dimension":

Given m equations in n unknowns (say $n \geq m$)

we can eliminate m of the unknowns

and have $n-m$ unknowns left over.

"Usually": space of solutions will be "$(n-m)$-dimensional".

Above example:

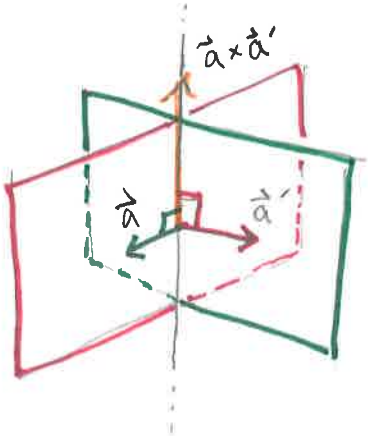
$$\underset{\text{unknowns}}{3} - \underset{\text{eqs.}}{2} = 1$$

← our space of solutions was a line which is "1-dimensional".

Another method to solve previous example:

$$\begin{cases} ax+by+cz=0 \\ a'x+b'y+c'z=0 \end{cases}$$

given two planes passing through origin



$\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\perp$ to 1^{st} plane

$\vec{a}' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ $\perp$ to 2^{nd} plane

Cross product $\vec{a}\times\vec{a}'$ will $\perp$ both $\vec{a}$, $\vec{a}'$, and lies on both planes.

Intersection of planes = the line $\{t(\vec{a}\times\vec{a}') \mid t\in\mathbb{R}\}$.

In our example,

$$\begin{cases} x+2y+3z=0 \\ x+y+z=0 \end{cases} \qquad \vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{a}' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{a}\times\vec{a}' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (2)(1)-(3)(1) \\ (3)(1)-(1)(1) \\ (1)(1)-(2)(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

So solutions are $\left\{t\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \mid t\in\mathbb{R}\right\}$, same as before!