Computing ν-invariants of Joyce's compact G_2-manifolds

Christopher Scaduto
University of Miami, Coral Gables, FL
c.scaduto@math.miami.edu

Abstract

Crowley and Nordström introduced an invariant of G_2-structures on the tangent bundle of a closed 7-manifold, taking values in the integers modulo 48. Using the spectral description of this invariant due to Crowley, Goette and Nordström, we compute it for many of the closed torsion-free G_2-manifolds defined by Joyce's generalized Kummer construction.

1 Introduction

In [CN15], Crowley and Nordström introduced an invariant $\nu(M, \phi) \in \mathbb{Z}/48$ for a closed 7-manifold M equipped with a G_2-structure ϕ on its tangent bundle, invariant under homotopies of ϕ. In the case that M has a metric g with holonomy contained in G_2 and associated G_2-structure ϕ_g, Crowley, Goette and Nordström [CGN18a] showed that this invariant has the spectral description

$$\nu(M, \phi_g) \equiv 3\eta(B_M) - 24\eta(D_M) + 24(1 + b_1(M)) \mod 48$$

Here $\eta(B_M)$ is the η-invariant of the odd signature operator B_M for the Riemannian manifold (M, g), and D_M is the associated spin Dirac operator. In fact, the authors show that

$$\bar{\nu}(M, g) := 3\eta(B_M) - 24\eta(D_M) \in \mathbb{Z}$$

is an invariant of the torsion-free G_2-manifold (M, g) which is locally constant on the moduli space of metrics on M with holonomy contained in G_2.

There are only a handful of methods available to construct closed G_2-manifolds. The first is the generalized Kummer construction of Joyce [Joy96, Joy00]. There is also the twisted connected sum method of Kovalev [Kov03], generalized by Corti–Haskins–Nordström–Pacini [CHNP15, CHNP13]. A further generalization, that of "extra-twisted" connected sums, is considered by Crowley–Goette–Nordström [CGN18a]; see also [Nor18]. More recently, Joyce and Karigiannis [JK18] introduced another construction, the input of which is a closed G_2-manifold with an involution.

Crowley and Nordström [CN15, Theorem 1.7] show $\nu \equiv 24$ (mod 48) for twisted connected sums. This is refined by Crowley–Goette–Nordström [CGN18a, Corollary 3], who show $\bar{\nu} = 0$ for these G_2-manifolds. More generally, the authors compute $\bar{\nu}$ for extra-twisted connected sums, [CGN18a, Theorem 1], producing many examples with non-vanishing $\bar{\nu}$.

Here we compute the ν-invariants for many of Joyce's original G_2-manifolds constructed in [Joy96]. Our investigation is by no means complete, and we largely focus on how a few observations allow one to use well-known "soft" properties of η-invariants in this setting. A more detailed undertaking may allow for the computation of the integer-valued $\overline{\nu}$-invariants.

The construction of a Joyce manifold (M, g) involves taking the resolution of orbifold singularities in the quotient $\mathcal{O} = T^7/\Gamma$ of a 7-torus T^7 by a finite group action Γ preserving the flat G_2-structure. Many of the examples considered in [Joy96] satisfy the following.

Hypothesis 1.1. *Each connected component of the singular set in $\mathcal{O} = T^7/\Gamma$ has a neighborhood isometric, for some finite subgroup $G \subset SU(2)$, to a neighborhood of the singular set in the orbifold*

$$T^3 \times \mathbb{C}^2/G \tag{1}$$

In general we write T^k for any quotient of $\mathbb{R}^k$ by a discrete full rank sublattice, not necessarily $\mathbb{Z}^k$. When the singular set is nice enough, the ν-invariant of the resulting torsion-free G_2-manifold M may be computed from invariants of the orbifold $\mathcal{O}$.

Theorem 1.2. *Let (M, g) be a compact G_2-manifold obtained from the generalized Kummer construction of Joyce using a flat orbifold $\mathcal{O} = T^7/\Gamma$ satisfying Hypothesis 1.1. Then*

$$\nu(M, \phi_g) \equiv 3\eta(B_{\mathcal{O}}) - 24\eta(D_{\mathcal{O}}) + 24(1 + b_1(\mathcal{O})) \mod 48 \tag{2}$$

We will see that this result also holds under weaker conditions than those imposed by Hypothesis 1.1; see Proposition 6.3. The right-hand side of (2) is straightforward to compute. The odd signature η-invariant of the orbifold $\mathcal{O} = T^7/\Gamma$ may be identified with the evaluation at $s = 0$ of

$$\eta(B_{\mathcal{O}})(s) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda) \cdot \dim(E_\lambda^\Gamma) \cdot |\lambda|^{-s} \tag{3}$$

where λ ranges over the non-zero eigenvalues of the odd signature operator for T^7 with corresponding eigenspaces E_λ and Γ-invariant subspaces $E_\lambda^\Gamma \subset E_\lambda$. In particular, we have

$$\eta(B_{\mathcal{O}}) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \eta_\gamma(B_{T^7}) \tag{4}$$

where $\eta_\gamma(B_{T^7})$ is the equivariant η-invariant, obtained from (3) by replacing $\dim(E_\lambda^\Gamma)$ with $\mathrm{Tr}(\gamma|E_\lambda)$ and evaluating at $s = 0$. Similar remarks hold for the η-invariant of the spin Dirac operator. We then compute these equivariant η-invariants using standard techniques, as in [APS75b, Don78, MP06]. We compute ν (mod 24) for all examples in Joyce's original papers [Joy96], and for a majority of these we compute ν (mod 48). Some results from our computations are:

- The two simply-connected torsion-free G_2-manifolds with $b_2 = 2$ and $b_3 = 10$ constructed by Joyce in [Joy96] have distinct ν invariants.
- For almost all examples considered, we compute $\nu \equiv 0$ (mod 24).
- For all of the examples considered, we compute $\nu \equiv 0$ (mod 3).

For general remarks on the possible values for ν, see e.g. [CGN18b]. The range of values ν takes for the examples in [Joy96] is small; see Table 1 and Figure 3. We expect that further computations, obtained in part by relaxing our assumptions on the singular set, will lead to a greater range of values. Indeed, our constraints on the singular set of $\mathcal{O}$ are not necessarily optimal for Theorem 1.2, and are mainly imposed by our methods.

The proof of Theorem 1.2 compares the invariants for the orbifold $\mathcal{O}$ and its resolution G_2-manifold through gluing formulae for η-invariants as described in [Bun95, KL04]; such formulae were used in [CGN18a]. However, our situation does not fit the hypotheses of these formulae: our gluing region is not isometric to an interval times a hypersurface. The central observation is that the invariants under consideration are insensitive to regions of the geometry locally isometric to a product in which one factor is a flat manifold, as is the case near the singularities in $\mathcal{O}$. We may then modify the geometry in these regions to satisfy the hypotheses of the gluing formulae.

A more careful analysis of the behavior of η-invariants under resolutions of orbifold singularities should lift Theorem 1.2 to the integers, and compute $\overline{\nu}$. This lift is achieved in the current article for η-invariants of the odd signature operator; the case of the spin Dirac operator is more delicate.

In the final section of the paper, we show how some of our computations can be recovered by decomposing Joyce's orbifolds along a flat 6-torus, similar to the decompositions of twisted connected sums.

Finally, we mention that a more analytical approach to computing $\overline{\nu}$ for Joyce's manifolds was studied in the PhD work of Nelvis Fornasin [For].

Acknowledgments. The author would like to thank Simon Donaldson for his support and encouragement, as well as Nelvis Fornasin, Sebastian Goette and Johannes Nordström for fruitful discussions. The "twisted connected sum" type decompositions in Section 8 were pointed out to the author by Sebastian Goette and Johannes Nordström. The author was supported by the Simons Collaboration Grant on *Special Holonomy in Geometry, Analysis and Physics*.

2 Hypotheses on the singular set

We first discuss Hypothesis 1.1 in the general context of Joyce's construction as described in [Joy00, Chapter 11]. Let Λ be a lattice in $\mathbb{R}^7$, isomorphic as an abelian group to $\mathbb{Z}^7$. Then the quotient $T^7 = \mathbb{R}^7/\Lambda$ is a 7-torus. Every point $x \in T^7$ may be written as $v + \Lambda$ for some $v \in \mathbb{R}^7$ and every tangent space T_xT^7 is naturally isomorphic to $\mathbb{R}^7$. The flat Euclidean G_2-structure on $\mathbb{R}^7$, described for example by the 3-form (17), descends to a flat G_2-structure on T^7.

Let Γ be a finite group acting on T^7 preserving its G_2-structure. Then $\mathcal{O} = T^7/\Gamma$ is a flat orbifold, with an inherited flat orbifold metric g^0. For a subgroup $A \subset \Gamma$ let Fix(A) denote the fixed points of A. Let $\mathcal{L}$ be the set of $F \subset T^7$ such that F is a connected component of Fix(A) for some subgroup $A \subset \Gamma$. Write $\mathcal{L} = \{F_i\}_{i\in I}$ where $0 \in I$ is the index such that $F_0 = T^7$. Then

$$\mathrm{Sing}(\mathcal{O}) = \bigcup_{i\in I\setminus\{0\}} F_i \Big/ \Gamma \subset \mathcal{O}$$

forms the singular set of $\mathcal{O}$. From [Joy00, Proposition 11.1.3], each F_i with $i \neq 0$ is either a 1-torus or a 3-torus. In general, we may have two distinct 3-tori F_i and F_j that intersect in a 1-torus F_k.

The normalizer $N(F)$ of a subset $F \subset T^7$ is the subgroup of $\gamma \in \Gamma$ such that $\gamma F = F$, and the centralizer $C(F)$ is the subgroup of $\gamma \in \Gamma$ that act as the identity on F. For $F_i \in \mathcal{L}$ define $A_i = C(F_i)$

and $B_i = N(F_i)/C(F_i)$. Then $F_i \subset \mathrm{Fix}(A_i)$. As γF_i is a component of $\mathrm{Fix}(\gamma A_i \gamma^{-1})$, there is an index denoted $\gamma \cdot i \in I$ with $\gamma F_i = F_{\gamma \cdot i}$. The part of $\mathrm{Sing}(\mathcal{O})$ coming from F_i is

$$\Gamma F_i/\Gamma = \bigcup_{\gamma \in \Gamma} F_{\gamma \cdot i} \Big/ \Gamma$$

Now suppose $F_i \cap F_{\gamma \cdot i} = \varnothing$ for each $\gamma \in \Gamma$ such that $\gamma \cdot i \neq i$. Then $\mathcal{O}$ is isomorphic near $\Gamma F_i/\Gamma$ to

$$Z_i := (F_i \times W_i/A_i)/B_i \tag{5}$$

Here, if V_i is the invariant subspace of the action of A_i lifted to $\mathbb{R}^7$, then W_i is the orthogonal complement of V_i. If B_i acts freely on F_i then the singular locus of (5) is the image of F_i. By [Joy00, Proposition 11.1.3], W_i/A_i is of the form $\mathbb{C}^2/G$ or $\mathbb{C}^3/G$ for G some subgroup of $SU(2)$ or $SU(3)$. Thus when B_i is trivial and $\dim F_i = 3$, the neighborhood (5) recovers the description (1). Most of our arguments go through under the following weakening of Hypothesis 1.1:

Hypothesis 2.1. *Each connected component of the singular set in $\mathcal{O} = T^7/\Gamma$ has a neighborhood isometric to a neighborhood of the singular set in the orbifold*

$$(T^{7-2n} \times \mathbb{C}^n/G)/B \tag{6}$$

where $n \in \{2, 3\}$, G is a finite subgroup of $SU(2)$ or $SU(3)$, the action of B preserves the two factors and acts freely on the torus. Furthermore, the torus T^{7-2n} admits an orientation-reversing isometry τ such that $\tau \times \mathrm{id}$ descends to define an isometry of (6).

Hypothesis 2.1 is satisfied if: (i) $F_i \cap F_{\gamma \cdot i} = \varnothing$ whenever $\gamma \cdot i \neq i$; (ii) each B_i acts freely on F_i; and (iii) each F_i admits an orientation-reversing isometry, which by extension to the identity on W_i/A_i induces an orientation-reversing isometry of Z_i. If B_i is trivial, then (iii) automatically holds. If B is $\mathbb{Z}/2$ and F_i is a 1-torus then (iii) also holds.

3 Flexibility of metric

Let (X, g) be a Riemannian manifold. Suppose for some open subset $U \subset X$ there is an isometry $\phi_U : (F, g_F) \times (V, g_V) \to (U, g|_U)$, where (F, g_F) is flat and of dimension $\geqslant 1$, and (V, g_V) is any Riemannian manifold. We say a metric h on X is related to g by a *flat factor move* if $g|_{X \setminus U} = h|_{X \setminus U}$ and $\phi_U^*(h|_U) = g_F + h_V$ for some open set $U \subset X$, isometry ϕ_U as above, and metric h_V on V. We say g and h are *flat factor equivalent* if they are related by a sequence of flat factor moves. Observe that it suffices in the definition to consider $(F, g_F) = (I, dt^2)$ for interals $I \subset \mathbb{R}$.

It was observed in [APS75b] that the odd signature η-invariant is invariant under conformal changes of the metric, and that the same is true, modulo $\mathbb{Z}$, for the reduced η-invariant of the spin Dirac type operator. Here the *reduced* η-invariant of an operator D is defined by

$$\xi(D) = \frac{1}{2}\left(\eta(D) + \dim \ker(D)\right)$$

The argument used there may be adapted to show the following:

Proposition 3.1. *Let X be a closed oriented odd-dimensional manifold, with flat factor equivalent metrics g and h. The odd signature η-invariants for (X, g) and (X, h) are equal. If X is given a spin structure, the same is true for the reduced spin Dirac η-invariants taken modulo $\mathbb{Z}$.*

Proof. It suffices to prove the claim when h and g are related by a flat factor move. Equip $[0,1]\times X$ with the metric $G = ds^2 + \lambda(s)h + (1-\lambda(s))g$ where $\lambda : [0,1] \to \mathbb{R}$ is a bump function equal to 0 for $s \in [0, 1/4]$ and equal to 1 for $s \in [3/4, 1]$. Write $B_{(M,g)}$ for the odd signature operator defined using the metric g. Then by the Atiyah-Patodi-Singer theorem [APS75a, Theorem 4.14] we have

$$\eta(B_{(M,h)}) - \eta(B_{(M,g)}) = \int_{[0,1]\times X} L(p(G))$$

where $L(p(G))$ is the Hirzebruch L-polynomial applied to the Pontryagin forms of the Riemannian metric G on $[0,1]\times X$. We use the following elementary property: the top degree term of $L(p(G))$ vanishes if the metric G is a non-trivial product metric with one of its factors a flat metric. On $X\backslash U$ the metrics g and h are equal, and thus $G = ds^2 + g|_{X\backslash U}$ has the flat factor $([0,1], ds^2)$. This implies that the integral of $L(p(G))$ over $[0,1]\times(X\backslash U)$ vanishes. Next,

$$\int_{[0,1]\times U} L(p(G)) = \int_{[0,1]\times F\times V} L(p(ds^2 + g_F + \lambda(s)h_V + (1-\lambda(s))g_V)) = 0,$$

as the metric appearing on the right, equal to $(\mathrm{id}\times\phi_U)^*(G)$, has the flat factor (F, g_F). The claim for the reduced η-invariant of the spin Dirac operator follows the same argument, using [APS75a, Theorem 4.2], with the $\widehat{A}$-polynomial in place of L. In this latter case, the index of the Dirac operator on $[0,1]\times X$ is not a topological invariant, and so the result holds only modulo $\mathbb{Z}$. □

In conclusion, the quantities $\eta(B_M)$ and $\xi(D_M)$ (mod $\mathbb{Z}$) are invariants of the equivalence class of a metric generated by conformal changes and flat factor equivalences. Note that these two notions are distinct. For example, $T^2\times(S^2\backslash\mathrm{pt})$ and $T^2\times\mathbb{R}^2$ with their standard metrics are flat factor equivalent but not conformally equivalent.

4 Comparison of odd signature η-invariants

Starting from a flat G_2-orbifold, the construction of Joyce proceeds in two steps. First, a smooth closed Riemannian 7-manifold (M, g^t) with a closed G_2-structure ϕ^t is constructed by choosing resolutions for the singularities and pasting structures together using a partition of unity. Then he shows that g^t and ϕ^t may be deformed into a torsion-free G_2-structure on M. In this section we compare the odd signature η-invariants of the flat orbifold $(\mathcal{O}, g^0)$ and (M, g^t).

Remark 4.1. *The parameter t is any small positive real number, and roughly represents the size of the glued-in resolution pieces. However, this will only be important in Section 6.*

We first recall the relevant aspects of Joyce's construction from [Joy96, Joy00]. We assume Hypothesis 2.1. We denote by $Z_i^\circ = (F_i\times D_i/A_i)/B_i$ the compact manifold with boundary obtained from $Z_i = (F_i\times W_i/A_i)/B_i$ of (5) by restricting to a small closed ball $D_i\subset W_i$ centered at the origin. Let M° be obtained from $\mathcal{O}$ by deleting neighborhoods of the singular set corresponding to the Z_i° and taking the closure. Then we have the decomposition

$$\mathcal{O} = M^\circ \cup \bigcup_{\Gamma i\in I/\Gamma} Z_i^\circ \tag{7}$$

Let $i\in I$ be such that F_i is a 3-torus. Choose an ALE Riemannian 4-manifold X_i with holonomy $SU(2)$ which is asymptotic to W_i/A_i, and a free isometric action of B_i on X_i such that $(F_i\times$

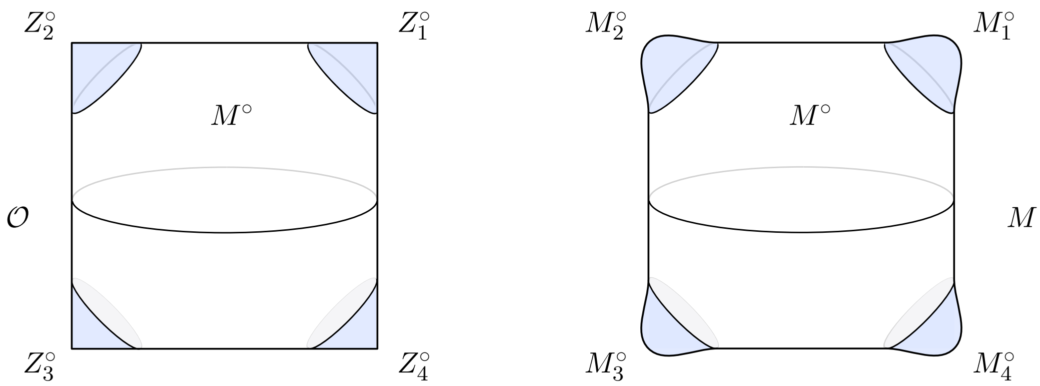

Figure 1: Schematic decompositions of $\mathcal{O}$ and M.

$X_i)/B_i$ is asymptotic to Z_i in the sense of [Joy00, Definition 11.2.2]. In particular, X_i is a non-compact 4-manifold with one end, which near infinity metrically resembles the end of W_i/A_i. When F_i is a 1-torus, we instead choose X_i to be a Quasi-ALE 6-manifold with holonomy $SU(3)$ as in [Joy00, Chapter 9]. We choose such data for each orbit Γi. Set $M_i = (F_i \times X_i)/B_i$. Then M_i is a smooth 7-manifold with one end, which may be truncated to obtain a 7-manifold with boundary, denoted M_i°. The resolution manifold M is then topologically

$$M = M^\circ \cup \bigcup_{\Gamma i \in I/\Gamma} M_i^\circ \tag{8}$$

The decompositions (7) and (8) are schematically depicted in Figure 1; there, $\mathcal{O}$ is replaced by a 2-dimensional orbifold with 4 singular points, and M is a corresponding resolution. Write $N_i = \partial Z_i^\circ = \partial M_i^\circ$ and $N = \bigcup N_i = \partial M^\circ$. For small ε and $t \in (0, \varepsilon]$, a closed G_2-structure ϕ^t and Riemannian metric g^t on M are constructed in [Joy96, Joy00] by patching together the torsion-free G_2-structures on the different pieces using a partition of unity.

Proposition 4.2. *If Hypothesis 2.1 holds, then* $\eta(B_{(M,g^t)}) = \eta(B_{(\mathcal{O},g^0)})$.

Before proceeding to the proof we make some remarks on the metrics involved, all of which are clear from Joyce's construction. We may choose the decompositions above such that $(M^o, g^t|_{M^o})$ is isometrically identified with $(M^o, g^0|_{M^\circ})$, and we may arrange that this holds for all $t \in (0, \varepsilon]$.

The metric g^t is constructed such that $(M_i^o, g^t|_{M_i^o})$ is locally the Riemannian product of metrics on F_i and X_i. Indeed, the flat metric g^0 has a compatible product structure near the boundary of M° and thus the partition of unity respects this structure. In particular, Hypothesis 2.1 guarantees that each of $(M_i^o, g^t|_{M_i^o})$ has an orientation-reversing isometry.

To prove Proposition 4.2 we invoke a gluing formula for odd signature η-invariants. As our application is similar to that of [CGN18a], we also refer the reader there for more details.

Theorem 4.3. [KL04, Theorem 8.12] *Let X be a closed, oriented odd-dimensional Riemannian manifold and $Y \subset X$ a hypersurface separating X into X_+ and X_-. Suppose the Riemannian metric on X is a product in a collar neighborhood of Y. Then*

$$\eta(B_M) = \eta_{\text{APS}}(B_{X_+}, V_+) + \eta_{\text{APS}}(B_{X_-}, V_-) + m(V_+, V_-; H^*(Y)) \tag{9}$$

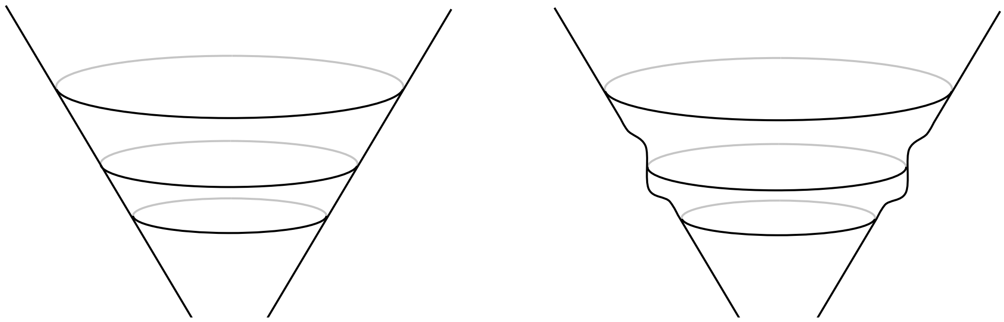

Figure 2: Modifying a metric near a collar neighborhood to a product metric.

with real coefficients in cohomology understood, where the last term is the Maslov index of the Lagrangian subspaces $V_\pm \subset H^*(Y)$ *defined by* $V_\pm = \operatorname{im}(H^*(X_\pm) \to H^*(Y))$.

To apply this result, we must modify our metrics. Suppose the ball $D_i \subset W_i$ has radius R. In a collar neighborhood of the boundary of M° inside $(\mathcal{O}, g^0)$, the metric is the quotient of a metric on $F_i \times W_i$ of the form $g_1 + dr^2 + r^2 g_2$, where g_1 is the metric on F_i and g_2 is the metric on the unit sphere in W_i. Here r is the radial coordinate in W_i and $r \in [R-\epsilon, R+\epsilon]$ for some small $\epsilon > 0$. This is not a product metric for the collar, but may be modified as follows.

Choose a smooth function $\rho : [R-\epsilon, R+\epsilon] \to \mathbb{R}$ such that $\rho(r) = r$ for $|R-r| > 2\epsilon/3$ and $\rho(r) = 1$ for $|R-r| < \epsilon/3$. Replace $g_1 + dr^2 + r^2 g_2$ with $g_1 + dr^2 + \rho(r)^2 g_2$; the effect of altering the metric $dr^2 + r^2 g_2$ on a collar neighborhood of the sphere $\partial D_i \subset W_i$ to a product metric is depicted in Figure 2. This replacement is compatible with the actions of A_i and B_i, and in this way we obtain a new metric g^0_c on $\mathcal{O}$ unaltered outside of small collar neighborhoods of the gluing boundaries. As the modification is radial in W_i, the manifold $(Z^o_i, g^0_c|_{Z^o_i})$ retains an orientation-reversing isometry. As F_i is a non-trivial flat manifold, the metric g^0_c is flat factor equivalent to g^0. By Proposition 3.1, we may compute $\eta(B_{\mathcal{O}})$ using g^0_c in place of g^0.

We make a similar modification of the metric g^t on M. In a gluing collar neighborhood as above, the metric g^t is of the form $g_1 + h^t$ where h^t is glued together from $dr^2 + r^2 g_2$ and a metric on X_i which may be arranged close to $dr^2 + r^2 g_2$. We alter g^t to equal $dr^2 + g_2$ on $|R-r| < \epsilon/3$, and so that it is unchanged for $|R-r| > 2\epsilon/3$ and outside of the gluing collar. This may all be done B_i-equivariantly, either directly or by averaging afterwards. We call the resulting metric g^t_c. By Proposition 3.1, we may compute $\eta(B_{(M,g^t)})$ using g^t_c in place of g^t. We may also arrange that $(M^\circ_i, g^t_c|_{M^\circ_i})$ retains an orientation-reversing isometry.

Proof of Proposition 4.2. As is evident from the proof of [KL04, Theorem 8.12], the statement of Theorem 4.3 works equally if X is a Riemannian orbifold isometric to a quotient by a finite group, as long as the hypersurface Y is disjoint from the singular set.

We compare the applications of (9) to $(\mathcal{O}, g^0_c)$ and (M, g^t_c) each with $X_+ = M^\circ$. The term $\eta_{\mathrm{APS}}(B_{X_+}, V_+)$ is the same in both applications. The term $\eta_{\mathrm{APS}}(B_{X_-}, V_-)$ vanishes in both cases, as $(Z^\circ_i, g^0_c|_{Z^\circ_i})$ and $(M^\circ_i, g^t_c|_{M^\circ_i})$ have orientation-reversing isometries preserving the Lagrangian subspace V_-. This vanishing argument is the same as in [CGN18a, Section 4.3].

Thus $\eta(B_M) = \eta(B_{\mathcal{O}})$ holds if we show that the Maslov index $m(V_+, V_-; H^*(Y))$ in the two cases are the same. According to [CGN18a, Remark 4.3], this Maslov index only depends on

$\mathrm{im}(H^3(X_\pm) \to H^3(Y))$. By additivity under disjoint union, it suffices to show that the images of $H^3(Z_i^\circ)$ and $H^3(M_i^\circ)$ in $H^3(N_i)$ are equal for each i.

There are two cases to consider. First suppose that F_i is a 1-torus. We may identify $\partial D_i \subset W_i$ with S^5. Note $H^*(S^5/A_i) = H^*(S^5)^{A_i} = H^*(S^5)$. Then

$$H^3(N_i) = H^3((S^1 \times S^5/A_i)/B_i) = H^3(S^1 \times S^5/A_i)^{B_i} = 0,$$

and there is nothing to check. Next suppose F_i is a 3-torus. We may identify $\partial D_i \subset W_i$ with S^3. Now $H^*(S^3/A_i)$ is isomorphic to $H^*(S^3)$. Thus $H^3(N_i)$ may be identified with the B_i-invariant subpace of $H^3(T^3) \oplus H^3(S^3/A_i)$. The map $H^3(Z_i^\circ) \to H^3(N_i)$ has image $H^3(T^3)^{B_i}$. Indeed, under these identifications it is the B_i-invariant image of the map

$$H^3(T^3) \oplus H^3(D^4/A_i) \to H^3(T^3) \oplus H^3(S^3/A_i). \tag{10}$$

The image of $H^3(M_i^\circ)$ is exactly the same; in (10), D^4/A_i is replaced by the truncation X_i° of X_i with boundary S^3/A_i, and $H^3(X_i^\circ) \to H^3(S^3/A_i)$ is zero by the long exact sequence of a pair. □

It is clear that Proposition 4.2 holds under more general conditions than stated. In particular, we have not used anything about G_2-structures. Similar computations may be done for resolutions of any orbifold T^n/Γ, where n is aribitrary, and the singular set behaves reasonably well, as in Hypothesis 2.1; the dimensions of the fixed point tori need not be 1 and 3. Note that the modification of metrics used above is valid even if $\dim F_i = 0$, for in this case the alteration is conformal.

5 Comparison of spin Dirac η-invariants

For the Dirac η-invariants, we follow the same strategy. We continue assuming Hypothesis 2.1 and use the setup of Section 4. We will apply the following mod $\mathbb{Z}$ gluing formula.

Theorem 5.1. [Bun95],[KL04, Theorem 5.9] *Let X be a closed, oriented odd-dimensional Riemannian manifold and $Y \subset X$ a hypersurface separating X into X_+ and X_-. Suppose the metric on X is a product in a neighborhood of Y, and the tangential operator on Y is invertible. Then*

$$\xi(D_X) = \xi_{\text{APS}}(D_{X_+}) + \xi_{\text{APS}}(D_{X_-}) \mod \mathbb{Z} \tag{11}$$

We may apply Theorem 5.1 to both (M, g_c^t) and $(\mathcal{O}, g_c^0)$, as the collar neighborhood in each case is a union of manifolds $[-\epsilon, \epsilon] \times N_i$, where N_i is a finite quotient of either $I \times T^3 \times S^3/A_i$ or $I \times S^1 \times S^5/A_i$. In each case, $Y = \cup N_i$ has a metric of positive scalar curvature, and so has no harmonic spinors. Consequently, the tangential operator is invertible.

We would like to show $\xi(D_{(M,g^t)}) \equiv \xi(D_{(\mathcal{O},g^0)})$ (mod $\mathbb{Z}$). From the gluing formula (11), it suffices to show that $\xi_{\text{APS}}(D_{X_-}) = 0$ (mod $\mathbb{Z}$) in the two cases of (M, g_c^t) and $(\mathcal{O}, g_c^0)$. In each case, Hypothesis 2.1 guarantees that X_- admits an orientation-reversing isometry. If this isometry is *spin*, then $\eta_{\text{APS}}(D_{X_-}) = 0$. This implies $\xi_{\text{APS}}(D_{X_-}) = h_{\text{APS}}(D_{X_-})/2$, where $h_{\text{APS}}(D_{X_-})$ denotes the dimension of the kernel of the Dirac operator D_{X_-} with APS boundary conditions.

Remark 5.2. *If the orientation-reversing isometries in Hypothesis 2.1 preserve the spin structure, we say that Hypothesis 2.1* with spin isometries *holds.*

Thus far we have established that $\xi(D_{(M,g^t)}) \equiv \xi(D_{(\mathcal{O},g^0)}) \pmod{\frac{1}{2}\mathbb{Z}}$ under the assumption of Hypothesis 2.1 with spin isometries.

Now assume Hypothesis 1.1. As X_- is isometric in each case to a product $T^3 \times V$ where V is some Riemannian manifold, the Künneth theorem for elliptic complexes implies $h_{\text{APS}}(D_{X_-}) = h(D_{T^3})h_{\text{APS}}(D_V)$, compare [Hit74, p.12]. Harmonic spinors on T^3 are in correspondence with constant vectors in the standard 2-dimensional representation of $\text{Spin}(3)$ via parallel transport. Thus $h(D_{T^3}) = 2$ and $h_{\text{APS}}(D_{X_-}) \equiv 0 \pmod 2$. It follows that $\xi_{\text{APS}}(D_{X_-}) = 0 \pmod{\mathbb{Z}}$. We conclude:

Proposition 5.3. *If Hypothesis 1.1 holds, then $\xi(D_{(M,g^t)}) \equiv \xi(D_{(\mathcal{O},g^0)}) \pmod{\mathbb{Z}}$. If only Hypothesis 2.1 with spin isometries holds, then this congruence holds modulo $\frac{1}{2}\mathbb{Z}$.*

However, in many cases the mod $\mathbb{Z}$ congruence still holds under weaker conditions than those given in Hypothesis 1.1. We next describe a common feature of many examples encountered in Section 7 for which the desired congruence continues to hold.

Situation 5.4. Suppose Hypothesis 2.1 holds. Let $\{F_i/B_i\}_{i\in J}$ be the connected components of the singular set of $\mathcal{O}$ which are not 3-tori. Here $J \subset I/\Gamma$. When defining the resolution manifold M, we replace a neighborhood Z_i° of each F_i/B_i with M_i°. We suppose that for each $i \in J$, there are an even number of $k \in J$ such that the resolution data M_k° is isomorphic to that of M_i°.

Proposition 5.5. *In Situation 5.4, $\xi(D_{(M,g^t)}) \equiv \xi(D_{(\mathcal{O},g^0)}) \pmod{\mathbb{Z}}$.*

Proof. Let us return to the argument for the proof of Proposition 5.3 given above, and focus on the case of $(\mathcal{O}, g^0)$. We may write $X_- = \cup Z_i^\circ$ where i ranges over the connected components of the singular set of $\mathcal{O}$. Then $h_{\text{APS}}(D_{X_-}) = \sum h_{\text{APS}}(D_{Z_i^\circ})$. The components with $F_i/B_i = T^3$ have $h_{\text{APS}}(D_{Z_i^\circ}) \equiv 0 \pmod 2$ as argued in the case of Hypothesis 1.1. Situation 5.4 allows us to gather the remaining $h_{\text{APS}}(D_{Z_i^\circ})$ into groups of equal terms of even cardinality, implying $h_{\text{APS}}(D_{X_-}) \equiv 0 \pmod 2$. The same holds for the case of (M, g^t). Then $\xi(D_{(M,g^t)}) \equiv \xi(D_{(\mathcal{O},g^t)}) \pmod{\mathbb{Z}}$ follows again from the gluing formula (11), as claimed. □

6 Flexibility of G_2-structure

Let (M, g) be a closed Riemannian 7-manifold. Suppose a spinor bundle SM over M is chosen, and $s \in \Gamma(SM)$ a non-vanishing spinor. Let g^{SM} and ∇^{SM} denote the metric and connection on SM induced by g and the Levi-Civita connection of g. Then Crowley–Goette–Nordström [CGN18a] define $\overline{\nu}$, the integer-valued extended ν-invariant of (M, g, s), as follows:

$$\overline{\nu}(M, g, s) = 2\int_M s^*\psi(\nabla^{SM}, g^{SM}) + 3\eta(B_M) - 24\eta(D_M) \in \mathbb{Z} \tag{12}$$

We describe the Mathai–Quillen current $s^*\psi(\nabla^{SM}, g^{SM})$ following [CGN18a]. The curvature R^{SM} is an element of $\Omega^2(M; \Lambda^2 SM)$, and $\nabla^{SM}s$ of $\Omega^1(M; SM)$. We have

$$s^*\psi(\nabla^{SM}, g^{SM}) = \int_0^\infty \int^B \frac{s}{2\sqrt{t}} e^{-R^{SM}+\sqrt{t}\nabla^{SM}s+t\|s\|^2}\,dt \tag{13}$$

Here $\int^B : \Omega^*(M; \Lambda^* SM) \to \Omega^*(M)$ denotes the Berezin integral, extracting a certain constant multiple of the top-degree component in $\Lambda^* SM$. In particular, $s^*\psi(\nabla^{SM}, g^{SM})$ is a differential form on M, not necessarily homogeneous.

As described in [CN15, §2.3], unit spinors in $\Gamma(SM)$ are in correspondence with G_2-structures on M. If ϕ is the G_2-structure corresponding to s, then by [CGN18a, Theorem 1.2], we have

$$\nu(M,\phi) \equiv \overline{\nu}(M,g,s) + 24h(D_M) \mod 48 \tag{14}$$

for any metric g. Here $h(D_M)$ is the dimension of the kernel of the spin Dirac operator D_M. When s and g are determined by a G_2-structure ϕ we write $\overline{\nu}(M,\phi) = \overline{\nu}(M,g,s)$. When s is g-parallel, or equivalently when the corresponding G_2-structure is compatible with g and torsion-free, then it is shown [CGN18a, Lemma 1.3] that $s^*\psi(\nabla^{SM}, g^{SM}) = 0$, leading to

$$\overline{\nu}(M,g) = 3\eta(B_M) - 24\eta(D_M) \equiv \nu(M,\phi) + 24(1+b_1(M)) \mod 48.$$

Here is used the fact that $h(D_M) = 1 + b_1(M)$ for a closed spin Riemannian manifold with holonomy contained in G_2. We presently determine some other conditions under which the term $\int_M s^*\psi(\nabla^{SM}, g^{SM})$ in (12) vanishes.

We say that (M,g,s) as above is *torsion-free up to flat factors* if there is an open set $U \subset M$ such that s is ∇^{SM}-parallel on the complement $M\backslash U$, and U is covered by open sets U_i each with an isometry $\phi_i : (F_i, g_{F_i}) \times (V_i, g_{V_i}) \to (U_i, g|_{U_i})$, where (F_i, g_{F_i}) is flat and of dimension $\geqslant 1$, and the spinor s is ∇^{SM}-parallel in the directions $(\phi_i)_*(v)$ where $v \in TF_i$. In short, (M,g) is a torsion-free G_2-manifold away from U with parallel spinor s, and on U, the metric g locally splits off a flat factor, and s is parallel with respect to this flat factor. As in the definition of flat factor equivalence, it suffices to consider $(F, g_F) = (I, dt^2)$ for intervals $I \subset \mathbb{R}$.

Proposition 6.1. *Suppose (M,g,s) is torsion-free up to flat factors. Then*

$$\overline{\nu}(M,g,s) = 3\eta(B_M) - 24\eta(D_M).$$

Proof. For simplicity we assume that the open covering $\{U_i\}$ consists only of U, and that (F, g_F) is isometric to an interval (I, dt^2). The computation is local on M and the general case easily follows. We let $v \in \Gamma(TM|_U)$ be the vector field on U induced by $\partial/\partial t$. Thus (M,g,s) has $\nabla^{SM}s = 0$ on $M\backslash U$, and $\nabla^{SM}_v s = 0$ on U. By (12) it suffices to show $\int_M s^*\psi(\nabla^{SM}, g^{SM}) = 0$. First,

$$\int_{M\backslash U} s^*\psi(\nabla^{SM}, g^{SM}) = 0$$

because $s^*\psi(\nabla^{SM}, g^{SM})|_{M\backslash U} = 0$, as explained in [CGN18a, Lemma 1.3]. The argument is as follows. In the expression (13), the terms R^{SM}, $\nabla^{SM}s$ and $\|s\|^2$ have their degrees with respect to Λ^*SM given by 2, 1 and 0, respectively. As $\nabla^{SM}s = 0$ on $M\backslash U$, it follows that the exponential in (13) is of even degree. This is multiplied by s, yielding an expression of odd degree in Λ^*SM. As $\operatorname{rank}(SM) = 8$, the Berezin integral vanishes, implying $s^*\psi(\nabla^{SM}, g^{SM})|_{M\backslash U} = 0$.

Next, we similarly claim that the Mathai–Quillen term vanishes over U:

$$\int_U s^*\psi(\nabla^{SM}, g^{SM}) = 0 \tag{15}$$

In contrast to the argument above, it is no longer necessarily true that $s^*\psi(\nabla^{SM}, g^{SM})|_U = 0$. However, as the contraction of R^{SM} with v is zero, and $\nabla^{SM}_v s = 0$, from (13) we easily see that the contraction of the differential form $s^*\psi(\nabla^{SM}, g^{SM})$ with v is zero. This implies that the top degree term of this differential form vanishes on U, from which (15) follows. □

The manifolds (M, g^t) defined by Joyce with closed G_2-structures ϕ^t, obtained by resolving an orbifold T^7/Γ satisfying Hypothesis 2.1, are torsion-free up to flat factors.

Corollary 6.2. *Let (M, g^t) for $t \in (0, \epsilon]$ be a resolution of a flat G_2-orbifold T^7/Γ satisfying Hypothesis 2.1, with closed G_2-structure ϕ^t as defined by Joyce. Then*

$$\overline{\nu}(M, \phi_t) = 3\eta(B_{(M,g^t)}) - 24\eta(D_{(M,g^t)}). \tag{16}$$

If (M, g) is a torsion-free G_2-manifold obtained from (M, g^t, ϕ^t) for $t \ll \epsilon$ then $\nu(M, g) \equiv \nu(M, \phi_t)$.

Proof. Equation (16) follows from Proposition 6.1 and the observation that (M, g^t, ϕ^t) is torsion-free up to flat factors. The G_2-structures ϕ^t for $t \in (0, \epsilon]$ are all homotopic, with homotopies given by the parameter t. As the ν-invariant is invariant under homotopies of G_2-structures, $\nu(M, \phi^t)$ is independent of $t \in (0, \epsilon]$. For small enough t, there exists a torsion-free G_2-structure (M, g, ϕ) with $\|\phi^t - \phi\|_{C^0} \leqslant Kt^{1/2}$ for some constant K independent of t, see [Joy00, Section 11.6]. Thus for small enough t, ϕ is homotopic to ϕ^t, and hence $\nu(M, \phi) \equiv \nu(M, \phi_t)$. □

Proof of Theorem 1.2. Combine Corollary 6.2, equation (14), Propositions 4.2 and 5.3, and the following well-known observation, which has already been mentioned above: for a closed Riemannian manifold M with holonomy contained in G_2, we have $h(D_M) = 1 + b_1(M)$. This is because the G_2-structure induces an identification of the spinor bundle with $\underline{\mathbb{R}} \oplus T^*M$, and for closed Ricci-flat manifolds, 1-forms are parallel if and only if they are harmonic. This readily adapts to the orbifold setting, so that $h(D_{\mathcal{O}}) = 1 + b_1(\mathcal{O})$. □

In fact, from Propositions 5.3 and 5.5 we have the following extension.

Proposition 6.3. *Under Hypothesis 2.1 with spin isometries, the congruence (2) of Theorem 1.2 holds modulo 24. In Situation 5.4, Theorem 1.2 continues to hold, i.e. (2) holds modulo 48.*

7 Computations

We now use Theorem 1.2 and its extensions to compute ν for many of Joyce's G_2-manifolds. Let $\mathcal{O} = T^7/\Gamma$ be a flat G_2-orbifold. To compute ν using (2) it suffices to compute $\eta(B_{\mathcal{O}})$ and $\eta(D_{\mathcal{O}})$ (mod $2\mathbb{Z}$), which are averages of the equivariant invariants $\eta_\gamma(B_{T^7})$ and $\eta_\gamma(D_{T^7})$, as seen in (4). A summary of our computations is given by Figure 3.

7.1 Examples with vanishing η-invariants

The first class of orbifolds considered by Joyce in [Joy96] are as follows. Let $T^7 = \mathbb{R}^7/\mathbb{Z}^7$. This has a flat G_2-structure induced by the 3-form

$$\phi = dx_{127} + dx_{136} + dx_{145} + dx_{235} - dx_{246} + dx_{347} + dx_{567} \tag{17}$$

where x_i are coordinates on $\mathbb{R}^7$ and $dx_{ijk} = dx_i dx_j dx_k$. Let α, β, γ be the involutions

$$\begin{aligned}
\alpha(x_1, \ldots, x_7) &= (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7)\\
\beta(x_1, \ldots, x_7) &= (b_1 - x_1, b_2 - x_2, x_3, x_4, -x_5, -x_6, x_7)\\
\gamma(x_1, \ldots, x_7) &= (c_1 - x_1, x_2, c_3 - x_3, x_4, c_5 - x_5, x_6, -x_7)
\end{aligned}$$

where b_1, b_2, c_1, c_3, $c_5 \in \{0, 1/2\}$ are fixed. These involutions preserve ϕ and generate a group isomorphic to $(\mathbb{Z}/2)^3$. Examples 1–5 of [Joy96] are obtained from resolutions of $\mathcal{O} = T^7/\Gamma$ for subgroups $\Gamma \subset \langle \alpha, \beta, \gamma \rangle$ with b_1, b_2, c_1, c_3, c_5 fixed constants. The orientation-reversing isometry $(x_1, \ldots, x_7) \mapsto (-x_1, \ldots, -x_7)$ commutes with α, β, γ and reflects the eigenspaces of the Dirac operator. Thus $\eta_g(B_{T^7}) = \eta_g(D_{T^7}) = 0$ for all $g \in \langle \alpha, \beta, \gamma \rangle$, implying $\eta(B_{\mathcal{O}}) = \eta(D_{\mathcal{O}}) = 0$. Example 6 of [Joy96] includes a generator sending $(x_1, \ldots, x_7)$ to $(\frac{1}{2} + x_1, x_2, \frac{1}{2} + x_3, \frac{1}{2} + x_4, \frac{1}{2} + x_5, x_6, x_7)$; this also commutes with the above reflection, so the same holds in this case.

For each of these examples, the singular set of $\mathcal{O}$ is a disjoint union of T^3 and $T^3/\mathbb{Z}/2$ where $\mathbb{Z}/2$ acts by $(y_1, y_2, y_3) \mapsto (\frac{1}{2} + y_1, -y_2, -y_3)$. Thus Hypothesis 2.1 with spin isometries is satisfied, and by Proposition 6.3, we conclude that Examples 1–6 of [Joy96] have $\nu \equiv 0 \pmod{24}$.

We next consider $\nu \pmod{48}$. Each neighborhood of T^3 is resolved using an Eguchi–Hanson space, and each neighborhood of $T^3/\mathbb{Z}/2$ is resolved using an Eguchi–Hanson space with $\mathbb{Z}/2$-action, of which there are two choices. Let ℓ be the number of resolutions of one of these distinguished choices. If ℓ is even, we are in Situation 5.4, and by Proposition 6.3, we conclude that Examples 1–6 [Joy96] with $\ell \equiv 0 \pmod 2$ have $\nu \equiv 24(1 + b_1) \pmod{48}$. Note that all examples have $b_1 = 0$ except for Examples 1 and 2, which have $b_1 = 3$ and $b_1 = 1$, respectively; these two examples have holonomy groups strictly smaller than G_2.

7.2 Donnelly's formula for $\eta(B_{\mathcal{O}})$

Before proceeding to the next examples, we make some remarks. The torus $T^7 = \mathbb{R}^7/\Lambda$ admits an orientation-reversing spin isometry, induced by negation on $\mathbb{R}^7$, so $\eta(B_{T^7}) = \eta(D_{T^7}) = 0$. Next, suppose $\gamma \in \Gamma$ has non-empty, proper fixed point set F. Then F is isometric to either a 1-torus or a 3-torus. In each of the examples we consider, it is easy to find an orientation-reversing isometry of F that extends to T^7 which commutes with γ. Thus we may write

$$\eta(B_{\mathcal{O}}) = \frac{1}{|\Gamma|} \sum_{\substack{\gamma \in \Gamma \\ \mathrm{Fix}(\gamma) = \varnothing}} \eta_\gamma(B_{T^7})$$

(In our examples this follows also from direct computation.) To compute each term $\eta_\gamma(B_{T^7})$ for our next set of examples, we describe a situation considered by Donnelly [Don78]. Assume T^7 is isometric to $T^6 \times S^1$, i.e. the lattice $\Lambda \subset \mathbb{R}^7$ defining $T^7 = \mathbb{R}^7/\Lambda$ splits off an orthogonal rank 1 summand, so that $\Lambda = \Lambda' \oplus \mathbb{Z}$ for some rank 6 lattice $\Lambda' \subset \mathbb{R}^6$. Suppose for $\gamma \in \Gamma$ we have

$$\gamma(x, y) = (Ax + c, y + d) \tag{18}$$

where $y \in S^1 = \mathbb{R}/\mathbb{Z}$ and $x \in T^6$. Here A acts linearly and orthogonally on $\mathbb{R}^6$ preserving Λ', conjugate to the direct sum of three rotation matrices with angles $\gamma_1, \gamma_2, \gamma_3$, while $c \in T^6$ and $d \in S^1$. If A has eigenvalues ± 1 or $d = 0 \in S^1$ then $\eta_\gamma(B_{T^7}) = 0$. Otherwise we have

$$\eta_\gamma(B_{T^7}) = \nu(\gamma) \cot(\pi d) \cot(\gamma_1/2) \cot(\gamma_2/2) \cot(\gamma_3/2)$$

where $\nu(\gamma)$ is the number of fixed points of the extension of γ to $T^6 \times D^2$. This result is only a slight extension of [Don78, Proposition 4.7], and follows by applying the equivariant Atiyah–Patodi–Singer theorem to γ acting on $T^6 \times D^2$. Then for $\mathcal{O} = T^7/\Lambda$ we may write

$$\eta(B_{\mathcal{O}}) = \frac{1}{|\Gamma|} \sum_{\substack{\gamma \in \Gamma \\ \mathrm{Fix}(\gamma) = \varnothing}} \nu(\gamma) \cot(\pi d) \prod_{i=1}^{3} \cot(\gamma_i/2). \tag{19}$$

Donnelly's method may be adapted to compute spin Dirac eta invariants, modulo $2\mathbb{Z}$, but not necessarily for the spin structure we want. We thus describe a more direct method which in addition computes the real number $\eta(D_{\mathcal{O}})$.

7.3 Direct computations for $\eta(D_{\mathcal{O}})$

Spinors on $\mathcal{O}$ are Γ-invariant spinors on T^7, which in turn are Λ-invariant spinors on $\mathbb{R}^7$. Following an observation which goes back to Friedrich [Fri84], the determination of eigenspinors for $\mathcal{O}$ quickly becomes representation-theoretic. A direct computation of $\eta(D_{\mathcal{O}})$ in this fashion is nearly contained in [MP06], which considers the case in which $\mathcal{O}$ is smooth, but the arguments carry over to the orbifold case without difficulty. Further, as we are not considering a twisted Dirac operator, and our spin structures are naturally induced by G_2-structures, our situation is considerably simpler. We review the key aspects, leaving only a few details to [MP06].

First, we describe eigenspaces of the Dirac operator acting on $T^7 = \mathbb{R}^7/\Lambda$. Let S denote the 8-dimensional complex spinor representation of Spin(7), equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle$ for which Clifford multiplication is skew-Hermitian. For $u \in \Lambda^*$ and $w \in S$ consider the spinor $f_{u,w} : T^7 \to S$ given by $f_{u,w}(x) = e^{2\pi i \langle u,x \rangle} w$. Write $D = D_{T^7}$ for the Dirac operator. Then

$$Df_{u,w}(x) = \sum_{j=1}^{7} e_j \cdot \frac{\partial}{\partial x_j} f_{u,w}(x) = 2\pi i u \cdot f_{u,w}(x).$$

The Clifford relation $u^2 = -|u|^2$ implies Clifford multiplication on S by u has eigenvalues $\pm i|u|$. Write $S_u^{\pm}$ for the $\mp i|u|$-eigenspaces. Then for $w \in S_u^{\pm}$ we have $Df_{u,w} = \pm 2\pi |u| f_{u,w}$. By the Stone–Weierstrass Theorem, the spinors $f_{u,w}$ for $u \in \Lambda^*$ and w ranging over bases of $S_u^{\pm}$ give a complete orthogonal system of $L^2(T^7; S)$. Thus the eigenspaces of D are $E_{\pm\mu}$ where

$$E_{\pm\mu} = \{ f_{u,w} : u \in \Lambda^*, \mu = 2\pi |u|, w \in S_u^{\pm} \}.$$

In fact, as the model case with u a multiple of e_1 shows, S_u^+ and S_u^- both have dimension 4 when $u \neq 0$, and we recover the fact that the spectrum of D is symmetric.

Now we turn to $\mathcal{O} = T^7/\Gamma$. Because the action of Γ on T^7 respects its G_2-structure, we have an induced action on S, to be described shortly, and a resulting action on spinors f defined by $(\gamma f)(x) = \gamma f(\gamma^{-1} x)$. Write $\Lambda_\mu^* = \{ u \in \Lambda^* : \mu = 2\pi |u| \}$ and $E_u^{\pm} = \{ f_{u,w} : w \in S_u^{\pm} \}$ so that $E_{\pm\mu} = \oplus_{u \in \Lambda_\mu^*} E_u^{\pm}$. Fix u and let $s_k^{\pm}$ be an orthonormal basis for $S_u^{\pm}$. Then for $\gamma \in \Gamma$ we compute

$$\mathrm{Tr}(\gamma|_{E_u^{\pm}}) \mathrm{vol}(T^7) = \sum_k \langle \gamma f_{u,s_k^{\pm}}, f_{u,s_k^{\pm}} \rangle_{L^2} = \sum_k \langle \gamma s_k^{\pm}, s_k^{\pm} \rangle \int_{T^7} e^{2\pi i \langle u, \gamma^{-1} x \rangle - 2\pi i \langle u, x \rangle} \tag{20}$$

Write $\gamma x = Bx + b$ where B is a linear map and $b \in T^7$. Noting $\gamma^{-1} x = B^{-1} x - B^{-1} b$, the integral on the right hand side of (20) is equal to $\delta_{Bu,u} \mathrm{vol}(T^7) e^{-2\pi i \langle u,b \rangle}$. From this we obtain

$$\mathrm{Tr}(\gamma|_{E_{\pm\mu}}) = \sum_{u \in (\Lambda_\mu^*)^B} \mathrm{Tr}(\gamma|_{E_u^{\pm}}) = \sum_{u \in (\Lambda_\mu^*)^B} e^{-2\pi i \langle u,b \rangle} \mathrm{Tr}(\gamma|_{S_u^{\pm}})$$

Now we focus on $\mathrm{Tr}(\gamma|_{S_u^{\pm}})$. We begin by describing S more explicitly. First, we recall that the 3-form ϕ defining the G_2-structure on $\mathbb{R}^7$ determines a cross-product $(u, v) \mapsto u \times v$ via the relation

$\phi(u,v,w) = \langle u \times v, w\rangle$. We extend the cross-product complex linearly to $\mathbb{C}^7$. We then define $S = (\mathbb{R} \oplus \mathbb{R}^7) \otimes \mathbb{C}$, and Clifford multiplication for $u \in \mathbb{R}^7$ and $(\lambda, v) \in \mathbb{C} \oplus \mathbb{C}^7 = S$ by

$$u \cdot (\lambda, v) = (-\langle v, u\rangle, \lambda u + u \times v),$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product on $S = \mathbb{C}^8$; compare [SW17, Section 10.2]. Let $e_1, \ldots, e_7$ be the standard basis for $\mathbb{R}^7$. With ϕ as in (17), we compute that

$$\begin{aligned} s_0^\pm &= (1, \pm i e_7)/\sqrt{2}, \quad s_1^\pm = (0, e_1 \pm i e_2)/\sqrt{2}, \\ s_2^\pm &= (0, e_3 \pm i e_4)/\sqrt{2}, \quad s_3^\pm = (0, e_5 \pm i e_6)/\sqrt{2} \end{aligned} \tag{21}$$

form an orthonormal basis for $S_u^\pm$ for $u \neq 0$ a real positive multiple of e_7. If u is a *negative* multiple of e_7 then these form bases for $S_u^\mp$. Note that $\gamma \in \Gamma$ acts on S by $\gamma(\lambda, v) = (\lambda, Bv)$. The rotation part of γ, written here as B, may be conjugated in Spin(7) to an element B' that fixes e_7 and rotates the (e_k, e_{k+1}) plane by an angle γ_k for $k \in \{1,2,3\}$, where $\gamma_1 + \gamma_2 + \gamma_3 \equiv 0 \pmod{2\pi}$. Also suppose that u, after the conjugation, is a real multiple of e_7, and let $\varepsilon_u = \operatorname{sign}(u/e_7) \in \{\pm 1\}$. Then $B' s_0^\pm = s_0^\pm$ and $B' s_k^\pm = e^{\mp i \gamma_k} s_k^\pm$ for $k \in \{1,2,3\}$, so that

$$\operatorname{Tr}(\gamma|_{S_u^+}) - \operatorname{Tr}(\gamma|_{S_u^-}) = \varepsilon_u \sum_{k=1}^{3} e^{-i\gamma_k} - e^{i\gamma_k} = -2i\varepsilon_u \sum_{k=1}^{3} \sin(\gamma_k).$$

Recalling that $\eta_\gamma(D_{T^7})(s) = \sum_{\pm\mu \neq 0} \pm \operatorname{Tr}(\gamma|_{E_{\pm\mu}})|\mu|^{-s}$ we obtain the formula

$$\eta_\gamma(D_{T^7})(s) = -2i(2\pi)^{-s} \sum_{u \in (\Lambda^* \backslash 0)^B} |u|^{-s} e^{-2\pi i \langle u, b\rangle} \varepsilon_u \sum_{k=1}^{3} \sin(\gamma_k). \tag{22}$$

In our next set of examples, every $u \in (\Lambda^* \backslash 0)^B$ for γ with $\operatorname{Fix}(\gamma) = \varnothing$ is an integral multiple of e_7, and this formula simplifies considerably. Note that $\varepsilon_{-u} = -\varepsilon_u$. We also mention that the same approach described here may be used to compute the odd signature η-invariant.

7.4 Dihedral examples

We now consider Examples 7–14 of [Joy96]. The general setup is as follows. Let z_1, z_2, z_3 be coordinates for $\mathbb{C}^3$ and x for $\mathbb{R}$. Let $\Lambda' \subset \mathbb{C}^3$ be a rank 6 lattice. Then the 7-torus $T^7 = \mathbb{C}^3 \times \mathbb{R}/\Lambda' \times \mathbb{Z}$ has a flat G_2-structure induced by the 3-form

$$\phi = \omega \wedge dx + \operatorname{Im}(\Omega)$$

where $\omega = \frac{i}{2}\sum_{k=1}^{3} dz_k d\overline{z}_k$ and $\Omega = dz_1 dz_2 dz_3$. Let u and v be complex roots of unity, and a the smallest positive integer such that $u^a = v^a = 1$. Let α and β be the isometries of $\mathbb{C}^3 \times \mathbb{R}$ defined by

$$\begin{aligned} \alpha(z_1, z_2, z_3, x) &= (u z_1, v z_2, \overline{uv} z_3, x + \tfrac{1}{a}) \\ \beta(z_1, z_2, z_3, x) &= (-\overline{z}_1, -\overline{z}_2, -\overline{z}_3, -x) \end{aligned}$$

If α and β preserve Λ', they descend to isometries on T^7 that preserve ϕ. They satisfy $\alpha^a = \beta^2 = 1$ and $\alpha\beta = \beta\alpha^{-1}$, and thus generate a dihedral group of order $2a$:

$$\Gamma := \langle \alpha, \beta \rangle = \{1, \alpha, \alpha^2, \ldots, \alpha^{a-1}, \beta, \beta\alpha, \beta\alpha^2, \ldots, \beta\alpha^{a-1}\} \tag{23}$$

The elements $\beta\alpha^j$ all have nonempty fixed point sets, while α^j for $j \not\equiv 0 \pmod a$ has no fixed points. If we write α as in (18), then $c = 0$ and $d = 1/a$, and the rotation angles $\theta_1, \theta_2, \theta_3$ of A are determined by $u = e^{i\theta_1}$, $v = e^{i\theta_2}$ and $\theta_1 + \theta_2 + \theta_3 \equiv 0 \pmod{2\pi}$. Upon computing $\nu(\alpha^j) = \det(1 - A^j) = 64 \prod_{k=1}^{3} \sin^2(j\theta_k/2)$, from (19) we obtain

$$\begin{aligned}\eta(B_{\mathcal{O}}) &= \frac{4}{a} \sum_{j=1}^{a-1} \cot(j\pi/a) \prod_{k=1}^{3} \sin(j\theta_k) \\ &= -\frac{1}{a} \sum_{j=1}^{a-1} \cot(j\pi/a) \sum_{k=1}^{3} \sin(2j\theta_k) = 2 \sum_{k=1}^{3} ((\theta_k/\pi)).\end{aligned} \tag{24}$$

The second equality follows from the elementary identity $-4 \prod_{k=1}^{3} \sin(\theta_k) = \sum_{k=1}^{3} \sin(2\theta_k)$, which holds whenever $\theta_1 + \theta_2 + \theta_3 \equiv 0 \pmod{2\pi}$, and the third equality follows from the classical identity of Eisenstein [Eis44], which says that $-\frac{1}{2a} \sum_{j=1}^{a-1} \cot(\pi j/a) \sin(2\pi jx/a) = ((x/a))$ where $((t)) = t - [t] - 1/2$ if $t \notin \mathbb{Z}$ and $((t)) = 0$ otherwise. Let us normalize the angles θ_j such that $\theta_j \in [0, 2\pi)$ for each j. Then equation (24) yields

$$\eta(B_{\mathcal{O}}) = \begin{cases} +1, & \theta_i < \pi \text{ for } i = 1,2,3 \\ -1, & \theta_i > \pi \text{ for some } i \\ 0, & \theta_i \in \{0, \pi\} \text{ for some } i \end{cases} \tag{25}$$

We now turn to the Dirac eta invariants. In (22), when $\gamma = \alpha^j$ every $u \in (\Lambda^*)^B$ is an integer multiple of e_7. We sum over $u = \pm n e_7$ with $n \in \mathbb{Z}_{>0}$ and $\varepsilon_u = \pm 1$ to obtain

$$\eta_{\alpha^j}(D_{T^7})(s) = -4(2\pi)^{-s} \sum_{n=1}^{\infty} \frac{1}{n^s} \sin(2\pi jn/a) \sum_{k=1}^{3} \sin(j\theta_k). \tag{26}$$

The function $\sum_{n=1}^{\infty} \sin(2\pi jn/a) n^{-s}$ is the imaginary part of the polylogarithm $L_s(z) = \sum_{n=1}^{\infty} z^n/n^s$ evaluated at $z = e^{2\pi ij/a}$. Using the identity $L_0(z) = z/(1-z)$, we evaluate (26) at $s = 0$:

$$\eta_{\alpha^j}(D_{T^7}) = -2\cot(\pi j/a) \sum_{k=1}^{3} \sin(j\theta_k).$$

Finally, taking the average over the equivariant η-invariants we obtain

$$\eta(D_{\mathcal{O}}) = -\frac{1}{a} \sum_{j=1}^{a-1} \cot(\pi j/a) \sum_{k=1}^{3} \sin(j\theta_k) = \begin{cases} -1, & \theta_i \not\equiv 0 \pmod{2\pi} \text{ for } i = 1,2,3 \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

where the second equality follows again from Eisenstein's identity. We may now compute the ν-invariants of the torsion-free G_2-manifolds (M, ϕ) as constructed in Examples 7–14 of [Joy96]. Each of these satisfies Hypothesis 1.1, and is either constructed from a dihedral orbifold as just described, or is a finite quotient thereof.

Example 7 of [Joy96]. This example has $a = 3$, $u = v = e^{2\pi i/3}$ and $\Lambda' = \mathbb{Z}^3 \oplus e^{2\pi i/3}\mathbb{Z}^3 \subset \mathbb{C}^3$. Thus $\theta_1 = \theta_2 = \theta_3 = 2\pi/3$. Then (25) and (27) yield $\eta(B_{\mathcal{O}}) = 1$ and $\eta(D_{\mathcal{O}}) = -1$. For the resulting

Ex. No.	$\|\Gamma\|$	π_1	b_2	b_3	$\eta(B_M)$	$\overline{\eta}(D_M)$	ν (mod 48)
7	6	0	5	13	1	-1	3
8	12	0	3	11	-1	-1	45
9	8	0	11	36	0	-1	0
10	16	$\mathbb{Z}/2$	6	21	0	-1	0
11	12	0	4	17	0	-1	0
12	24	$\mathbb{Z}/2$	2	11	0	-1	0
13	14	0	2	10	-1	-1	45
14	18	0	2	10	$1/3$	$-1/3$	33

Table 1: ν invariants for Examples 7–14 of [Joy96]. Here $\overline{\eta} := \eta \pmod{2\mathbb{Z}}$.

resolution torsion-free G_2-manifold (M, ϕ) we have $\nu(M, \phi) \equiv 3 \pmod{48}$.

Example 8 of [Joy96]. Here $a = 6$, $u = v = e^{\pi i/3}$ and Λ' is as in Example 7. Thus $\theta_1 = \theta_2 = \pi/3$ and $\theta_3 = 4\pi/3$. Then $\eta(B_{\mathcal{O}}) = \eta(D_{\mathcal{O}}) = -1$, and $\nu(M, \phi) \equiv 45 \pmod{48}$.

Example 9 of [Joy96]. Here $a = 4$, $u = v = i$ and $\Lambda' = \mathbb{Z}^3 \oplus i\mathbb{Z}^3 \subset \mathbb{C}^3$. Thus $\theta_1 = \theta_2 = \pi/2$ and $\theta_3 = \pi$. Then $\eta(B_{\mathcal{O}}) = 0$, $\eta(D_{\mathcal{O}}) = -1$, and $\nu(M, \phi) \equiv 0 \pmod{48}$.

Example 10 of [Joy96]. This is obtained from Example 9 by incorporating the involution γ defined by $(z_1, z_2, z_3, x) \mapsto (z_1, z_2, z_3 + \frac{1+i}{2}, x)$. Thus the computations are slightly different: when averaging over the equivariant η-invariants, in addition to the terms associated to α^j are those associated to $\gamma\alpha^j$. It is easily verified that $\eta_{\gamma\alpha^j}(B_{T^7}) = \eta_{\alpha^j}(B_{T^7})$, and similarly for the Dirac operator. As the size of our group has doubled, we obtain the same results as in Example 9.

Example 11 of [Joy96]. This example has $a = 6$, $u = e^{\pi i/3}$, $v = e^{2\pi i/3}$ with corresponding lattice $\Lambda' = (\mathbb{Z} \oplus e^{2\pi i/3}\mathbb{Z}) \oplus (\mathbb{Z} \oplus e^{2\pi i/3}\mathbb{Z}) \oplus (\mathbb{Z} \oplus i\mathbb{Z}) \subset \mathbb{C}^3$. Thus $\theta_1 = \pi/3$, $\theta_2 = 2\pi/3$, $\theta_3 = \pi$. We have $\eta(B_{\mathcal{O}}) = 0$, $\eta(D_{\mathcal{O}}) = -1$ and $\nu(M, \phi) \equiv 0 \pmod{48}$.

Example 12 of [Joy96]. This is obtained from Example 11 by incorporating the involution defined in Example 10, and yields the same results as in Example 11, similar to Example 10.

Example 13 of [Joy96]. This has $a = 7$, $u = e^{2\pi i/7}$, $v = u^2$ and

$$\Lambda' = \langle (u^j, u^{2j}, u^{4j}) \in \mathbb{C}^3 : j = 1, 2, 3, 4, 5, 6 \rangle.$$

Thus $\theta_1 = 2\pi/7$, $\theta_2 = 4\pi/7$, $\theta_3 = 8\pi/7$; thus $\eta(B_{\mathcal{O}}) = \eta(D_{\mathcal{O}}) = -1$, and $\nu(M, \phi) \equiv 45 \pmod{48}$.

Example 14 of [Joy96]. Consider the dihedral group $\langle \alpha, \beta \rangle$ defined in Example 7, and adjoin to it $\gamma(z_1, z_2, z_3, x) = (e^{2\pi i/3}z_1, e^{4\pi i/3}z_2, z_3 + i/\sqrt{3}, x)$. Then α and γ commute, while $\gamma\beta = \beta\gamma^{-1}$. Thus $\Gamma = \langle \alpha, \beta, \gamma \rangle$ is a group of order 18. The elements with fixed points are $\beta\alpha^j\gamma^k$. In addition, the elements $g = \alpha^j\gamma^k$ with $k \in \{1, 2\}$ have $\eta_g(B_{\mathcal{O}}) = 0$ and $\eta_g(D_{T^7}) \equiv 0 \pmod{2\mathbb{Z}}$, as each such g has more than one $+1$ eigenvalue in its associated rotation matrix A from (18). Thus the computations of the η-invariants differ from Example 7 only in that $|\Gamma| = 18$ instead of $|\Gamma| = 6$. We have $\eta(B_{\mathcal{O}}) = 1/3$ and $\eta(D_{\mathcal{O}}) = -1/3$, implying $\nu \equiv 33 \pmod{48}$.

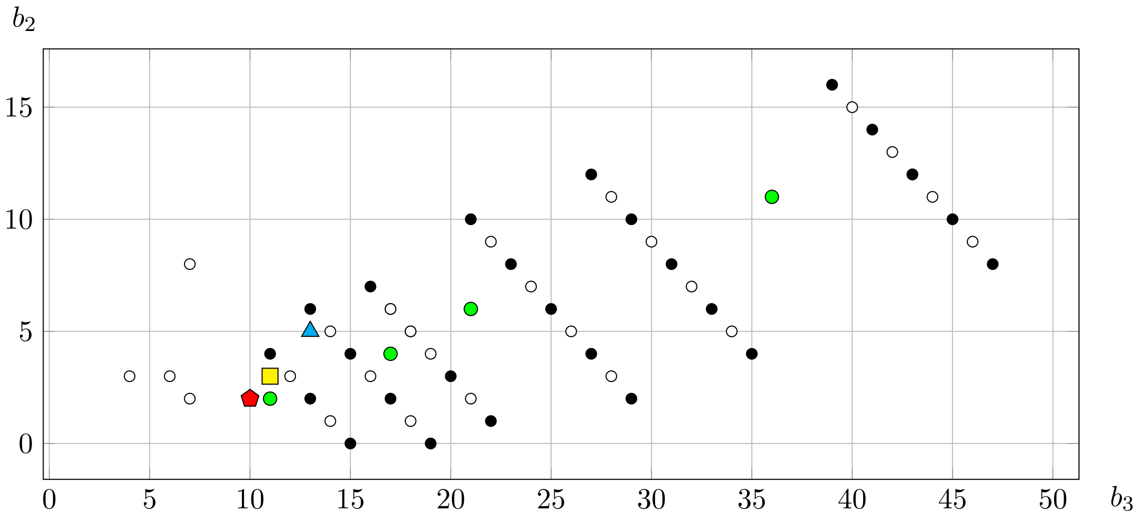

Figure 3: This is a variant of [Joy96, Table 2] listing the betti numbers b_2, b_3 of all the closed manifolds constructed in [Joy96] with holonomy G_2. Our computations are: ● : $\nu \equiv 0 \pmod{48}$; ○ : $\nu \equiv 0 \pmod{24}$; ● : $\nu \equiv 24 \pmod{48}$; ▲ : $\nu \equiv 3 \pmod{48}$; ■ : $\nu \equiv 45 \pmod{48}$; ⬟ : there are two G_2-manifolds here, with $\nu \equiv 45$ and $\nu \equiv 33 \pmod{48}$. Some nodes have more than one manifold; apart from ⬟, the values of ν hold for all G_2-manifolds from [Joy96] at the given node.

7.5 A few more examples

The remaining examples in [Joy96] satisfy Hypothesis 2.1 with spin isometries, and thus by Proposition 6.3 we may compute $\nu \pmod{24}$. In fact, the orientation-reversing isometry $(x_1, \ldots, x_7) \mapsto (-x_1, \ldots, -x_7)$ is well-defined on $\mathcal{O} = T^7/\Lambda$ for each of Exs. 15–18 of [Joy96], and commutes with Γ in each case. Thus $\eta_\gamma(B_\mathcal{O}) = 0$ and $\eta_\gamma(D_\mathcal{O}) \equiv 0 \pmod{\mathbb{Z}}$ for each $\gamma \in \Gamma$, the latter holding because the isometry reverses the Dirac spectrum. Thus $\nu \equiv 0 \pmod{24}$ for Exs. 15–18 of [Joy96]. A summary of all our computations is represented in Figure 3.

8 Twisted connected sum type decompositions

Here we explain another route to some of the above computations. The starting point is as follows: suppose we can realize a given Joyce orbifold $\mathcal{O} = T^7/\Gamma$ as a union

$$\mathcal{O} = \mathcal{O}_+ \cup \mathcal{O}_- \tag{28}$$

where $\mathcal{O}_\pm$ are orbifolds with (smooth) boundaries whose collar neighborhoods are isometric to $T^6 \times [-\epsilon, \epsilon]$, and $\mathcal{O}$ is obtained by gluing along the T^6 boundaries. See Figure 4. This is similar to the twisted connected sum picture, with cross-section T^6 replacing $K3 \times T^2$. Assume $b_1(\mathcal{O}) = 0$, and that $\mathcal{O}$ satisfies Hypothesis 1.1 or one of its variations. Then we have shown that the resolution G_2-holonomy manifold (M, g) as constructed by Joyce has ν-invariant given by

$$\nu(M, \phi_g) \equiv 3\eta(B_\mathcal{O}) - 24\eta(D_\mathcal{O}) + 24 \mod 48$$

Instead of computing the terms directly for $\mathcal{O}$ as before, we may apply gluing formulas for $\eta(B_\mathcal{O})$ and $\eta(D_\mathcal{O})$ using the decomposition (28). We first explain this for the signature operator term.

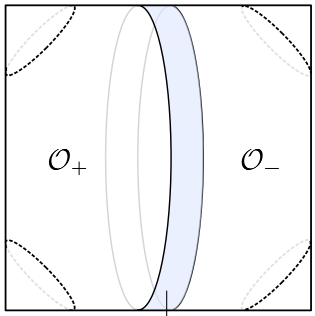

Figure 4: A twisted connected sum type decomposition of $\mathcal{O}$.

Write $L_\pm \subset H^3(T^6)$ for the two Lagrangians given by the images of $H^3(\mathcal{O}_\pm) \to H^3(T^6)$. Assume $\mathcal{O}_\pm$ each admit an orientation-reversing isometry. Then Theorem 4.3 yields

$$\eta(B_{\mathcal{O}}) = m(L_+, L_-; H^3(T^3))$$

as the relative η-invariants vanish by spectral symmetry. The Maslov index is computed as follows, see [CGN18a, Section 4.2]. Let $A_\pm$ be the isometries of $H^3(T^6)$ which anticommute with the Hodge star whose 1-eigenspaces are $L_\pm$. Let $E_- \subset H^3(T^6; \mathbb{C})$ be the $(-i)$-eigenspace of the Hodge star. Write the eigenvalues of $-A_+A_-|_{E_-}$ as $e^{i\phi_1}, \ldots, e^{i\phi_{10}}$ where $\phi_j \in (-\pi, \pi]$. Then:

$$m(L_+, L_-; H^3(T^6)) = -\sum_{\phi_j \neq \pi} \frac{\phi_j}{\pi} \tag{29}$$

We apply this to some of the dihedral examples from Section 7.4, so that $\mathcal{O} = T^7/\Gamma$ where Γ is the dihedral group (23). Here we have a decomposition as in (28) with

$$\mathcal{O}_+ = \left(T^6 \times [-\tfrac{1}{4a}, \tfrac{1}{4a}]\right)/\beta, \qquad \mathcal{O}_- = \left(T^6 \times [\tfrac{1}{4a}, \tfrac{3}{4a}]\right)/\alpha\beta$$

This was pointed out to the author by Sebastian Goette and Johannes Nordström. Write the coordinates of T^6 as (z_1, z_2, z_3) as in Section 7.4. We may identify $H^3(T^6) = H^3(T^6; \mathbb{R})$ with the Hodge groups $H^{3,0}(T^6) \oplus H^{2,1}(T^6) \subset H^3(T^6; \mathbb{C})$. Concretely,

$$H^3(T^6) = \bigoplus_{jk\ell \in S} \mathbb{R} \cdot \mathrm{Re}(dz_{jk\ell}) \oplus \mathbb{R} \cdot \mathrm{Im}(dz_{jk\ell})$$

where $S = \{123, \bar{1}23, 1\bar{2}3, 12\bar{3}, 1\bar{1}2, 1\bar{1}3, 12\bar{2}, 2\bar{2}3, 13\bar{3}, 23\bar{3}\}$. The Lagrangian L_+ is the subspace of $H^3(T^6)$ invariant under β. We compute

$$L_+ = \bigoplus_{jk\ell \in S} \mathbb{R} \cdot \mathrm{Im}(dz_{jk\ell})$$

Similarly, L_- is the subspace invariant under $\alpha\beta$. It is spanned by $\mathrm{Im}(dz_{123})$, the 3 elements

$$\mathrm{Im}(e^{-i\theta_j} dz_{\bar{j}k\ell}) = \cos(\theta_j)\mathrm{Im}(dz_{\bar{j}k\ell}) - \sin(\theta_j)\mathrm{Re}(dz_{\bar{j}k\ell})$$

where $j, k, \ell \in \{1, 2, 3\}$ are distinct, and the 6 elements

$$\mathrm{Im}(e^{i\theta_k/2} dz_{j\bar{j}k}) = \cos(\theta_k/2)\mathrm{Im}(dz_{j\bar{j}k}) + \sin(\theta_k/2)\mathrm{Re}(dz_{j\bar{j}k})$$

where $j, k \in \{1, 2, 3\}$ are distinct. Note each of $L_\pm$ is 10-dimensional, coherent with the fact that they are Lagrangian subspaces inside the 20-dimensional space $H^3(T^6)$.

Turning to the Maslov index, a computation from the above description shows that the 10 eigenvalues of $-A_+A_-|_{E_-}$ are -1, $-e^{-2i\theta_j}$, where $j \in \{1, 2, 3\}$, and 3 conjugate pairs. Thus (29) is

$$-\sum_{j=1}^{3} \frac{\phi_j}{\pi}$$

where ϕ_j is the value in $(-\pi, \pi]$ equal to $\pi - 2\theta_j$ modulo 2π, as long as all $\theta_j \notin \{0, \pi\}$; otherwise (29) is zero. This is the same as expression (25), our previous computation of $\eta(B_{\mathcal{O}})$. In particular, we recover $\eta(B_{\mathcal{O}})$ for Examples 7, 8, 9, 11, 13 of [Joy96], previously computed in Table 1.

The same procedure can be carried out for the spin Dirac invariant $\eta(D_{\mathcal{O}})$. Assume, as is the case in the above examples, that $\mathcal{O}_\pm$ admit orientation-reversing spin isometries. Then the relative η-invariants vanish, and the gluing formula [Bun95, Theorem 1.8] yields

$$\eta(D_{\mathcal{O}}) \equiv m(S_+, S_-; S_{T^6}) \mod \mathbb{Z}$$

Here S_{T^6} is the space of harmonic spinors on the cross-section T^6, which may be identified with parallel spinors on $T^6 \times (-\epsilon, \epsilon)$. The subspaces $S_\pm$ consist of those spinors that extend to (orbifold) harmonic spinors over $\mathcal{O}_\pm$. The space S_{T^6} may be identified with the $\mathrm{Spin}(6)$ representation $\mathbb{R} \oplus \mathbb{R}^7$ obtained by restriction from the $\mathrm{Spin}(7)$ representation in Section 7.3. As before, e_i are the standard basis vectors for $\mathbb{R}^7$. Clifford multiplication by e_7 plays the role of the Hodge star here. The Lagrangian S_+ is the β-invariant subspace, which is spanned by

$$(1, 0), \quad (0, e_2), \quad (0, e_4), \quad (0, e_6).$$

Next, S_- is the $\alpha\beta$-invariant subspace, spanned by $(1, 0) \in S_{T^6} = \mathbb{R} \oplus \mathbb{R}^7$ and the 3 elements

$$\begin{aligned}
&(0, -\sin(\theta_1/2)e_1 + \cos(\theta_1/2)e_2),\\
&(0, -\sin(\theta_2/2)e_3 + \cos(\theta_2/2)e_4),\\
&(0, -\sin(\theta_3/2)e_5 + \cos(\theta_3/2)e_6).
\end{aligned}$$

The $(-i)$-eigenspace $E_- \subset S_{T^6} \otimes \mathbb{C}$ in this case is spanned by s_j^+ for $j \in \{1, 2, 3, 4\}$, from (21). The matrix $-A_+A_-|_{E_-}$ has the 4 eigenvalues -1 and $e^{i(\pi - \theta_j)}$ for $j \in \{1, 2, 3\}$. Then

$$m(S_+, S_-; S_{T^6}) = -\sum_{j=1}^{3} \frac{\pi - \theta_j}{\pi} = -1$$

as long as the θ_j are not all in $2\pi\mathbb{Z}$, and otherwise it vanishes. This recovers (27) modulo $\mathbb{Z}$, although we see that the answers agree as integers as well, which suggests that the mod $\mathbb{Z}$ restriction on the gluing formula may not be neccessary in this setting.

References

[APS75a] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69. MR0397797 ↑5

[APS75b] ———, *Spectral asymmetry and Riemannian geometry. II*, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 3, 405–432. MR0397798 ↑2, 4

[Bun95] Ulrich Bunke, *On the gluing problem for the η-invariant*, J. Differential Geom. **41** (1995), no. 2, 397–448. MR1331973 ↑3, 8, 19

[CGN18a] Diarmuid Crowley, Sebastion Goette, and Johannes Nordström, *An analytic invariant of G_2-manifolds* (2018). https://arxiv.org/abs/1505.02734. ↑1, 3, 6, 7, 9, 10, 18

[CGN18b] ———, *Dinstinguishing G_2-manifolds* (2018). https://arxiv.org/abs/1808.05585. ↑3

[CHNP13] Alessio Corti, Mark Haskins, Johannes Nordström, and Tommaso Pacini, *Asymptotically cylindrical Calabi-Yau 3-folds from weak Fano 3-folds*, Geom. Topol. **17** (2013), no. 4, 1955–2059. MR3109862 ↑1

[CHNP15] ———, *G_2-manifolds and associative submanifolds via semi-Fano 3-folds*, Duke Math. J. **164** (2015), no. 10, 1971–2092. MR3369307 ↑1

[CN15] Diarmuid Crowley and Johannes Nordström, *New invariants of G_2-structures*, Geom. Topol. **19** (2015), no. 5, 2949–2992. MR3416118 ↑1, 10

[Don78] Harold Donnelly, *Eta invariants for G-spaces*, Indiana Univ. Math. J. **27** (1978), no. 6, 889–918. MR511246 ↑2, 12

[Eis44] G. Eisenstein, *Aufgaben und Lehrsätze*, J. Reine Angew. Math. **27** (1844), 281–284. MR1578398 ↑15

[For] Nelvis Fornasin, *η invariants under degeneration to cone-edge singularities*, Doctoral Dissertation, Albert-Ludwigs-Universität Freiburg. ↑3

[Fri84] Th. Friedrich, *Zur Abhängigkeit des Dirac-operators von der Spin-Struktur*, Colloq. Math. **48** (1984), no. 1, 57–62. MR750754 ↑13

[Hit74] Nigel Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55. MR0358873 ↑9

[JK18] Dominic Joyce and Spiro Karigiannis, *A new construction of compact torsion-free G_2-manifolds by gluing families of Eguchi–Hanson spaces* (2018). https://arxiv.org/pdf/1707.09325.pdf. ↑1

[Joy00] Dominic D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. MR1787733 ↑1, 3, 4, 5, 6, 11

[Joy96] ———, *Compact Riemannian 7-manifolds with holonomy G_2. I, II*, J. Differential Geom. **43** (1996), no. 2, 291–328, 329–375. MR1424428 ↑1, 2, 3, 5, 6, 11, 12, 14, 15, 16, 17, 19

[KL04] Paul Kirk and Matthias Lesch, *The η-invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary*, Forum Math. **16** (2004), no. 4, 553–629. MR2044028 ↑3, 6, 7, 8

[Kov03] Alexei Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math. **565** (2003), 125–160. MR2024648 ↑1

[MP06] Roberto J. Miatello and Ricardo A. Podestá, *The spectrum of twisted Dirac operators on compact flat manifolds*, Trans. Amer. Math. Soc. **358** (2006), no. 10, 4569–4603. MR2231389 ↑2, 13

[Nor18] Johannes Nordström, *Extra-twisted connected sum G_2-manifolds* (2018). https://arxiv.org/abs/1809.09083. ↑1

[SW17] Dietmar A. Salamon and Thomas Walpuski, *Notes on the octonions*, Proceedings of the Gökova Geometry-Topology Conference 2016, 2017, pp. 1–85. MR3676083 ↑14