AN ODD KHOVANOV HOMOTOPY TYPE

SUCHARIT SARKAR, CHRISTOPHER SCADUTO, AND MATTHEW STOFFREGEN

ABSTRACT. For each link $L \subset S^3$ and every quantum grading j, we construct a stable homotopy type $\mathcal{X}_o^j(L)$ whose cohomology recovers Ozsváth-Rasmussen-Szabó's odd Khovanov homology, $\widetilde{H}^i(\mathcal{X}_o^j(L)) = Kh_o^{i,j}(L)$, following a construction of Lawson-Lipshitz-Sarkar of the even Khovanov stable homotopy type. Furthermore, the odd Khovanov homotopy type carries a $\mathbb{Z}/2$ action whose fixed point set is a desuspension of the even Khovanov homotopy type. We also construct a $\mathbb{Z}/2$ action on an even Khovanov homotopy type, with fixed point set a desuspension of $\mathcal{X}_o^j(L)$.

CONTENTS

1. Introduction

1.1. **Khovanov homologies.** In [Kho00] Khovanov categorified the Jones polynomial: to a link diagram L, he associated a bigraded chain complex, whose graded Euler characteristic is (a certain normalization of) the Jones polynomial of L, and whose (graded) chain homotopy type is an invariant of the underlying link. Several generalizations were soon constructed, such as invariants for tangles [Kho02, BN05], various perturbations [Lee05, BN05], versions for other polynomials [KR08a, KR08b], and many others. The categorified invariant carried structure that was not visible at the decategorified level. To wit, to a link cobordism in $\mathbb{R}^3 \times [0, 1]$, there is an associated map of Khovanov chain complexes [Jac04, Kho06, BN05, CMW]; and this map, along with Lee's perturbation, was used by Rasmussen in [Ras10] to define a numerical concordance invariant s and to give a combinatorial proof of a theorem due to Kronheimer and Mrowka [KM93] on the four-ball genus of torus knots (popularly known as the Milnor conjecture). Khovanov homology itself turns out to be a more powerful invariant than the Jones polynomial. Indeed, Khovanov homology is known to detect the unknot [KM11], while the corresponding question for the Jones polynomial remains wide open.

In [OSz05], Ozsváth and Szabó constructed the first relation between Khovanov homology and Floer-theoretic invariants—Heegaard Floer homology [OSz04] to be specific—in the form of a spectral sequence from reduced Khovanov homology of a link to the Heegaard Floer homology of its

branched double cover. The spectral sequence was originally constructed over $\mathbb{Z}/2$, but it was soon realized that its integral lift does not start from the usual Khovanov homology, but rather from a different homology theory which has the same $\mathbb{Z}/2$ reduction; this new chain complex was constructed by Ozsváth-Rasmussen-Szabo [ORSz13] and is usually called the odd Khovanov complex, and this is the version that seems closely related to Heegaard Floer type invariants. (The even theory was later discovered to be related to certain other Floer theories, such as instanton Floer homology [KM11] and symplectic Khovanov homology [SS06].) The two versions were combined by a pullback into a single unified theory by Putyra [Put14], cf. [PS16]: the unified Khovanov complex is a chain complex over $\mathbb{Z}[\xi]/(1 - \xi^2)$ which recovers the even (respectively, odd) Khovanov chain complex upon setting $\xi = 1$ (respectively, $\xi = -1$).

In this paper, we will typically decorate the objects from the even theory by the subscript e, the ones from the odd theory by the subscript o, and the ones from the unified theory by the subscript u. In particular, we will denote the even, odd, and the unified Khovanov complexes as $Kc_e(L)$, $Kc_o(L)$, and $Kc_u(L)$, respectively.

1.2. **Khovanov homotopy types.** In [LS14a], Lipshitz and Sarkar associated to a link diagram L a finite CW spectrum $\mathcal{X}_e(L)$, whose reduced cellular cochain complex is isomorphic to the Khovanov complex $Kc_e(L)$, taking the (non-basepoint) cells of $\mathcal{X}_e(L)$ to the standard generators of $Kc_e(L)$. The (stable) homotopy type of $\mathcal{X}_e(L)$ is an invariant of the underlying link; specifically, Reidemeister moves from the diagram L to a diagram L' induce stable homotopy equivalences $\mathcal{X}_e(L) \to \mathcal{X}_e(L')$. A different construction of an even Khovanov homotopy type was constructed independently by Hu-Kriz-Kriz [HKK16], and the two versions were later shown to be equivalent [LLS]. A stable homotopy refinement of Khovanov homology endowed it with extra structure, such as an action by the Steenrod algebra [LS14c], which was then used to construct a family of additional s-type concordance invariants [LS14b], as well as to show that Khovanov homotopy type is a strictly stronger invariant than Khovanov homology [See].

One could ask for a spectrum invariant $\mathcal{X}_o(L)$ satisfying analogous properties, but with Khovanov homology replaced with odd Khovanov homology. The original Lipshitz-Sarkar contruction using the Cohen-Jones-Segal framed flow categories machine from [CJS95] does not seem to admit an easy generalization: on account of the signs that appear in the definition of odd Khovanov homology, there is no framed flow category for the odd theory covering the framed cube flow category. However, in [LLS], Lawson-Lipshitz-Sarkar provided several more abstract constructions of $\mathcal{X}_e(L)$—similar to the one from [HKK16]—in order to understand the behavior of the Khovanov spectrum under disjoint union and connected sum. In this paper, we will give a slight generalization of their machinery to construct a finite $\mathbb{Z}_2$-equivariant CW spectrum $\mathcal{X}_o(L) = \bigvee_j \mathcal{X}_o^j(L)$ for each oriented link diagram L (Definition 5.2).

Theorem 1.1. *The (stable) homotopy type of the odd Khovanov spectrum $\mathcal{X}_o(L) = \bigvee_j \mathcal{X}_o^j(L)$ from Definition 5.2 is independent of the choices in its construction and is an invariant of the isotopy class of the link corresponding to L. Its reduced cellular cochain complex agrees with the odd Khovanov complex $Kc_o(L)$,*

$$\widetilde{C}^i_{\mathrm{cell}}(\mathcal{X}_o^j(L)) = Kc_o^{i,j}(L),$$

with the cells mapping to the distinguished generators of $Kc_o(L)$.

We also construct a reduced theory: a finite $\mathbb{Z}_2$-equivariant CW spectrum $\widetilde{\mathcal{X}}_o(L, p) = \bigvee_j \widetilde{\mathcal{X}}_o^j(L, p)$ for each oriented link diagram L with basepoint p (Definition 5.7).

Theorem 1.2. *The (stable) homotopy type of the reduced odd Khovanov spectrum $\widetilde{\mathcal{X}}_o(L, p) = \bigvee_j \widetilde{\mathcal{X}}_o^j(L, p)$ from Definition 5.7 is independent of the choices in its construction and is an invariant of the isotopy class of the pointed link corresponding to (L, p). Its reduced cellular cochain complex agrees with the reduced odd Khovanov complex $\widetilde{Kc}_o(L)$,*

$$\widetilde{C}_{\mathrm{cell}}^i(\widetilde{\mathcal{X}}_o^j(L, p)) = \widetilde{Kc}_o^{i,j}(L),$$

with the cells mapping to the distinguished generators of $\widetilde{Kc}_o(L)$. There is a cofibration sequence

$$\widetilde{\mathcal{X}}_o^{j-1}(L, p) \to \mathcal{X}_o^j(L) \to \widetilde{\mathcal{X}}_o^{j+1}(L, p).$$

We introduce concordance invariants built from this construction, in analogy with [LS14b]. To do so, we show that associated to a cobordism of links, there exists a map of odd Khovanov spectra (we do not attempt to show that the map is well-defined); the map induces a map on the odd Khovanov chain complex, and reduces mod-2 to the usual cobordism map on $Kc(L; \mathbb{Z}_2)$. Therefore:

Theorem 1.3. *The Khovanov cobordism map $Kh(L; \mathbb{Z}_2) \to Kh(L'; \mathbb{Z}_2)$ associated to a link cobordism $L \to L'$ from [Jac04, Kho06, BN05] is a map of $\mathcal{A}^\sigma$-modules, where $\mathcal{A}^\sigma$ is the free product of two copies of the mod-2 Steenrod algebra, and the first (respectively, second) copy acts on the mod-2 Khovanov homology by viewing it as the mod-2 cohomology of the even (respectively, odd) Khovanov homotopy type.*

Moreover, in Definition 5.4, we construct an even stable homotopy type $\mathcal{X}_e'(L)$ (that is, a finite CW spectrum whose cellular chain complex is the even Khovanov chain complex), equipped with a $\mathbb{Z}_2$-action. This $\mathbb{Z}_2$-action is not visible from the Burnside functor constructed in [LLS], so in some sense this $\mathbb{Z}_2$-action arises from the odd theory. We conjecture that the even space constructed here is stable homotopy equivalent to the construction of [LS14a].

Theorem 1.4. *The (stable) homotopy type of the even Khovanov spectrum $\mathcal{X}_e'(L)$ from Definition 5.4 is independent of the choices in its construction and is an invariant of the isotopy class of L. Its reduced cellular cochain complex agrees with the odd Khovanov complex $Kc_e(L)$,*

$$\widetilde{C}_{\mathrm{cell}}^i(\mathcal{X}_e'^j(L)) = Kc_e^{i,j}(L),$$

with the cells mapping to the distinguished generators of $Kc_o(L)$.

Finally, similar to unified Khovanov homology, we combine $\mathcal{X}_e(L)$ and $\mathcal{X}_o(L)$ into a single finite $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant CW spectrum $\mathcal{X}_u(L) = \bigvee_j \mathcal{X}_u^j(L)$, which we think of as a 'unified Khovanov spectrum' (Definition 5.5):

Theorem 1.5. *The (stable) homotopy type of the unified Khovanov spectrum $\mathcal{X}_u(L)$ from Definition 5.5 is independent of the choices in its construction and is an invariant of the isotopy class of L. Its reduced cellular cochain complex agrees with the unified Khovanov complex $Kc_u(L)$,*

$$\widetilde{C}_{\mathrm{cell}}^i(\mathcal{X}_u^j(L)) = Kc_u^{i,j}(L),$$

with the cells mapping to the distinguished generators of $Kc_u(L)$, and the two $\mathbb{Z}_2$ actions correspond to multiplication by ξ and $-\xi$, respectively.

There is also a reduced unified spectrum $\widetilde{\mathcal{X}}_u(L)$ for which the analogue of Theorem 1.2 (Proposition 5.9) holds.

The different spectra and the different actions admit the following relationship:

Theorem 1.6. *Let L be a link diagram.*

(1) *The action of the two $\mathbb{Z}_2$-factors is free away from the basepoint on $\mathcal{X}_u(L)$: $\mathcal{X}_e(L)$ is the geometric quotient under the action of the first factor (sometimes called $\mathbb{Z}_2^+$) and $\mathcal{X}_o(L)$ is the geometric quotient under the second factor (sometimes called $\mathbb{Z}_2^-$); moreover, the $\mathbb{Z}_2$-action on $\mathcal{X}_o(L)$ is the quotient of the $\mathbb{Z}_2^+$-action on $\mathcal{X}_u(L)$.*

(2) *The geometric fixed-point set of $\mathcal{X}_o^j(L)$ under $\mathbb{Z}_2$ is precisely $\Sigma^{-1}\mathcal{X}_e^j(L)$, and quotienting by the fixed point set produces $\mathcal{X}_u(L)$. The induced $\mathbb{Z}_2$-action on $\mathcal{X}_u(L)$ agrees with the $\mathbb{Z}_2^+$-action. This produces a cofibration sequence*

$$\Sigma^{-1}\mathcal{X}_e(L) \to \mathcal{X}_o(L) \to \mathcal{X}_u(L),$$

and the induced long exact sequence on cohomology agrees with the one constructed in [PS16].

(3) *The Puppe map $\mathcal{X}_u(L) \to \mathcal{X}_e(L)$ from the previous cofibration sequence is homotopic to the quotient map $\mathcal{X}_u(L) \to \mathcal{X}_u(L)/\mathbb{Z}_2^+$.*

(4) *The geometric fixed-point set of $\mathcal{X}_e'^j(L)$ under $\mathbb{Z}_2$ is precisely $\Sigma^{-1}\mathcal{X}_o^j(L)$, and quotienting by the fixed point set produces $\mathcal{X}_u(L)$. The induced $\mathbb{Z}_2$-action on $\mathcal{X}_u(L)$ agrees with the $\mathbb{Z}_2^-$-action. This produces a cofibration sequence*

$$\Sigma^{-1}\mathcal{X}_o(L) \to \mathcal{X}_e'(L) \to \mathcal{X}_u(L),$$

and the induced long exact sequence of cohomology agrees with the one constructed in [PS16].

(5) *The Puppe map $\mathcal{X}_u(L) \to \mathcal{X}_o(L)$ from the previous cofibration sequence is homotopic to the quotient map $\mathcal{X}_u(L) \to \mathcal{X}_u(L)/\mathbb{Z}_2^-$.*

1.3. **Burnside categories and functors.** This paper uses the machinery of Burnside functors from [HKK16, LLS]. There, the dual of the Khovanov chain complex of a link diagram with n (ordered) crossings is viewed as a diagram of abelian groups:

$$\mathfrak{F}_e \colon (\underline{2}^n)^{\mathrm{op}} \to \mathbb{Z}\text{-Mod},$$

where $\underline{2}^n$ is the category with objects elements of $\{0,1\}^n$ and a unique arrow $a \to b$ if $a \geq b$.

In order to construct a stable homotopy type, one considers a certain 2-category $\mathscr{B}$, *the Burnside category*, whose objects are finite sets and whose 1-morphisms are finite correspondences. The 2-category $\mathscr{B}$ naturally comes with a forgetful functor to abelian groups $\mathscr{B} \to \mathbb{Z}\text{-Mod}$ by sending a set S to the free abelian group $\mathbb{Z}\langle S \rangle$ generated by S. The Khovanov stable homotopy type arises

from a lift:

$$
\begin{array}{ccc}
 & & \mathscr{B} \\
 & \nearrow^{F_e} & \downarrow \\
\underline{2}^n & \xrightarrow{\ \mathfrak{F}_e^{\mathrm{op}}\ } & \mathbb{Z}\text{-Mod}
\end{array}
$$

The *realization* of any functor $F_e \colon \underline{2}^n \to \mathscr{B}$ is then defined as a finite CW spectrum $|F_e|$ associated to F_e. Indeed, as shown in [LLS17], the stable equivalence class of the functor F_e, modulo shifting by the number of negative crossings n_-, is itself an invariant (which recovers the even Khovanov homotopy type $\mathcal{X}_e$—the appropriately shifted stable homotopy type of $|F_e|$).

In this paper, we first review, in §2, the odd Khovanov chain complex, viewing it as a diagram:

$$\mathfrak{F}_o \colon (\underline{2}^n)^{\mathrm{op}} \to \mathbb{Z}\text{-Mod}.$$

Indeed, $\mathfrak{F}_o$ and $\mathfrak{F}_e$ can be combined by a pullback into a unified functor

$$\mathfrak{F}_u \colon (\underline{2}^n)^{\mathrm{op}} \to \mathbb{Z}_u\text{-Mod}.$$

where $\mathbb{Z}_u = \mathbb{Z}[\xi]/(\xi^2 - 1)$, and $\mathfrak{F}_e$ (respectively, $\mathfrak{F}_o$) is obtained by setting $\xi = +1$ (respectively, $\xi = -1$).

Then in §3, we move to some slight generalizations of the Burnside 2-category, $\mathscr{B}_\sigma$, the *signed Burnside category* in order to take account of the signs appearing in odd Khovanov homology, and $\mathscr{B}_\xi$, the *free $\mathbb{Z}_2$-equivariant Burnside category* in order to take account of the ξ-action.

In §4 we show how the realization construction of [LLS] generalizes to $\mathscr{B}_\sigma$ and $\mathscr{B}_\xi$. Roughly, the realization process of a functor to $\mathscr{B}_\sigma$ is comparable to the realization process of a functor to $\mathscr{B}$, except that where a sign appears, the corresponding cell is glued in by a fixed orientation-reversing homeomorphism.

And then in §5 we construct lifts

where the arrows among the various versions of Burnside categories are those from Figure 1. Note that the lift

$$F_o \colon \underline{2}^n \to \mathscr{B}_\sigma$$

recovers all the other lifts; it decomposes along quantum gradings $F_o = \amalg_j F_o^j$, and its equivariant equivalence class, after shifting by n_-, recovers all the (correctly shifted) stable homotopy types obtained by the above-mentioned realization procedure, and is itself an invariant:

Theorem 1.7. *The equivariant equivalence class of the shifted functor $\Sigma^{-n_-} F_o^j$ from Definition 5.1 is independent of all the choices in its construction and is a link invariant.*

2. Khovanov homologies

In this section we review the definitions and basic properties of three versions of Khovanov homology for an oriented link L: ordinary or *even* Khovanov homology $Kh(L) = Kh_e(L)$, defined by Khovanov [Kho00]; *odd* Khovanov homology $Kh_o(L)$ defined by Ozsváth, Rasmussen and Szabó [ORSz13]; and $Kh_u(L)$, the *unified* theory of Putyra and Putyra-Shumakovitch [Put14, PS16], which generalizes the previous two theories. These three homological invariants will be upgraded to Burnside functors in §5.

2.1. **The cube category.** We first recall the cube category. Call $\underline{2} = \{0, 1\}$ the one-dimensional cube, viewed as a partially ordered set by setting $1 > 0$, or as a category with a single non-identity morphism from 1 to 0.

Call $\underline{2}^n = \{0, 1\}^n$ the n-dimensional cube, with the partial order given by

$$u = (u_1, \ldots, u_n) \geq v = (v_1, \ldots, v_n) \text{ if and only if } \forall\, i\ (u_i \geq v_i).$$

It has the categorical structure induced by the partial order, where $\text{Hom}_{\underline{2}^n}(u, v)$ has a single element if $u \geq v$ and is empty otherwise. Write $\phi_{u,v}$ for the unique morphism $u \to v$ if it exists. The cube carries a grading given by $|v| = \sum_i v_i$. Write $u \geq_k v$ if $u \geq v$ and $|u| - |v| = k$. When $u \geq_1 v$, call the corresponding morphism $\phi_{u,v}$ an *edge*.

Definition 2.1. The *standard sign assigment* s is the following function from edges of $\underline{2}^n$ to $\mathbb{Z}_2$. For $u \geq_1 v$, let k be the unique element in $\{1, \ldots, n\}$ with $u_k > v_k$. Then

$$s_{u,v} := \sum_{i=1}^{k-1} u_i \bmod 2.$$

Note that s may be viewed as a 1-cochain in $C^*_{\text{cell}}([0, 1]^n; \mathbb{Z}_2)$. In general, $s + c$ is called a *sign assignment* for any 1-cocycle c in $C^*_{\text{cell}}([0, 1]^n; \mathbb{Z}_2)$.

2.2. **Some rings and modules.** We will often write $\mathbb{Z}_2$ multiplicatively as $\{1, \xi\}$. The integral group ring of $\mathbb{Z}_2$ then has the presentation $\mathbb{Z}[\xi]/(\xi^2 - 1)$, which we abbreviate to $\mathbb{Z}_u$. There are two basic $\mathbb{Z}_u$-modules $\mathbb{Z}_e$ and $\mathbb{Z}_o$ obtained from $\mathbb{Z}_u$ by setting $\xi = +1$ and $\xi = -1$, which fit into the

following diagram:

(2.2)

$$
\begin{array}{ccc}
 & \mathbb{Z}_u & \\
\xi = +1 \swarrow & & \searrow \xi = -1 \\
\mathbb{Z}_e & & \mathbb{Z}_o \\
\searrow & & \swarrow \\
 & \mathbb{Z}_2 &
\end{array}
$$

The modules $\mathbb{Z}_e$ and $\mathbb{Z}_o$ are both infinite cyclic groups, for which $\xi \in \mathbb{Z}_u$ acts as $+1$ on $\mathbb{Z}_e$, and by -1 on $\mathbb{Z}_o$. All maps in the above diagram are surjections, and in fact $\mathbb{Z}_u$ is a pull-back for this diagram in the category of rings. Equivalently, $\mathbb{Z}_u$ is isomorphic to the subring of $\mathbb{Z} \oplus \mathbb{Z}$ consisting of pairs (a, b) with $a \equiv b \bmod 2$. Note that the kernel of the map $\xi = +1$ (resp. $\xi = -1$) in the diagram is isomorphic to $\mathbb{Z}_o$ (resp. $\mathbb{Z}_e$). In particular, we have a short exact sequence $0 \to \mathbb{Z}_e \to \mathbb{Z}_u \to \mathbb{Z}_o \to 0$, and an analogous exact sequence with e and o swapped.

Now let S be any finite set, and let $T(S)$ be the tensor algebra generated by S over $\mathbb{Z}_u$. Let I be the two-sided ideal of $T(S)$ generated by elements $x \otimes x$ and $x \otimes y - \xi y \otimes x$ where $x, y \in S$.

Definition 2.3. Given a finite set S, we define the $\mathbb{Z}_u$-module $\Lambda_u(S) := T(S)/I$.

We will abuse notation and write $x_1 \otimes \cdots \otimes x_n \in \Lambda_u(S)$ for the equivalence class of the element $x_1 \otimes \cdots \otimes x_n \in T(S)$, whenever each $x_i \in S$. We have the fundamental relation

$$
x_1 \otimes \cdots \otimes x_n \; = \; \xi^{\mathrm{sign}(\sigma)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}
$$

for any permutation σ of length n. Upon setting $\xi = +1$, we recover $\Lambda_e(S)$, the symmetric algebra on the set S modulo the ideal generated by squares of elements in S. If we set $\xi = -1$, we recover $\Lambda_o(S)$, the usual exterior algebra on the set S. If we write $\Lambda_2(S)$ for the $\mathbb{Z}_2$-exterior algebra on the set S, then these four algebras fit into a pull-back diamond analogous to Diagram (2.2).

2.3. **Three Khovanov homology theories.** We will now recall the definition of odd Khovanov homology, as well as the definition of the unified theory (in the spirit of odd Khovanov homology). Let L be a link diagram with n ordered crossings. Each crossing $\times$ can be resolved as the 0-*resolution* $)($ or the 1-*resolution* $\asymp$. We assume that L is decorated by an *orientation of crossings*, i.e., a choice of an arrow at each crossing, $\nrightarrow$ or $\nleftarrow$, connecting the two arcs of the 0-resolution at that crossing. Rotating the arrows $90°$ degrees clockwise (this requires choosing an orientation of the plane) produces an arrow joining the two arcs of the 1-resolution at that crossing as well. That is, a crossing $\nrightarrow$ (respectively, $\nleftarrow$) has 0-resolution $\mapsto$ (respectively, $\mapsfrom$) and 1-*resolution* $\curlyvee$ (respectively, $\curlywedge$).

We will recall the 'top-down' construction of three functors

$$
\mathfrak{F}_u : (\underline{2}^n)^{\mathrm{op}} \longrightarrow \mathbb{Z}_u\text{-Mod}, \qquad \mathfrak{F}_e : (\underline{2}^n)^{\mathrm{op}} \longrightarrow \mathbb{Z}\text{-Mod}, \qquad \mathfrak{F}_o : (\underline{2}^n)^{\mathrm{op}} \longrightarrow \mathbb{Z}\text{-Mod},
$$

by first defining the unified functor $\mathfrak{F}_u$, and then defining $\mathfrak{F}_e$ and $\mathfrak{F}_o$ by restricting scalars $\xi = +1$ and $\xi = -1$, respectively. (Alternatively, one can also carry out a bottom-up approach, defining $\mathfrak{F}_e$ and $\mathfrak{F}_o$, and then defining $\mathfrak{F}_u$ as the pullback.)

For each $v \in \underline{2}^n$, let L_v be the complete resolution diagram formed by taking the 0-resolution at the ith crossing if $v_i = 0$, or the 1-resolution otherwise. The diagram L_v is a planar diagram of embedded circles and oriented arcs. Write $Z(L_v)$ for the set of circles in L_v.

For objects $v \in \underline{2}^n$, set $\mathfrak{F}_u(v) = \Lambda_u(Z(L_v))$. For morphisms, first consider the following assignment $\mathfrak{F}'_u$ on the edges; the actual functor $\mathfrak{F}_u$ will be a slight modification of this assignment. Let $\phi_{v,w}$ be an edge in $\underline{2}^n$, so that its reverse $\phi^{\text{op}}_{w,v}$ is a morphism in the opposite category. Suppose $\phi^{\text{op}}_{w,v}$ corresponds to a split cobordism, so that some circle $a \in Z(L_w)$ splits into two circles $a_1, a_2 \in Z(L_v)$, and that the other elements of these two sets of circles are naturally identified. Suppose further that the arc in L_v associated to this splitting is pointing from a_1 to a_2. Define

$$\mathfrak{F}'_u(\phi^{\text{op}}_{w,v})(x) \;=\; (a_1 + \xi a_2) \otimes x$$

where $\Lambda_u(Z(L_w))$ is viewed embedded in $\Lambda_u(Z(L_v))$ by sending a to either a_1 or a_2. Now suppose instead we have a merge cobordism, so that two circles $a_1, a_2 \in Z(L_w)$ merge into one circle $a \in Z(L_v)$, and that the other elements in these two sets of circles are naturally identified. Define $\mathfrak{F}'_u(\phi^{\text{op}}_{w,v})$ to be the $\mathbb{Z}_u$-algebra map $\Lambda_u(Z(L_w)) \to \Lambda_u(Z(L_v))$ determined by sending a_1 and a_2 to a, and by the identity map on other circle generators. The assignment $\mathfrak{F}'_u$ on the edges does not commute across the 2-dimensional faces; rather, it does so only up to possible multiplication by ξ. We correct the assignment $\mathfrak{F}'_u$ on morphisms as follows.

The two-dimensional configurations can be divided into four categories as follows (with unoriented arcs being orientable in either direction).

(2.4)

For the type-A faces, $\mathfrak{F}'_u$ commutes after multiplication by ξ, for the type-C faces $\mathfrak{F}'_u$ commutes directly, while for the type-X and type-Y faces, $\mathfrak{F}'_u$ commutes, both directly, and after multiplication by ξ. Define ψ_X (respectively, ψ_Y), an element of $C^2_{\text{cell}}([0,1]^n; \{1, \xi\})$, to be ξ for the type-A or -X faces (respectively, type-A or -Y faces), and 1 for the type-C or -Y faces (respectively, type-C or -X faces).

Definition 2.5. A type-X (respectively, type-Y) *edge assignment* for the diagram L with oriented crossings is a (multiplicative) cochain $\epsilon \in C^1_{\text{cell}}([0,1]^n; \{1, \xi\})$ such that $\delta\epsilon = \psi_X$ (respectively, $= \psi_Y$).

Fix an edge assignment ϵ, either of type-X or type-Y. For an edge $\phi^{\text{op}}_{w,v}$, set

$$\mathfrak{F}_u(\phi^{\text{op}}_{w,v}) = \epsilon(\phi^{\text{op}}_{w,v})\mathfrak{F}'_u(\phi^{\text{op}}_{w,v})$$

and this defines the functor $\mathfrak{F}_u$. Setting $\xi = +1$ and $\xi = -1$ throughout the above construction defines the even and odd functors $\mathfrak{F}_e$ and $\mathfrak{F}_o$, respectively.

The three Khovanov homology theories are then defined from these functors as follows. First, we define for $\bullet \in \{u, e, o\}$ a chain complex:

$$Kc_\bullet(L) \;=\; \bigoplus_{v \in \underline{2}^n} \mathfrak{F}_\bullet(v), \qquad \partial_\bullet \;=\; \sum_{v \geqslant_1 w} (-1)^{s_{v,w}} \, \mathfrak{F}_\bullet(\phi_{w,v}^{\mathrm{op}})$$

(Here s is the standard sign assignment from Definition 2.1.) This complex depends on the ordering of the crossings, the choice of crossing orientations, and the choice of edge assignment, as does the corresponding functor. However, these choices do not affect the resulting chain homotopy type [Kho00, Theorem 1], [ORSz13, Theorem 1.3], [Put14, §7].

Definition 2.6. For $\bullet \in \{u, e, o\}$ define $Kh_\bullet(L) = H_*(Kc_\bullet(L), \partial_\bullet)$.

The unified homology group $Kh_u(L)$ is a $\mathbb{Z}_u$-module, while the even and odd theories are abelian groups. Each theory has a bigrading, defined in the usual way, as follows. Let n_- be the number of negative crossings $\diagdown\kern-0.6em\diagup$ in the diagram L. The homological and quantum gradings, denoted i and j respectively, are defined on $x = a_1 \otimes \cdots \otimes a_k \in \Lambda_\bullet(Z(L_v))$ with $a_1, \ldots, a_k \in Z(L_v)$ to be

$$i(x) \;=\; |v| - n_-, \qquad j(x) \;=\; |Z(L_v)| - 2k + |v| + n - 3n_-.$$

We write $Kh_\bullet^{i,j}(L)$ for the corresponding bigraded module. We note that the unified theory in [Put14] is called the *covering homology*, and is more specfically obtained from Example 10.7 of loc. cit. by setting $X = Z = 1$ and $Y = \xi$. The terminology *unified* is used in [PS16].

Remark 2.7. Our definition of an edge assignment is non-standard, but the standard type-X (respectively, type-Y) edge assignment from [ORSz13, Put14] may be obtained from our type-X (respectively, type-Y) edge assignment by multiplying by $(-1)^{s_{v,w}}$.

2.4. **Relations between the theories.** From the definitions, it is clear that the chain complexes $Kc_\bullet(L)$ above fit into a pull-back diagram just as in Diagram (2.2),

(2.8)
$$
\begin{array}{ccc}
 & Kc_u(L) & \\
\scriptstyle \xi = +1 \swarrow & & \searrow \scriptstyle \xi = -1 \\
Kc_e(L) & & Kc_o(L) \\
\searrow & & \swarrow \\
 & Kc_2(L) &
\end{array}
$$

where $Kc_2(L)$ denotes Khovanov chain complex with $\mathbb{Z}_2$ coefficients. Indeed, we may define $Kc_u(L)$ to be the pullback of $Kc_e(L)$ and $Kc_o(L)$ over $Kc_2(L)$,

$$Kc_u(L) = \{(a, b) \in Kc_e(L) \oplus Kc_o(L) \mid a \equiv b \bmod 2\},$$

which then naturally inherits a $\mathbb{Z}_2$-action $\xi(a, b) = (a, -b)$.

We also have a short exact sequence of chain complexes $0 \to Kc_e(L) \to Kc_u(L) \to Kc_o(L) \to 0$. This may be viewed as arising from tensoring the short exact sequence $0 \to \mathbb{Z}_e \to \mathbb{Z}_u \to \mathbb{Z}_o \to 0$ by the unified chain complex $Kc_u(L)$ over $\mathbb{Z}_u$. There is a similar exact sequence with e and o swapped. Passing to homology yields the following long exact sequences, cf. [PS16]:

(2.9)
$$\cdots \to Kh_e^{i,j}(L) \longrightarrow Kh_u^{i,j}(L) \longrightarrow Kh_o^{i,j}(L) \xrightarrow{\phi_{oe}} Kh_e^{i+1,j}(L) \to \cdots$$

(2.10) $$\cdots \to Kh_o^{i,j}(L) \longrightarrow Kh_u^{i,j}(L) \longrightarrow Kh_e^{i,j}(L) \xrightarrow{\phi_{eo}} Kh_o^{i+1,j}(L) \to \cdots$$

2.5. **Reduced theories.** Let p be a basepoint on the planar diagram L. Then the reduced complex $\widetilde{Kc}_\bullet(L, p)$ for $\bullet \in \{u, e, o\}$ is the subcomplex of $Kc_\bullet(L)$ consisting of elements of the form $a \otimes y$, where a is a circle containing p in a resolution diagram, and y is any other element. The complex $\widetilde{Kc}_\bullet(L)$ is homologically graded as a subcomplex of $Kc_\bullet(L)$, but there is a shift in its quantum grading, which is defined as *one more* than the formula for $j(x)$ above. The reduced functors $\widetilde{\mathfrak{F}}_\bullet$ are defined in the same way.

The chain homotopy type depends on the isotopy type of L and which component of the link p lies in. In the odd case, the basepoint does not matter, and the chain complex is a direct sum [ORSz13, Prop 1.7]:

(2.11) $$Kc_o^{*,j}(L) \;=\; \widetilde{Kc}_o^{*,j-1}(L) \oplus \widetilde{Kc}_o^{*,j+1}(L)$$

In contrast to the odd case, the unified and even theories do not split into a direct sum of their reduced theories. Instead, for $\bullet \in \{u, e\}$, there is a short exact sequence of chain complexes

$$0 \longrightarrow \widetilde{Kc}_\bullet^{*,j+1}(L, p) \longrightarrow Kc_\bullet^{*,j}(L) \longrightarrow \widetilde{Kh}_\bullet^{*,j-1}(L, p) \longrightarrow 0.$$

The reduced Khovanov homology is defined as

Definition 2.12. For $\bullet \in \{u, e, o\}$ define $\widetilde{Kh}_\bullet^{i,j}(L, p) = H_i(\widetilde{Kc}_\bullet^{*,j}(L, p), \partial_\bullet)$.

2.6. **Khovanov generators.** In the sequel, we will need to fix bases of these chain complexes to facilitate the construction of the various Khovanov spectra. For even Khovanov homology, there is a natural basis of generators: the elements $a_1 \otimes \cdots \otimes a_k \in \Lambda_e(Z(L_v))$ where each $a_i \in Z(L_v)$ is distinct. Since $\Lambda_e(Z(L_v))$ is the quotient of a symmetric algebra on these generators, their order does not matter. For the unified and odd cases, order matters, however: recall that $a_1 \otimes a_2 = \xi a_2 \otimes a_1$ in the unified case, and $a_1 \otimes a_2 = -a_2 \otimes a_1$ in the odd case. To fix generators, we will thus fix at every vertex $v \in \underline{2}^n$ a total ordering $>$ of the set $Z(L_v)$. Once this is done, we write

$$Kg(v) \;=\; Kg_\bullet(v) \;:=\; \{a_1 \otimes \cdots \otimes a_k : a_i \in Z(L_v),\ a_1 > \cdots > a_k\} \qquad \bullet \in \{u, e, o\}$$

for the set of *Khovanov generators at* v. As indicated, we will often omit the subscript $\bullet$ from the notation, as each of the three sets for a fixed v are naturally identified (once the circles in $Z(L_v)$ are totally ordered). Note that in the unified case, the set of Khovanov generators over all $v \in \underline{2}^n$ gives a $\mathbb{Z}_u$-basis for the chain complex. On the other hand, a $\mathbb{Z}$-basis for the unified chain complex at $v \in \underline{2}^n$ is given by

$$Kg(v) \ \amalg \ \xi Kg(v)$$

where $\xi Kg(v)$ is the set of ξx with $x \in Kg(v)$. Note that $Kg(v)$ has $2^{|Z(L_v)|}$ elements. Given a basepoint p on our diagram L, we can also form the set of reduced generators $\widetilde{Kg}(v; p)$ at the vertex v, the subset of $Kg(v)$ whose elements each include the circle containing the basepoint. This set has half the number of elements of $Kg(v)$, and, running over all $v \in \underline{2}^n$, forms a basis for the reduced complex $\widetilde{Kc}_\bullet(L, p)$ where $\bullet \in \{e, o\}$, and a $\mathbb{Z}_u$-basis for $\widetilde{Kc}_u(L, p)$.

3. Burnside categories and functors

In this section, we review the definition of the Burnside category $\mathscr{B}$ from [LLS, LLS17], and define some slight modifications: the signed Burnside category $\mathscr{B}_\sigma$ and the $\mathbb{Z}_2$-equivariant Burnside category $\mathscr{B}_\xi$. We then discuss functors from the cube category $\underline{2}^n$ to these Burnside categories.

3.1. The Burnside category.
Given finite sets X and Y, a correspondence from X to Y is a triple (A, s, t) for a finite set A, where s, t are set maps $s\colon A \to X$ and $t\colon A \to Y$; s and t are called the *source* and *target* maps, respectively. The correspondence (A, s, t) is depicted:

$$
\begin{array}{ccc}
 & A & \\
{}^{s_A}\swarrow & & \searrow{}^{t_A} \\
X & & Y
\end{array}
$$

For correspondences (A, s_A, t_A) and (B, s_B, t_B) from X to Y and Y to Z, respectively, define the composition $(B, s_B, t_B) \circ (A, s_A, t_A)$ to be the correspondence (C, s, t) from X to Z given by the fiber product $C = B \times_Y A = \{(b, a) \in B \times A \mid t(a) = s(b)\}$ with source and target maps $s(b, a) = s_A(a)$ and $t(b, a) = t_B(b)$. There is also the identity correspondence from a set X to itself, i.e., $(X, \mathrm{Id}_X, \mathrm{Id}_X)$ from X to X. Given correspondences (A, s_A, t_A), (B, s_B, t_B) from X to Y, a morphism of correspondences (A, s_A, t_A) to (B, s_B, t_B) is a bijection $f\colon A \to B$ commuting with the source and target maps. There is also the identity morphism from a correspondence to itself.

Composition (of set maps) gives the set of correspondences from X to Y the structure of a category. Define the *Burnside category* $\mathscr{B}$ to be the weak 2-category whose objects are finite sets, morphisms are finite correspondences, and 2-morphisms are maps of correspondences.

Recall that in a weak 2-category, that arrows need only be associative up to an equivalence, and similarly the identity axiom holds only after composing with a 2-morphism. To be explicit, for finite sets X, Y and (A, s, t) a correspondence from X to Y, neither $(Y, \mathrm{Id}_Y, \mathrm{Id}_Y) \circ (A, s, t)$, nor $(A, s, t) \circ (X, \mathrm{Id}_X, \mathrm{Id}_X)$, equals (A, s, t). Rather, there are natural 2-morphisms, left and right unitors,

$$
\lambda\colon Y \times_Y A \to A, \qquad \rho\colon A \times_X X \to A
$$

given by $\lambda(y, a) = a$ and $\rho(a, x) = a$. Further, the composition $C \circ (B \circ A)$, for A from W to X, B from X to Y, and C from Y to Z, is not identical to $(C \circ B) \circ A$, rather there is an associator

$$
\alpha\colon (C \times_Y B) \times_X A \to C \times_Y (B \times_X A)
$$

given by $\alpha((c, b), a) = (c, (b, a))$. The categories to follow are slight variations of this one. The total diagram of Burnside categories that we will consider in this article is depicted in Figure 1.

3.2. The signed Burnside category.
Given sets X and Y, a *signed correspondence* is a correspondence (A, s_A, t_A) equipped with a map $\sigma_A\colon A \to \{+1, -1\}$, regarded as a tuple (A, s_A, t_A, σ_A);

we call σ_A the "sign" or "sign map" of the signed correspondence:

$$\{+1, -1\}$$

$$\sigma_A \uparrow$$

$$s_A \swarrow \quad A \quad \searrow t_A$$

$$X \qquad\qquad\qquad Y$$
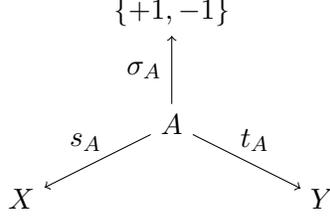
In the sequel we often write "correspondence" for "signed correspondence", where it will not cause any confusion. We define a composition $(B, s_B, t_B, \sigma_B) \circ (A, s_A, t_A, \sigma_A)$ of signed correspondences (A, s_A, t_A, σ_A) from X to Y, and (B, s_B, t_B, σ_B) from Y to Z by (C, s, t, σ), where (C, s, t) is the composition $(B, s_B, t_B) \circ (A, s_A, t_A)$ and $\sigma(b, a) = \sigma_B(b)\sigma_A(a)$. Also, we define the identity (signed) correspondence by $(X, \mathrm{Id}_X, \mathrm{Id}_X, 1)$ (i.e., the identity correspondence takes value 1 on all elements).

We define maps of signed correspondences $f \colon (A, s_A, t_A, \sigma_A) \to (B, s_B, t_B, \sigma_B)$ to be morphisms of correspondences $f \colon (A, s_A, t_A) \to (B, s_B, t_B)$ such that $\sigma_B \circ f = \sigma_A$. We may then define the *signed Burnside category* $\mathscr{B}_\sigma$ to be the weak 2-category with objects finite sets, morphisms given by signed correspondences, and 2-morphisms given by maps of signed correspondences. The structure maps λ, ρ, α of §3.1 are easily seen to respect the sign, confirming that $\mathscr{B}_\sigma$ is indeed a weak 2-category. There is a forgetful 2-functor $\mathcal{F} \colon \mathscr{B}_\sigma \to \mathscr{B}$ which forgets signs. There is also an inclusion-induced 2-functor $\mathcal{I} \colon \mathscr{B} \to \mathscr{B}_\sigma$. We will usually refer to such 2-functors simply as functors.

3.3. The $\mathbb{Z}_2$-equivariant Burnside category.
We let $\mathscr{B}_\xi$ denote the 2-category whose objects are finite, free $\mathbb{Z}_2$-sets, with $\mathbb{Z}_2$-equivariant correspondences, and 2-morphisms $\mathbb{Z}_2$-equivariant bijections of correspondences. (Recall that we write $\mathbb{Z}_2 = \{1, \xi\}$.) The 2-category $\mathscr{B}_\xi$ is a subcategory of the Burnside 2-category for the group $\mathbb{Z}_2$.

There is a forgetful functor $\mathcal{F} \colon \mathscr{B}_\xi \to \mathscr{B}$. There is also a "quotient" functor $\mathcal{Q} \colon \mathscr{B}_\xi \to \mathscr{B}$, which simply takes the quotient by the action of $\mathbb{Z}_2$ on objects, 1- and 2-morphisms.

There is also a strictly unitary "doubling" 2-functor $\mathcal{D} \colon \mathscr{B}_\sigma \to \mathscr{B}_\xi$ consisting of the following data.

(1) For each object X of $\mathscr{B}_\sigma$, we need to specify an object $\mathcal{D}(X)$ of $\mathscr{B}_\xi$. Define $\mathcal{D}(X) = \{1, \xi\} \times X$, with the $\mathbb{Z}_2$-action on $\{1, \xi\} \times X$ being $\xi(1, x) = (\xi, x)$ for all $x \in X$.

(2) For any 2 objects X, Y of $\mathscr{B}_\sigma$, we need to specify a functor, also denoted $\mathcal{D}$, from $\mathrm{Hom}_{\mathscr{B}_\sigma}(X, Y)$ to $\mathrm{Hom}_{\mathscr{B}_\xi}(\mathcal{D}(X), \mathcal{D}(Y))$ that sends Id_X to $\mathrm{Id}_{\mathcal{D}(X)}$. This functor sends a signed correspondence A from X to Y to the correspondence $\{1, \xi\} \times A$ from $\{1, \xi\} \times X$ to $\{1, \xi\} \times Y$ (the $\mathbb{Z}_2$-action is similar). The source and target maps on $\{1\} \times A$ are defined as
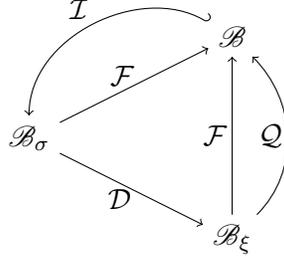
$$s(1, a) = (1, s(a))$$
$$t(1, a) = (\sigma(a), t(a))$$

FIGURE 1. **The Burnside categories and some functors between them.** We have the relations $\mathcal{Q} \circ \mathcal{D} = \mathcal{F}$ and $\mathcal{F} \circ \mathcal{I} = \mathrm{Id}$. The $\mathcal{F}$'s are forgetful functors, $\mathcal{I}$ is a subcategory inclusion, $\mathcal{Q}$ is a quotient functor, and $\mathcal{D}$ stands for doubling.

(The sign $\sigma(a)$ takes value in $\mathbb{Z}_2 = \{+1, -1\}$, which has been identitified with $\mathbb{Z}_2 = \{1, \xi\}$.) The source and target maps are then extended equivariantly to $\{\xi\} \times A$. Finally, $\mathcal{D}$ sends a 2-morphism $f \colon A \to B$ to the 2-morphism $(\mathrm{Id}, f) \colon \{1, \xi\} \times A \to \{1, \xi\} \times B$.

(3) Finally, for any correspondences A from X to Y and B from Y to Z in $\mathscr{B}_\sigma$, we need to specify a 2-morphism in $\mathscr{B}_\xi$ from $\mathcal{D}(B) \circ \mathcal{D}(A) = (\{1, \xi\} \times B) \times_{(\{1,\xi\} \times Y)} (\{1, \xi\} \times A)$ to $\mathcal{D}(B \circ A) = \{1, \xi\} \times (B \times_Y A)$ that is natural in A and B. Define it to be

$$((\sigma(a), b), (1, a)) \mapsto (1, (b, a)),$$

extended $\mathbb{Z}_2$-equivariantly. It is not hard to check that these maps satisfy the required coherence relations with the structure maps λ, ρ, α of $\mathscr{B}_\sigma$ and $\mathscr{B}_\xi$, as described in [Bén67, Definition 4.1 and Remark 4.2]. Therefore, the above is indeed a 2-functor.

3.4. Functors from Burnside categories.
For $\bullet \in \{\varnothing, \sigma\}$ we define a functor $\mathcal{A} \colon \mathscr{B}_\bullet \to \mathbb{Z}\text{-Mod}$ by sending a set $X \in \mathscr{B}_\bullet$ to the free abelian group generated by X, denoted $\mathcal{A}(X)$. For a signed correspondence $\phi = (A, s, t, \sigma)$ from X to Y, we define $\mathcal{A}(\phi) \colon \mathcal{A}(X) \to \mathcal{A}(Y)$ by

$$\tag{3.1} \mathcal{A}(\phi)(x) = \sum_{a \in A \,|\, s(a) = x} \sigma(a) t(a)$$

for elements $x \in X$, extended linearly over $\mathbb{Z}$. Similarly, we have a functor $\mathcal{A} \colon \mathscr{B}_\xi \to \mathbb{Z}_u\text{-Mod}$ that sends a free $\mathbb{Z}_2$-set X to $\mathcal{A}(X)$, which is a free $\mathbb{Z}_u$-module; for a $\mathbb{Z}_2$-equivariant correspondence ϕ, we use the same formula as Equation (3.1), but exclude σ, and extend linearly over $\mathbb{Z}_u$.

3.5. Functors to Burnside categories.
We now consider functors from the cube category $\underline{2}^n$ to the Burnside categories thus far introduced. We let $\mathscr{B}_\bullet$ be one of the Burnside categories introduced above, appearing in Figure 1, so that $\bullet \in \{\varnothing, \sigma, \xi\}$. The functors $F \colon \underline{2}^n \to \mathscr{B}_\bullet$ we consider will be strictly unitary 2-functors; that is, they will consist of the following data:

(1) For each vertex v of $\underline{2}^n$, an object $F(v)$ of $\mathscr{B}_\bullet$.
(2) For any $u \geq v$, a 1-morphism $F(\phi_{u,v})$ in $\mathscr{B}_\bullet$ from $F(u)$ to $F(v)$, such that $F(\phi_{u,u})$ is the identity morphism $\mathrm{Id}_{F(u)}$.

(3) Finally, for any $u \geq v \geq w$, a 2-morphism $F_{u,v,w}$ in $\mathscr{B}_\bullet$ from $F(\phi_{v,w}) \circ F(\phi_{u,v})$ to $F(\phi_{u,w})$ that agrees with λ (respectively, ρ) when $v = w$ (respectively, $u = v$), and that satisfies, for any $u \geq v \geq w \geq z$,

$$F_{u,w,z} \circ_2 (\mathrm{Id} \circ F_{u,v,w}) = (F_{v,w,z} \circ \mathrm{Id}) \circ_2 F_{u,v,z}$$

(with $\circ$ denoting composition of 1-morphisms and $\circ_2$ denoting composition of 2-morphisms; and we have suppressed the associator α).

We will usually use the characterization of these functors in the lemma to follow.

Lemma 3.2. *Let $\mathscr{B}_\bullet$ be any of the Burnside categories with $\bullet \in \{\varnothing, \sigma, \xi\}$. Consider the data of objects $F(v)$ for $v \in \underline{2}^n$, 1-morphisms $F(\phi_{v,w})$ for edges $v \geqslant_1 w$, and 2-morphisms $F_{u,v,v',w} \colon F(\phi_{v,w}) \circ F(\phi_{u,v}) \to F(\phi_{v',w}) \circ F(\phi_{u,v'})$ for each 2d face described by $u \geqslant_1 v, v' \geqslant_1 w$, such that the following compatibility conditions are satisfied:*

(1) For any 2d face u, v, v', w as above, $F_{u,v,v',w} = F_{u,v',v,w}^{-1}$;
(2) For any 3d face in $\underline{2}^n$ on the left, the hexagon on the right commutes:
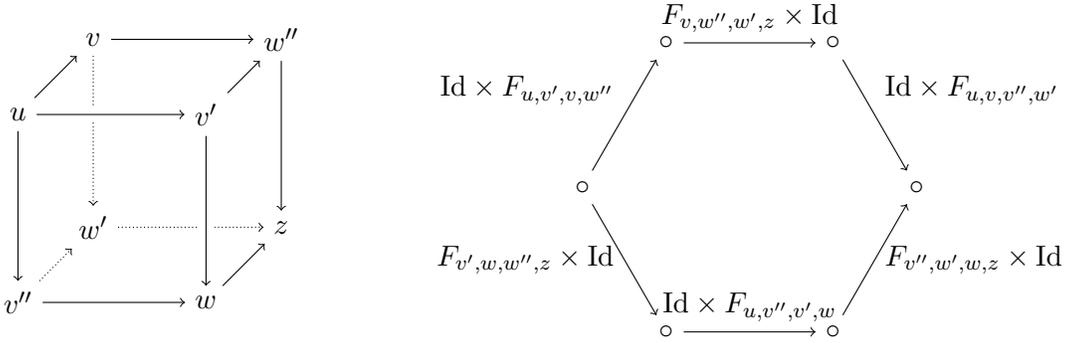
This collection of data can be extended to a strictly unitary functor $F \colon \underline{2}^n \to \mathscr{B}_\bullet$, uniquely up to natural isomorphism, so that $F_{u,v,v',w} = F_{u,v',w}^{-1} \circ_2 F_{u,v,w}$.

Proof. The proof is same as that of [LLS, Lemma 2.12] and [LLS17, Lemma 4.2]. □

3.6. **Totalization.** Given a functor $F \colon \underline{2}^n \to \mathscr{B}_\bullet$ we construct a chain complex denoted $\mathrm{Tot}(F)$, and called the *totalization* of the functor F. The underlying chain group of $\mathrm{Tot}(F)$ is given by

$$\mathrm{Tot}(F) = \bigoplus_{v \in \underline{2}^n} \mathcal{A}(F(v)).$$

We set the homological grading of the summand $\mathcal{A}(F(v))$ to be $|v|$. The differential is given by defining the components $\partial_{u,v}$ from $\mathcal{A}(F(u))$ to $\mathcal{A}(F(v))$ by

$$\partial_{u,v} = \begin{cases} (-1)^{s_{u,v}} \mathcal{A}(F(\phi_{u,v})) & \text{if } u \geqslant_1 v \\ 0 & \text{otherwise.} \end{cases}$$

Note that for a functor $F \colon \underline{2}^n \to \mathscr{B}_\xi$, the totalization $\mathrm{Tot}(F)$ is a chain complex over $\mathbb{Z}_u$.

3.7. **Natural transformations.** The following will serve as the basic relation between functors from the cube to a Burnside category. As before, $\bullet \in \{\varnothing, \sigma, \xi\}$.

Definition 3.3. A *natural transformation* $\eta\colon F_1 \to F_0$ between 2-functors $F_1, F_0\colon \underline{2}^n \to \mathscr{B}_\bullet$ is a strictly unitary 2-functor $\eta\colon \underline{2}^{n+1} \to \mathscr{B}_\bullet$ so that $\eta|_{\{1\}\times\underline{2}^n} = F_1$ and $\eta|_{\{0\}\times\underline{2}^n} = F_0$.

A natural transformation (functorially) induces a chain map between the chain complexes of Burnside functors, which we write as $\mathrm{Tot}(\eta)\colon \mathrm{Tot}(F_1) \to \mathrm{Tot}(F_0)$.

Many of the natural transformations we will encounter will be sub-functor inclusions or quotient functor surjections. Given a functor $F\colon \underline{2}^n \to \mathscr{B}_\bullet$, a *sub-functor* (respectively, *quotient functor*) $G\colon \underline{2}^n \to \mathscr{B}_\bullet$ is a functor that satisfies:

(1) $G(v) \subset F(v)$ for all $v \in \underline{2}^n$; if $\bullet = \xi$, the $\mathbb{Z}_2$-action on $G(v)$ is induced from the $\mathbb{Z}_2$-action on $F(v)$.
(2) $G(\phi_{u,v}) \subset F(\phi_{u,v})$ for all $u \geq v$, with the source and target maps (and the sign map if $\bullet = \sigma$ or the $\mathbb{Z}_2$-action if $\bullet = \xi$) being restrictions of the corresponding ones on $F(\phi_{u,v})$.
(3) $s^{-1}(x) \subset G(\phi_{u,v})$ (respectively, $t^{-1}(y) \subset G(\phi_{u,v})$) for all $u \geq v$ and for all $x \in G(u)$ (respectively, $y \in G(v)$).

If G is a sub (respectively, quotient) functor of F, then there is a natural transformation $G \to F$ (respectively, $F \to G$), and the induced chain map $\mathrm{Tot}(G) \to \mathrm{Tot}(F)$ (respectively, $\mathrm{Tot}(F) \to \mathrm{Tot}(G)$) is an inclusion (respectively, a quotient map).

Definition 3.4. If G is a sub-functor of $F\colon \underline{2}^n \to \mathscr{B}_\bullet$, then the functor H defined as $H(v) = F(v) \setminus G(v)$ and $H(\phi_{u,v}) = F(\phi_{u,v}) \setminus G(\phi_{u,v})$ is a quotient functor of F (and vice-versa). Such a sequence

$$G \to F \to H$$

is called a *cofibration sequence* of Burnside functors; it induces the short exact sequence

$$0 \to \mathrm{Tot}(G) \to \mathrm{Tot}(F) \to \mathrm{Tot}(H) \to 0$$

of chain complexes.

The following is another particular example of a natural transformation which will appear later. Suppose we are given a functor $F\colon \underline{2}^n \to \mathscr{B}_\sigma$. For each object $v \in \underline{2}^n$, choose a function $\zeta_v\colon F(v) \to \{+1, -1\}$. Define a new functor $F'\colon \underline{2}^n \to \mathscr{B}_\sigma$ that is equal to F except that in the correspondence $F(\phi_{v,w}) = (A, s, t, \sigma)$ for $v \geqslant_1 w$ we change the sign function σ to be $\sigma'(x) = \zeta_v(s(x))\zeta_w(t(x))\sigma(x)$. There is a naturally induced natural transformation $\eta\colon F \to F'$:

Definition 3.5. A *sign reassignment* η of $F\colon \underline{2}^n \to \mathscr{B}_\sigma$ is a natural transformation $\eta\colon F \to F'$ as described above, induced by a function $\zeta_v\colon F(v) \to \{\pm 1\}$ for each $v \in \underline{2}^n$.

In the context of Morse theory, a sign reassignment as above corresponds to changing the orientation on the stable tangent bundle to a critical point in Morse theory. In the sequel, the appearance of sign reassignments will be specific to odd Khovanov homology. Such reassignments are not necessary in the (even) Khovanov setting, since in that case there is a preferred choice of signs: the (even) Khovanov complex comes equipped with a choice of generators for which all signs in the differentials, apart from the standard sign assignment, are positive, cf. §2.6. In the odd

Khovanov setting, in which there are generally no such positive bases, sign reassignments inevitably appear.

3.8. **Stable equivalence of functors.** In the sequel, we will be interested not just in functors $F\colon \underline{2}^n \to \mathscr{B}_\bullet$, but *stable* functors, which are pairs (F, r) with $r \in \mathbb{Z}$. We will write $\Sigma^r F$ for the pair (F, r); its totalization is defined to be $\Sigma^r \mathrm{Tot}(F) = \mathrm{Tot}(F)[r]$, that is, the chain complex $\mathrm{Tot}(F)$ shifted up by r. In this section we describe when two such stable functors are equivalent, only slightly modifying [LLS17, Definition 5.9] by allowing $\bullet \in \{\varnothing, \sigma, \xi\}$.

A *face inclusion* ι is a functor $\underline{2}^n \to \underline{2}^N$ that is injective on objects and preserves the relative gradings. We remark that self-equivalences $\iota\colon \underline{2}^n \to \underline{2}^n$ are face inclusions. Now consider a face inclusion $\iota\colon \underline{2}^n \to \underline{2}^N$ and a functor $F\colon \underline{2}^n \to \mathscr{B}_\bullet$. The induced functor $F_\iota\colon \underline{2}^N \to \mathscr{B}_\bullet$ is uniquely determined by requiring that $F = F_\iota \circ \iota$, and such that for $v \in \underline{2}^N / \iota(\underline{2}^n)$, we have $F_\iota(v) = \varnothing$. For a face inclusion ι, we define $|\iota| = |\iota(v)| - |v|$ for any $v \in \underline{2}^n$, which is independent of v because ι is assumed to preserve relative gradings. As observed in [LLS17, §5], for any face inclusion ι and functor F as above,

$$\mathrm{Tot}(F_\iota) \;\cong\; \Sigma^{|\iota|} \mathrm{Tot}(F)$$

where the isomorphism is natural up to certain sign choices. With this background, we state the relevant notion of equivalence for stable functors.

Definition 3.6. Two stable functors $(E_1\colon \underline{2}^{m_1} \to \mathscr{B}_\bullet, q_1)$ and $(E_2\colon \underline{2}^{m_2} \to \mathscr{B}_\bullet, q_2)$ are *stably equivalent* if there is a sequence of stable functors $\{(F_i\colon \underline{2}^{n_i} \to \mathscr{B}_\bullet, r_i)\}$ $(0 \le i \le \ell)$ with $\Sigma^{q_1} E_1 = \Sigma^{r_0} F_0$ and $\Sigma^{q_2} E_2 = \Sigma^{r_\ell} F_\ell$ such that for each pair $\{\Sigma^{r_i} F_i, \Sigma^{r_{i+1}} F_{i+1}\}$, one of the following holds:

(1) $(n_i, r_i) = (n_{i+1}, r_{i+1})$ and there is a natural transformation $\eta\colon F_i \to F_{i+1}$ or $\eta\colon F_{i+1} \to F_i$ such that the induced map $\mathrm{Tot}(\eta)$ is a chain homotopy equivalence.
(2) There is a face inclusion $\iota\colon \underline{2}^{n_i} \hookrightarrow \underline{2}^{n_{i+1}}$ such that $r_{i+1} = r_i - |\iota|$ and $F_{i+1} = (F_i)_\iota$; or a face inclusion $\iota\colon \underline{2}^{n_{i+1}} \hookrightarrow \underline{2}^{n_i}$ such that $r_i = r_{i+1} - |\iota|$ and $F_i = (F_{i+1})_\iota$.

We call such a sequence, along with the arrows between $\Sigma^{r_i} F_i$, a *stable equivalence* between the stable functors $\Sigma^{q_1} E_1$ and $\Sigma^{q_2} E_2$. If $\bullet = \sigma$, and if the sequence is such that the maps η satisfy $\mathrm{Tot}(\mathcal{D}\eta)$ are chain homotopy equivalences over $\mathbb{Z}_u$ (where $\mathcal{D}\colon \mathscr{B}_\sigma \to \mathscr{B}_\xi$ is from Figure 1), we call it a *equivariant (stable) equivalence*, and say that $\Sigma^{q_i} E_i$ are *equivariantly equivalent*.

We note that a stable equivalence from $\Sigma^{q_1} E_1$ to $\Sigma^{q_2} E_2$ induces a chain homotopy equivalence $\mathrm{Tot}(\Sigma^{q_1} E_1) \to \mathrm{Tot}(\Sigma^{q_2} E_2)$, well-defined up to choices of inverses of the chain homotopy equivalences involved in its construction, and an overall sign. Note that for $\mathscr{B}_\xi$, the category of chain complexes under consideration is over $\mathbb{Z}_u$.

We will also need the notion of a map of Burnside functors:

Definition 3.7. A *map* $\Sigma^{q_1} E_1 \to \Sigma^{q_2} E_2$ of Burnside functors $\Sigma^{q_i} E_i\colon \underline{2}^{m_i} \to \mathscr{B}_\bullet$ consists of a sequence of stable functors $\{(F_i\colon \underline{2}^{n_i} \to \mathscr{B}_\bullet, r_i)\}$ $(0 \le i \le \ell)$ with $\Sigma^{q_1} E_1 = \Sigma^{r_0} F_0$ and $\Sigma^{q_2} E_2 = \Sigma^{r_\ell} F_\ell$ along with the following:

(1) For i even, a stable equivalence from $\Sigma^{r_i} F_i$ to $\Sigma^{r_{i+1}} F_{i+1}$.
(2) For i odd, a natural transformation $F_i \to F_{i+1}$, and we require $r_i = r_{i+1}$.

Similarly, for $\bullet = \sigma$, an *equivariant map* $\Sigma^{q_1} E_1 \to \Sigma^{q_2} E_2$ will consist of the same information, but where the stable equivalences are required to be equivariant equivalences.

3.9. **Coproducts.** Finally, let us describe the elementary coproduct operation on functors $\underline{2}^n \to \mathscr{B}_\bullet$, generalizing that from [LLS17]. [LLS17] also define a product operation, but we have no need for that.

Given two 2-functors $F_1, F_2 \colon \underline{2}^n \to \mathscr{B}_\bullet$ the *coproduct* 2-functor $F_1 \amalg F_2 \colon \underline{2}^n \to \mathscr{B}_\bullet$ is defined as follows. On objects and 1-morphisms, $F_1 \amalg F_2$ is just the disjoint union: $(F_1 \amalg F_2)(v) = F_1(v) \amalg F_2(v)$ for $v \in \underline{2}^n$, with $\mathbb{Z}_2$-action defined component-wise if $\bullet = \xi$, and $(F_1 \amalg F_2)(\phi_{u,v}) = F_1(\phi_{u,v}) \amalg F_2(\phi_{u,v})$ for $u \geq v$, with the source map, target map, and sign map if $\bullet = \sigma$, and $\mathbb{Z}_2$-action if $\bullet = \xi$, defined component-wise. For $u \geq v \geq w$, the associated 2-morphism may be viewed as a map

$$(F_1 \amalg F_2)_{u,v,w} \colon \coprod_{i=1,2} F_i(\phi_{v,w}) \times_{F_i(v)} F_i(\phi_{u,v}) \longrightarrow \coprod_{i=1,2} F_i(\phi_{u,w})$$

and it is defined component-wise using the bijections $(F_i)_{u,v,w}$ for $i = 1, 2$. We have the following immediate property for chain complexes:

$$\mathrm{Tot}(F_1 \amalg F_2) = \mathrm{Tot}(F_1) \oplus \mathrm{Tot}(F_2).$$

4. Realizations of Burnside functors

In this section, given a functor $F \colon \underline{2}^n \to \mathscr{B}_\bullet$ to any of the previously defined Burnside categories, we construct an essentially well-defined finite CW spectrum $|F|$, which in the $\mathbb{Z}_2$-equivariant case $\bullet = \xi$ is a $\mathbb{Z}_2$-equivariant spectrum. As a first step, we construct finite CW complexes $\|F\|_k$ for sufficiently large k, so that increasing the parameter k corresponds to suspending the CW complex $\|F\|_k$. The finite CW spectrum $|F|$ is then defined from this sequence of spaces. The construction of $\|F\|_k$ depends on some auxiliary choices, but its stable homotopy type does not. Moreover, the spectra constructed from two stably equivalent Burnside functors will be homotopy equivalent.

For signed Burnside functors, i.e., when $\bullet = \sigma$, we can carry out our construction with a $\mathbb{Z}_2$-action. For ordinary Burnside functors, i.e., when $\bullet = \varnothing$, our construction recovers the "little boxes" realization of [LLS, §5], cf. [LLS17, §7]; but if it comes from a signed Burnside functor, we produce an alternate constructon with an extra $\mathbb{Z}_2$-action.

After reviewing the notion of "box maps" used in the little box realization construction of [LLS17, §5], we introduce "signed box maps." After providing the necessary background on homotopy colimits, we then use signed box maps to construct the realization $|\cdot|$ for functors to the signed Burnside category. This is all that is needed to construct the odd Khovanov homotopy type. We then indicate the modifications needed to define the other realization functors and to construct the various extra $\mathbb{Z}_2$-actions.

4.1. **Signed box maps.** We start with the construction of (ordinary) box maps, following [LLS, §2.10]. Fix an identification $S^k = [0,1]^k / \partial$ which we maintain though the sequel, and view S^k as a pointed space. Let B be a box in $\mathbb{R}^k$ with edges parallel to the coordinate axes, that is, $B = [a_1, b_1] \times \cdots \times [a_k, b_k]$ for some a_i, b_i. Then there is a standard homeomorphism from B to $[0,1]^k$, via $(x_1, \ldots, x_k) \to (\frac{x_1 - a_1}{b_1 - a_1}, \ldots, \frac{x_k - a_k}{b_k - a_k})$. Then we have an identification $S^k \cong B/\partial B$.

Given a collection of sub-boxes $B_1, \ldots, B_\ell \subset B$ with disjoint interiors, there is an induced map

$$(4.1) \qquad S^k = B/\partial B \to B/(B\backslash(\mathring{B}_1 \cup \cdots \cup \mathring{B}_\ell)) = \bigvee_{a=1}^{\ell} B_a/\partial B_a = \bigvee_{a=1}^{\ell} S^k \to S^k.$$

The last map is the identity on each summand, so that the composition has degree ℓ. As pointed out in [LLS], this construction is continuous in the position of the sub-boxes. We let $E(B, \ell)$ denote the space of boxes with disjoint interiors in B, and have a continuous map $E(B, \ell) \to \mathrm{Map}(S^k, S^k)$.

We can generalize the above procedure to associate a map of spheres to a map of sets $A \to Y$, as follows. Say we have chosen sub-boxes $B_a \subset B$ with disjoint interiors, for $a \in A$. Then we have a map:

$$(4.2) \qquad S^k = B/\partial B \to B/(B\backslash(\bigcup_{a\in A} \mathring{B}_a)) = \bigvee_{a\in A} B_a/\partial B_a = \bigvee_{a\in A} S^k \to \bigvee_{y\in Y} S^k$$

where the last map is built using the map of sets $A \to Y$.

More generally, we can also assign a box map to a correspondence of sets, as follows. Fix a correspondence A from X to Y with source map s and target map t. Say that we also have a collection of boxes B_x for $x \in X$. Finally, we also choose a collection of sub-boxes $B_a \subset B_{s(a)}$ with disjoint interiors, for $a \in A$. We then have an induced map

$$(4.3) \qquad \bigvee_{x\in X} S^k \to \bigvee_{y\in Y} S^k,$$

by applying, on B_x, the map associated to the set map $s^{-1}(x) \to Y$. A map as in Equation (4.3) is said to *refine* the correspondence A. Let $E(\{B_x\}, s)$ be the space of collections of labeled sub-boxes $\{B_a \subset B_{s(a)} \mid a \in A\}$ with disjoint interiors. Then, choosing a correspondence (A, s, t) (so that A and s are those appearing in the definition of $E(\{B_x\}, s)$—note that the definition of $E(\{B_x\}, s)$ does not involve the target map t)—Equation (4.3) gives a map $E(\{B_x\}, s) \to \mathrm{Map}(\vee_{x\in X} S^k, \vee_{y\in Y} S^k)$. We write

$$(4.4) \qquad \Phi(e, A) \in \mathrm{Map}(\bigvee_{x\in X} S^k \to \bigvee_{y\in Y} S^k)$$

for the map associated to $e \in E(\{B_x\}, s)$ and a compatible correspondence (A, s, t). The main point is that, for any box map $\Phi(e, A)$ refining A, the induced map on the kth homology agrees with the abelianization map

$$\mathcal{A}(A)\colon \mathcal{A}(X) = \widetilde{H}_k(\vee_{x\in X} S^k) \to \mathcal{A}(Y) = \widetilde{H}_k(\vee_{y\in Y} S^k).$$

We now indicate a further generalization of box maps to cover signed correspondences. Fix a signed correspondence (A, s, t, σ) from X to Y, and let B_x, $x \in X$ be some collection of boxes. Fix a collection of sub-boxes $B_a \subset B_{s(a)}$ for $a \in A$. There is an induced map just as in Equation (4.3), but whose construction depends on the sign map σ, as follows. For $x \in X$, we have a set map $s^{-1}(x) \to Y$, along with signs for each element of $s^{-1}(x)$. We modify the box map refining $s^{-1}(x) \to Y$ (without sign) by precomposing with $\mathfrak{r}$, reflection in the first coordinate, in boxes with

non-trivial sign:

$$S^k = B/\partial B \to B/(B\backslash(\bigcup_{a\in A} \mathring{B}_a)) = \bigvee_{a\in A} B_a/\partial B_a \xrightarrow{\vee \mathfrak{r}_a} \bigvee_{a\in A} B_a/\partial B_a = \bigvee_{a\in A} S^k \to \bigvee_{y\in Y} S^k.$$

Here $\mathfrak{r}_a = \mathfrak{r}$ if $\sigma(a) = -1$ and $\mathfrak{r}_a = \mathrm{Id}$ if $\sigma(a) = +1$. This defines the map on the factor on the left of Equation (4.3) indexed by the element $x \in X$. We say that a map constructed this way *refines* the signed correspondence (A, s, t, σ). As before, we can regard it as a map

$$\Phi(e, A) \in \mathrm{Map}(\bigvee_{x\in X} S^k, \bigvee_{y\in Y} S^k),$$

where $e \in E(\{B_x\}, s)$, and (A, s, t, σ) is a compatible signed correspondence. Once again, the induced map on the kth homology agrees with the abelianization map.

Similarly, for a signed correspondence (A, s, t, σ), we can consider box maps refining the (unsigned) correspondence, and then precompose by $\mathfrak{d}$, the reflection in the first two coordinates, if the sign is nontrivial. We call a map obtained this way a *doubly signed* refinement of the tuple (A, s, t, σ), and denote it $\widehat{\Phi}(e, A)$.

We next record two lemmas from [LLS] about the spaces $E(\{B_x\}, s)$ that we need later. For a map $s\colon A \to X$ define $E_{\mathrm{sym}}(\{B_x\}, s)$ to be the subspace of $E(\{B_x\}, s)$ in which we require the box B_a to lie symmetrically in $B_{s(a)}$ with respect to reflection in the first coordinate $\mathfrak{r}$, for all $a \in A$, and $E_{2\mathrm{sym}}(\{B_x\}, s)$ by requiring symmetry in the first two coordinates.

Lemma 4.5. *Consider $s\colon A \to X$. If the boxes are k-dimensional then $E(\{B_x\}, s)$ is $(k - 2)$-connected and $E_{\mathrm{sym}}(\{B_x\}, s)$ is $(k - 3)$-connected and $E_{2\mathrm{sym}}(\{B_x\}, s)$ is $(k - 4)$-connected.*

Proof. The first statement is [LLS, Lemma 2.29], and the second statement follows from the first since the space of symmetric k-dimensional boxes, $E_{\mathrm{sym}}^k(\{B_x\}, s)$, is homotopy equivalent to the space of $(k-1)$-dimensional boxes, $E^{k-1}(\{B_x\}, s)$, from which the third statement also follows. □

Lemma 4.6. *If $e \in E(\{B_x\}, s_A)$ is compatible with a signed correspondence A from X to Y, and $f \in E(\{B_y\}, s_B)$ is compatible with a signed correspondence B from Y to Z, then there is a unique $f \circ_{\sigma_A} e \in E(\{B_x\}, s_{B\circ A})$ compatible with $B \circ A$, depending only on e, f, and the sign map σ_A, so that $\Phi(f \circ e, B \circ A) = \Phi(f, B) \circ \Phi(e, A)$; we will sometimes drop the subscript σ_A (as we did just now). Similarly, there is a unique $f \,\widehat{\circ}_{\sigma_A}\, e \in E(\{B_x\}, s_{B\circ A})$ compatible with $B \circ A$, so that $\widehat{\Phi}(f \,\widehat{\circ}\, e, B \circ A) = \widehat{\Phi}(f, B) \circ \widehat{\Phi}(e, A)$. Both of these assignments $\circ, \widehat{\circ}\colon E(\{B_y\}, s_B) \times E(\{B_x\}, s_A) \to E(\{B_x\}, s_{B\circ A})$ are continuous and send $E_{\mathrm{sym}}(\{B_y\}, s_B) \times E_{\mathrm{sym}}(\{B_x\}, s_A)$ to $E_{\mathrm{sym}}(\{B_x\}, s_{B\circ A})$ (and similarly for $E_{2\mathrm{sym}}$). Moreover, $\circ$ agrees with $\widehat{\circ}$ when restricted to $E_{2\mathrm{sym}}(\{B_y\}, s_B) \times E_{2\mathrm{sym}}(\{B_x\}, s_A)$.*

Proof. This 'composition map' $E(\{B_y\}, s_B) \times E(\{B_x\}, s_A) \to E(\{B_x\}, s_{B\circ A})$, as discussed in [LLS, §2.10], is constructed in the unsigned as follows: Given sub-boxes $B_b \subset B_{s_B(b)}$ for all $b \in B$ and sub-boxes $B_a \subset B_{s_A(a)}$ for all $a \in A$, define, for all $(b, a) \in B \times_Y A$, the sub-box $B_{(b,a)} \subset B_{s_A(a)}$ to be the 'sub-box of the sub-box', $B_b \subset B_{s_B(b) = t_A(a)} \cong B_a \subset B_{s_A(a)}$.

In the signed case, one needs to precompose with a reflection if $\sigma_A(a) = -1$. That is, define $B_{(b,a)}$ to be sub-box $\mathfrak{r}_a(B_b) \subset B_{s_B(b)} \cong B_a$ of the sub-box $B_a \subset B_{s_A(a)}$, where as before, $\mathfrak{r}_a = \mathfrak{r}$ if $\sigma_A(a) = -1$ and $\mathfrak{r}_a = \mathrm{Id}$ if $\sigma_A(a) = +1$.

It is clear that $\Phi(f \circ e, B \circ A) = \Phi(f, B) \circ \Phi(e, A)$. The slight subtlety lies in the case when $(b, a) \in B \circ A$ has sign $+1$, but each of a and b has sign -1; then each of the box maps $\Phi(e, A)$ and $\Phi(f, B)$ requires reflecting along the first coordinate, and so their composition does not. The case for $\widehat{\Phi}$ is similar.

It is clear that both $\circ$ and $\widehat{\circ}$ send $E_{\text{sym}}(\{B_y\}, s_B) \times E_{\text{sym}}(\{B_x\}, s_A)$ to $E_{\text{sym}}(\{B_x\}, s_{B \circ A})$ and $E_{2\text{sym}}(\{B_y\}, s_B) \times E_{2\text{sym}}(\{B_x\}, s_A)$ to $E_{2\text{sym}}(\{B_x\}, s_{B \circ A})$. In both cases, the definition of $\circ$ does not require any reflections, while in the latter case, the definition of $\widehat{\circ}$ does not. So in the latter case, $\circ$ and $\widehat{\circ}$ agree. $\qquad\square$

4.2. **Homotopy coherence.** In this section, we briefly review homotopy colimits and homotopy coherent diagrams following [LLS, §2.9]. Let Top_* be the category of well-based topological spaces; we will usually work with finite CW complexes. A *weak equivalence* $X \to Y$ is a map that induces isomorphisms on all homotopy groups; typically our spaces are all simply connected, when the definition reduces to being isomorphisms on all homology groups. A homotopy equivalence is a special case of weak equivalence, and for CW complexes (the case at hand), the two notions are equivalent.

We will sometimes also work with spaces equipped with an action by a fixed finite group G (which is $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ in our case), and all maps are G-equivariant, forming a category $G\text{-Top}_*$. We also require that the inclusions of fixed points $X^H \to X^{H'}$, for all subgroups $H' < H$ of G, are cofibrations; in our case, all the spaces will carry CW structures so that the actions are CW actions—that is, each group element simply permutes the cells and respects the attaching maps. A map $X \to Y$ is called a *weak equivalence* if the induced map $X^H \to Y^H$ is a weak equivalence for all subgroups H of G. A homotopy equivalence in $G\text{-Top}_*$ induces a weak equivalence. For G-CW complexes (the case at hand), the two notions are equivalent by the G-Whitehead theorem, see [GM95, Theorem 2.4]. For G-CW complexes, a weak equivalence $X \to Y$ induces a weak equivalence between quotients of fixed points, $X^{H'}/X^H \to Y^{H'}/Y^H$, for all subgroups $H' < H$ of G, and between orbit spaces, $X/H \to Y/H$, for all subgroups H of G.

Now we recall the notion of a homotopy coherent diagram, which is the data from which a homotopy colimit is constructed. Fix a finite group G (typically 0 or $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$). A homotopy coherent diagram is intuitively a diagram $F \colon \mathscr{C} \to G\text{-Top}_*$ which is not commutative, but commutative up to homotopy, and the homotopies themselves commute up to higher homotopy, and so on, and for which all the homotopies and higher homotopies are viewed as part of the data of the diagram. Precisely, we have the following.

Definition 4.7 ([Vog73, Definition 2.3]). A homotopy coherent diagram $F \colon \mathscr{C} \to G\text{-Top}_*$ assigns to each $x \in \mathscr{C}$ a space $F(x) \in G\text{-Top}_*$, and for each $n \geq 1$ and each sequence

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$$

of composable morphisms in $\mathscr{C}$ a continuous G-map

$$F(f_n, \ldots, f_1) \colon [0, 1]^{n-1} \times F(x_0) \to F(x_n)$$

with $F(f_n, \ldots, f_1)([0,1]^{n-1} \times \{*\}) = *$. These maps are required to satisfy the following compatibility conditions:

$$F(f_n, \ldots, f_1)(t_1, \ldots, t_{n-1}) =$$

(4.8)
$$\begin{cases} F(f_n, \ldots, f_2)(t_2, \ldots, t_{n-1}), & f_1 = \mathrm{Id} \\ F(f_n, \ldots, \hat{f}_i, \ldots, f_1)(t_1, \ldots, t_{i-1} \cdot t_i, \ldots, t_{n-1}), & f_i = \mathrm{Id}, 1 < i < n \\ F(f_{n-1}, \ldots, f_1)(t_1, \ldots, t_{n-2}), & f_n = \mathrm{Id} \\ F(f_n, \ldots, f_{i+1})(t_{i+1}, \ldots, t_{n-1}) \circ F(f_i, \ldots, f_1)(t_1, \ldots, t_{i-1}), & t_i = 0 \\ F(f_n, \ldots, f_{i+1} \circ f_i, \ldots, f_1)(t_1, \ldots, \hat{t}_i, \ldots, t_{n-1}), & t_i = 1. \end{cases}$$

When $\mathscr{C}$ does not contain any non-identity isomorphisms, homotopy coherent diagrams may be defined only in terms of non-identity morphisms and the last two compatibility conditions.

Given a homotopy coherent diagram, we can define its *homotopy colimit* in G-Top$_*$, quite concretely, as follows:

Definition 4.9 ([Vog73, §5.10]). Given a homotopy coherent diagram $F \colon \mathscr{C} \to G$-Top$_*$ the *homotopy colimit* of F is defined by

$$(4.10) \qquad \mathrm{hocolim}\, F = \{*\} \amalg \coprod_{n \geq 0} \coprod_{x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n} [0,1]^n \times F(x_0)/\sim,$$

where the equivalence relation $\sim$ is given as follows:

$$(f_n, \ldots, f_1; t_1, \ldots, t_n; p) \sim \begin{cases} (f_n, \ldots, f_2; t_2, \ldots, t_n; p), & f_1 = \mathrm{Id} \\ (f_n, \ldots, \hat{f}_i, \ldots, f_1; t_1, \ldots, t_{i-1} \cdot t_i, \ldots, t_n; p), & f_i = \mathrm{Id}, i > 1 \\ (f_n, \ldots, f_{i+1}; t_{i+1}, \ldots, t_n; F(f_i, \ldots, f_1)(t_1, \ldots, t_{i-1}, p)), & t_i = 0 \\ (f_n, \ldots, f_{i+1} \circ f_i, \ldots, f_1; t_1, \ldots, \hat{t}_i, \ldots, t_n; p), & t_i = 1, i < n \\ (f_{n-1}, \ldots, f_1; t_1, \ldots, t_{n-1}; p), & t_n = 1 \\ *, & p = *. \end{cases}$$

When $\mathscr{C}$ does not contain any non-identity isomorphisms, homotopy colimits may be defined only in terms of non-identity morphisms and the last four equivalence relations.

We will need the following properties, most of which are immediate consequences of the above formulas:

(ho-1) Suppose that $F_0, F_1 \colon \mathscr{C} \to G$-Top$_*$ are homotopy coherent diagrams and $\eta \colon F_1 \to F_0$ is a natural transformation, that is, a homotopy coherent diagram

$$\eta \colon \underline{2} \times \mathscr{C} \to \mathrm{Top}_*$$

with $\eta|_{\{i\} \times \mathscr{C}} = F_i$, $i = 0, 1$. Then η induces a map $\mathrm{hocolim}\, \eta \colon \mathrm{hocolim}\, F_1 \to \mathrm{hocolim}\, F_0$. If $\eta(x)$ is a weak equivalence for each $x \in \mathscr{C}$—we will call such an η a weak equivalence $F_1 \to F_0$—then $\mathrm{hocolim}\, \eta$ is a weak equivalence as well.

When the spaces involved are G-CW complexes (the case at hand), a weak equivalence $\eta\colon F_1 \to F_0$ is also a homotopy equivalence [Vog73, Proposition 4.6], that is, there exists $\zeta, \zeta'\colon F_0 \to F_1$ and
$$\mathfrak{h}, \mathfrak{h}'\colon \{2 \to 1 \to 0\} \times \mathscr{C} \to G\text{-Top}_*,$$
with $\mathfrak{h}|_{\{2\to1\}\times\mathscr{C}} = \eta$, $\mathfrak{h}|_{\{1\to0\}\times\mathscr{C}} = \zeta$, $\mathfrak{h}|_{\{2\to0\}\times\mathscr{C}} = \mathrm{Id}_{F_0}$, and $\mathfrak{h}'|_{\{2\to1\}\times\mathscr{C}} = \zeta'$, $\mathfrak{h}'|_{\{1\to0\}\times\mathscr{C}} = \eta$, $\mathfrak{h}'|_{\{2\to0\}\times\mathscr{C}} = \mathrm{Id}_{F_1}$.

(ho-2) Suppose that $F_0, F_1\colon \mathscr{C} \to G\text{-Top}_*$ are diagrams and that $F_0 \vee F_1\colon \mathscr{C} \to G\text{-Top}_*$ is the diagram obtained by wedge sum: $(F_0 \vee F_1)(x) = F_0(x) \vee F_1(x)$ for all $x \in \mathscr{C}$, and
$$(F_0 \vee F_1)(f_n, \ldots, f_1)(t_1, \ldots, t_{n-1}, p) = F_i(f_n, \ldots, f_1)(t_1, \ldots, t_{n-1}, p)$$
for all $i = 0, 1$, $x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$, and $p \in F_i(x_0)$. Then $\mathrm{hocolim}(F_0 \vee F_1)$ and $(\mathrm{hocolim}F_0) \vee (\mathrm{hocolim}F_1)$ are naturally homeomorphic.

(ho-3) For any normal subgroup H of G, define the fixed point diagram $F^H\colon \mathscr{C} \to G/H\text{-Top}_*$ by setting $F^H(x)$ to be the fixed points $F(x)^H$. Define the quotient diagram $F/F^H\colon \mathscr{C} \to G\text{-Top}_*$ by setting $(F/F^H)(x)$ to be the quotient $F(x)/\{p \sim *$ for all $p \in F^H(x)\}$. Then there are natural homeomorphisms

$$
\begin{array}{ccccc}
\mathrm{hocolim}(F)^H & \longrightarrow & \mathrm{hocolim}F & \longrightarrow & \mathrm{hocolim}(F)/\mathrm{hocolim}(F)^H \\
\downarrow{\cong} & & \Vert & & \downarrow{\cong} \\
\mathrm{hocolim}(F^H) & \longrightarrow & \mathrm{hocolim}F & \longrightarrow & \mathrm{hocolim}(F/F^H)
\end{array}
$$

with the arrows on the bottom row being induced from the natural transformations $F^H \to F \to F/F^H$.

For diagrams of G-CW complexes, the arrows on the top row form a cofibration sequence since $(\mathrm{hocolim}F, \mathrm{hocolim}(F)^H)$ form a CW-pair; moreover, a weak equivalence $F_1 \to F_0$ induces weak equivalences $F_1^H \to F_0^H$ and $F_1/F_1^H \to F_0/F_0^H$ by the G-Whitehead theorem.

(ho-4) For any normal subgroup H of G, let $F/H\colon \mathscr{C} \to G/H\text{-Top}_*$ be the orbit diagram, obtained by setting $(F/H)(x)$ to be the orbit $F(x)/\{p \sim h(p)$ for all $p \in F(x), h \in H\}$. Then $\mathrm{hocolim}(F)/H$ and $\mathrm{hocolim}(F/H)$ are naturally homeomorphic, and the map $\mathrm{hocolim}F \to \mathrm{hocolim}(F/H)$ induced from the natural transformation $F \to F/H$ is identified with the quotient map $\mathrm{hocolim}F \to \mathrm{hocolim}(F)/H$.

For diagrams of G-CW complexes, a weak equivalence $F_1 \to F_0$ induces a weak equivalence $F_1/H \to F_0/H$ by the G-Whitehead theorem.

4.3. **Little boxes refinement.** With this background, we are ready to review the little box realization construction of [LLS, §5] and generalize for the other kinds of Burnside functors introduced here, i.e., functors to $\mathscr{B}_\bullet$ for $\bullet \in \{\varnothing, \sigma, \xi\}$.

Definition 4.11. Fix a small category $\mathscr{C}$ and a strictly unitary 2-functor $F\colon \mathscr{C} \to \mathscr{B}_\bullet$. A k-dimensional *spatial refinement* of F is a homotopy coherent diagram $\widetilde{F}_k\colon \mathscr{C} \to \mathrm{Top}_*$ such that

(1) For any $u \in \mathscr{C}$, $\widetilde{F}_k(u) = \vee_{x \in F(u)} S^k$, where $S^k = [0,1]^k/\partial$. When $\bullet = \xi$, this space has an additional $\mathbb{Z}_2$-action—denoted ξ—induced from the $\mathbb{Z}_2$-action on the set $F(u)$.

(2) For any sequence of morphisms $u_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} u_n$ in $\mathscr{C}$ and any $(t_1, \ldots, t_{n-1}) \in [0,1]^{n-1}$ the map

$$\widetilde{F}_k(f_n, \ldots, f_1)(t_1, \ldots, t_{n-1}): \bigvee_{x \in F(u_0)} S^k \to \bigvee_{x \in F(u_n)} S^k$$

is a box map refining the (possibly signed) correspondence $F(f_n \circ \cdots \circ f_1)$, which is naturally isomorphic to $F(f_n) \times_{F(u_{n-1})} \cdots \times_{F(u_1)} F(f_1)$; when $\bullet = \xi$, we require each $\widetilde{F}_k$ to be ξ-equivariant.

Note that for $\bullet = \xi$, the ξ-action is a CW action, with the CW complex structure given by the boxes.

This definition reduces to [LLS, Definition 5.1] when $\bullet = \varnothing$. The additions here are the signed box map refinements introduced in §4.1, which are needed for functors to $\mathscr{B}_\sigma$ with non-trivial signs in the correspondences, and the equivariant conditions.

The following is the main technical result in this section. As usual, $\bullet \in \{\varnothing, \sigma, \xi\}$ in the statement. For $\bullet = \varnothing$, the ordinary Burnside category, this reduces to [LLS, Proposition 5.2].

Proposition 4.12. *Let $\mathscr{C}$ be a small category in which every sequence of composable non-identity morphisms has length at most n, and let $F: \mathscr{C} \to \mathscr{B}_\bullet$ be a strictly unitary 2-functor.*

(1) If $k \geq n$, there is a k-dimensional spatial refinement of F (which is ξ-equivariant if $\bullet = \xi$).
(2) If $k \geq n + 1$, then any two k-dimensional spatial refinements of F are weakly equivalent (ξ-equivariantly if $\bullet = \xi$).
(3) If $\widetilde{F}_k$ is a k-dimensional spatial refinement of F, then the result of suspending each $\widetilde{F}_k(u)$ and $\widetilde{F}_k(f_n, \ldots, f_1)$ gives a $(k+1)$-dimensional spatial refinement of F.

Proof. The proof is parallel to that of [LLS, Proposition 5.2]. For all values of $\bullet$, the third point is clear.

The case $\bullet = \sigma$ is an immediate generalization of the case $\bullet = \varnothing$ as worked out in [LLS]. We sketch the main details. For point (1), given $u \in \mathscr{C}$, set $\widetilde{F}_k(u) = \vee_{x \in F(u)} S^k$, where the x-summand is B_x/∂. We choose for each $f: u \to v$ in $\mathscr{C}$ a signed box map $\widetilde{F}_k(f)$ refining the signed correspondence $F(f)$; for the identity morphism Id_u, choose $\widetilde{F}_k(\mathrm{Id}_u)$ to be the identity map $\mathrm{Id}_{\widetilde{F}_k(u)}$, which indeed is a box map refining the identity correspondence $\mathrm{Id}_{F(u)} = F(\mathrm{Id}_u)$. Let $e_f \in E(\{B_x \mid x \in F(u)\}, s_{F(f)})$ be the collection of little boxes corresponding to $F(f)$.

This gives a definition of $\widetilde{F}_k$ on vertices and arrows. We now need to define the appropriate coherences among these maps, which will be done inductively. Assume for all sequences $v_0 \xrightarrow{f_1} \cdots \xrightarrow{f_m} v_m$ with $m \leq \ell$, we have defined continuous maps

$$e_{f_m, \ldots, f_1}: [0,1]^{m-1} \to E(\{B_x \mid x \in F(v_0)\}, s_{F(f_m \circ \cdots \circ f_1)}),$$

and that these maps satisfy Equation (4.8) (with the composition map from Lemma 4.6 playing the role $\circ$). Then for the induction step, given $v_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{\ell+1}} v_{\ell+1}$, we have a continuous map

$$S^{\ell-1} = \partial([0,1]^\ell) \to E(\{B_x \mid x \in F(v_0)\}, s_{F(f_{\ell+1} \circ \cdots \circ f_1)})$$

defined by

$$(4.13) \quad \begin{aligned} (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_\ell) &\mapsto e_{f_{\ell+1}, \ldots, f_{i+1}}(t_{i+1}, \ldots, t_\ell) \circ e_{f_i, \ldots, f_1}(t_1, \ldots, t_{i-1}) \\ (t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_\ell) &\mapsto e_{f_{\ell+1}, \ldots, f_{i+1} \circ f_i, \ldots, f_1}(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_\ell). \end{aligned}$$

By Lemma 4.5, this map extends to a map, call it $e_{f_{\ell+1}, \ldots, f_1}$, from $[0,1]^\ell$. By definition, the maps

$$\Phi(e_{f_m, \ldots, f_1}) \colon [0,1]^{m-1} \times \bigvee_{x \in F(v_0)} S^k \to \bigvee_{x \in F(v_m)} S^k$$

assemble to form a homotopy coherent diagram.

Next, we address point (2). Say that we have spatial refinements $\widetilde{F}_k^0$ and $\widetilde{F}_k^1$ of F. Consider the functor $G \colon \underline{2} \times \mathscr{C} \to \mathscr{B}_\bullet$ defined as the composition $\underline{2} \times \mathscr{C} \xrightarrow{\pi_2} \mathscr{C} \xrightarrow{F} \mathscr{B}_\bullet$. We need only define a spatial refinement $\widetilde{G}_k$ of G that restricts to $\widetilde{F}_k^i$ at $\{i\} \times \mathscr{C}$ for $i = 0, 1$. The construction of $\widetilde{G}_k$ proceeds uneventfully by induction as before. Then for each $u \in \mathscr{C}$, $G(\phi_{1,0} \times \mathrm{Id}_u)$ refines the identity correspondence $\mathrm{Id}_{F(u)}$ (and indeed, may be chosen to be the identity map), and hence is a weak equivalence; therefore, by (ho-1), $\widetilde{G}_k \colon \widetilde{F}_k^1 \to \widetilde{F}_k^0$ is a weak equivalence as well.

Next, we consider the case $\bullet = \xi$. Construct a functor $G \colon \mathscr{C} \to \mathscr{B}_\sigma$ so that $\mathcal{D} \circ G$ is naturally isomorphic to F by choosing a section of the $\mathbb{Z}/2$-quotient map $\mathrm{II}_{u \in \mathscr{C}} F(u) \to \mathrm{II}_{u \in \mathscr{C}} F(u)/\mathbb{Z}_2$; we leave the details to the reader. Then construct a box map refinement of G; let B_x be the box assigned to x, for $x \in \mathrm{II}_{u \in \mathscr{C}} G(u)$, and let $e_{f_m, \ldots, f_1} \colon [0,1]^{m-1} \to E(\{B_x \mid x \in G(v_0), s_{G(f_m \circ \cdots \circ f_1)}\})$ denote the $[0,1]^{m-1}$-parameter family chosen for the sequence $v_0 \xrightarrow{f_1} \cdots \xrightarrow{f_m} v_m$ during the construction.

We may then define $B_{\{1\} \times x} = \{1\} \times B_x$ and $B_{\{\xi\} \times x} = \{\xi\} \times B_x$, for $x \in \mathrm{II}_{u \in \mathscr{C}} G(u)$. For any $v_0 \xrightarrow{f_1} \cdots \xrightarrow{f_m} v_m$ in $\mathscr{C}$, and any $(t_1, \ldots, t_{m-1}) \in [0,1]^{m-1}$, the configuration of disjoint boxes $e_{f_m, \ldots, f_1}(t_1, \ldots, t_{m-1}) \in E(\{B_x \mid x \in G(v_0), s_{G(f_m \circ \cdots \circ f_1)}\})$ refining the signed correspondence $G(f_m \circ \cdots \circ f_1)$ can be doubled to get a configuration of disjoint boxes

$$d_{f_m, \ldots, f_1}(t_1, \ldots, t_{m-1}) \in E(\{B_y \mid y \in F(v_0) = \{1, \xi\} \times G(v_0), s_{F(f_m \circ \cdots \circ f_1)} = \mathrm{Id}_{\{1, \xi\}} \times s_{G(f_m, \circ \cdots \circ f_1)}\})$$

refining the unsigned correspondence $F(f_m \circ \cdots \circ f_1) = \{1, \xi\} \times G(f_m \circ \cdots \circ f_1)$. Use these $d_{f_m, \ldots, f_1}$'s to construct the refinement, which is automatically ξ-equivariant. $\square$

4.4. Realization of cube-shaped diagrams.

Finally in this section we will discuss how to construct a CW complex $\|F\|_k$, and then a CW spectrum $|F|$, from a given diagram $F \colon \underline{2}^n \to \mathscr{B}_\bullet$. Let $\underline{2}_+$ be the category with objects $\{0, 1, *\}$ and unique non-identity morphisms $1 \to 0$ and $1 \to *$, and let $\underline{2}_+^n = (\underline{2}_+)^n$.

Let $\widetilde{F}_k \colon \underline{2}^n \to \mathrm{Top}_*$ be a spatial refinement of F using k-dimensional boxes, and let $\widetilde{F}_k^+ \colon \underline{2}_+^n \to \mathrm{Top}_*$ be the diagram obtained from $\widetilde{F}_k$ by setting $\widetilde{F}_k^+(x)$ to be a point for all $x \in \underline{2}_+^n \setminus \underline{2}^n$. Let $\|F\|_k$ be the homotopy colimit of $\widetilde{F}_k^+$. We call $\|F\|_k$ a *realization* of $F \colon \underline{2}^n \to \mathscr{B}_\bullet$ for $\bullet = \{\varnothing, \sigma, \xi\}$.

Corollary 4.14. *If $k \geq n+1$, then $\|F\|_k$ is well-defined up to weak equivalence in Top_* (or $\mathbb{Z}_2\text{-}\mathrm{Top}_*$ if $\bullet = \xi$). In each case, $\|F\|_{k+1} = \Sigma\|F\|_k$.*

Proof. As in [LLS, Corollary 5.6], this follows from Proposition 4.12 and properties of homotopy colimits (ho-1). $\square$

The homotopy colimit $\|F\|_k$ may be given several CW structures. First, from Definition 4.9, there is the *standard* CW structure, with cells $[0,1]^m \times B_x$, parameterized by tuples $(f_m, \ldots, f_1)$ subject to some relations.

We have a second CW structure on $\|F\|_k$, the *fine* structure, which is obtained from the standard structure by subdividing each cell $[0,1]^m \times B_x$ along the central $(m + k - 1)$-dimensional box $[0,1]^m \times B_x^{\mathfrak{r}}$, where $B_x^{\mathfrak{r}} \subset B_x$ is the fixed-point set of the reflection $\mathfrak{r} \colon B_x \to B_x$ along the first coordinate. The fine CW complex structure will be of relevance in §4.5.

There is also the *coarse* cell structure of [LLS, Section 6]. There they construct a CW structure on $\|F\|_k$ for F an unsigned Burnside functor, with cells formed from unions of standard cells, so that there is exactly one (non-basepoint) cell $\mathcal{C}(x)$ for each $x \in \amalg_u F(u)$. In more detail, if F_x denotes the Burnside sub-functor of F generated by x, then the subcomplex $\|F_x\|_k$ of $\|F\|_k$ is the image of the cell $\mathcal{C}(x)$. The construction generalizes without changes to give a CW structure on $\|F\|_k$ for $F \colon \underline{2}^n \to \mathscr{B}_\bullet$, with the same set of cells; when $\bullet = \xi$, this produces a $\mathbb{Z}_2$-CW complex. Unless otherwise specified, this is the default CW complex structure that we consider on $\|F\|_k$.

Lemma 4.15. *A cofibration sequence $G \to F \to H$ of functors $\underline{2}^n \to \mathscr{B}_\bullet$ (cf. Definition 3.4), upon realization, induces a cofibration sequence of spaces. In general, any natural transformation $\eta \colon F_1 \to F_0$ of Burnside functors $\underline{2}^n \to \mathscr{B}_\bullet$ induces a map on the realizations.*

Proof. Consider the standard CW complex structures. For the first statement, a spatial refinement $\widetilde{F}_k$ of F produces spatial refinements $\widetilde{G}_k$ of G and $\widetilde{H}_k$ of H; working with those refinements, it is an immediate consequence of the definitions that $\|G\|_k$ is a CW subcomplex of $\|F\|_k$ with quotient complex $\|H\|_k$.

For the second part, if $\eta \colon \underline{2}^{n+1} \to \mathscr{B}_\bullet$ is the natural transformation, then $(F_0)_{\iota_0}$ is a subfunctor and $(F_1)_{\iota_1}$ is the corresponding quotient functor, where $\iota_i \colon \underline{2}^n \to \underline{2}^{n+1}$ is the face inclusion to $\{i\} \times \underline{2}^n$. Therefore, we get a cofibration sequence

$$\|(F_0)_{\iota_0}\|_k \to \|\eta\|_k \to \|(F_1)_{\iota_1}\|_k.$$

However, $\|(F_0)_{\iota_0}\|_k = \|F_0\|_k$, while $\|(F_1)_{\iota_1}\|_k = \Sigma\|F_1\|_k$ since $\|F_1\|$ is constructed as a hocolim over $\underline{2}_+^n$, while $\|(F_1)_{\iota_1}\|$ is constructed as a hocolim over $\underline{2}_+^{n+1}$. Therefore, the Puppe map

$$\|(F_1)_{\iota_1}\|_k = \Sigma\|F_1\|_k = \|F_1\|_{k+1} \to \|(F_0)_{\iota_0}\|_k = \Sigma\|F_0\|_k = \|F_0\|_{k+1}$$

is the required map. $\square$

Proposition 4.16. *If $F \colon \underline{2}^n \to \mathscr{B}_\bullet$, then its shifted reduced cellular complex $\widetilde{C}_{\mathrm{cell}}(\|F\|_k)[-k]$ is isomorphic to the totalization $\mathrm{Tot}(F)$ with the cells mapping to the corresponding generators. If $\eta \colon F_1 \to F_0$ is a natural transformation of Burnside functors, then the map $\|F_1\|_k \to \|F_0\|_k$ is cellular, and the induced cellular chain map agrees with $\mathrm{Tot}(\eta)$.*

Proof. The first statement is an immediate generalization of the corresponding statement for unsigned Burnside functors from [LLS, Theorem 6]. The second statement is also clear from the form of the map constructed in Lemma 4.15, using similar arguments. $\square$

We can then package all these spaces together to construct a *finite CW spectrum*, by which we mean a pair (X, r) (sometimes written $\Sigma^r X$), where X is a finite CW complex and $r \in \mathbb{Z}$; one may view it as an object in the Spanier-Whitehead category, or as $\Sigma^r(\Sigma^\infty X)$, the rth suspension of the suspension spectrum of the finite CW complex X. One can take the reduced cellular chain complex of a finite CW spectrum, whose chain homotopy type is an invariant of the (stable) homotopy type of the CW spectrum. Then, for a stable Burnside functor $(F \colon \underline{2}^n \to \mathscr{B}_\bullet, r)$, after fixing a k-dimensional spatial refinement $\widetilde{F}_k$, we may define its *realization* as the finite CW spectrum $|\Sigma^r F| = (\|F\|_k, r - k)$.

Lemma 4.17. *Let $\Sigma^{r_1} F_1 \to \Sigma^{r_2} F_2$ be a map of Burnside functors F_1, F_2. Then there is an induced map of realizations*

$$|\Sigma^{r_1} F_1| \to |\Sigma^{r_2} F_2|$$

well-defined up to homotopy equivalence. If the map of Burnside functors is a stable equivalence, then the induced map is a homotopy equivalence. If $\bullet = \xi$, the induced map and the homotopy equivalence may be taken $\mathbb{Z}_2$-equivariant.

Proof. First, associated to a natural transformation of Burnside functors, there is a well-defined space map by Lemma 4.15. Since all maps of Burnside functors are obtained as compositions of natural transformations and stable equivalences, we need only show that there is a well-defined (up to homotopy) homotopy equivalence of the realizations associated to a stable equivalence.

For this, we must check that maps as in the items of Definition 3.6 preserve the stable homotopy type of $\|F\|_k$. For a natural transformation $\eta \colon F_i \to F_{i+1}$, Proposition 4.16 produces a cellular map $\|F_i\|_k \to \|F_{i+1}\|_k$, which induces the map $\text{Tot}(\eta)$ on the cellular chain complex. The condition that $\text{Tot}(\eta)$ is a chain homotopy equivalence implies that the map of spaces is a homotopy equivalence by Whitehead's theorem (we assume k is sufficiently large so that all relevant spaces are simply connected), so has a homotopy inverse well-defined up to homotopy. When $\bullet = \xi$, the $\mathbb{Z}_2$-action is free (away from the basepoint), so by the G-Whitehead theorem, the homotopy equivalences may be taken equivariant. $\square$

4.5. **Equivariant constructions.** We now explain how to make the constructions of the previous sections equivariant. Recall that each $S^k = [0,1]^k/\partial$ carries a natural $\mathbb{Z}_2$-action by $\mathfrak{r}$—reflection in the first coordinate, as well as a $\mathbb{Z}_2$-action by $\mathfrak{d}$—composition of the reflections in the first two coordinates (that is, a $180°$ rotation in the first coordinate plane).

Definition 4.18. Fix a small category $\mathscr{C}$ and a strictly unitary 2-functor $F \colon \mathscr{C} \to \mathscr{B}_\bullet$. A *reflection-equivariant or $\mathfrak{r}$-equivariant k-dimensional spatial refinement* of F is a $\mathbb{Z}_2$-equivariant k-dimensional spatial refinement $\widetilde{F}_k \colon \mathscr{C} \to \text{Top}_*$ such that for any sequence of morphisms $u_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} u_n$ in $\mathscr{C}$ and any $(t_1, \ldots, t_{n-1}) \in [0,1]^{n-1}$ the map

$$\widetilde{F}_k(f_n, \ldots, f_1)(t_1, \ldots, t_{n-1}) \colon \bigvee_{x \in F(u_0)} S^k \to \bigvee_{x \in F(u_n)} S^k$$

is $\mathfrak{r}$-equivariant. When $\bullet = \xi$, we require each $\widetilde{F}_k$ to be ξ-equivariant as well; in this case, $\widetilde{F}_k$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant diagram, with the first factor (denoted $\mathbb{Z}_2^+$) acting by ξ and the second factor (denoted $\mathbb{Z}_2^-$) acting by $\mathfrak{r} \circ \xi$.

Definition 4.19. Fix a small category $\mathscr{C}$ and a strictly unitary 2-functor $F\colon \mathscr{C} \to \mathscr{B}_\sigma$. A *doubly-equivariant k-dimensional spatial refinement* of F is a $\mathbb{Z}_2$-equivariant k-dimensional *doubly signed* spatial refinement $\widehat{F}_k\colon \mathscr{C} \to \mathrm{Top}_*$ such that for any sequence of morphisms $u_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} u_n$ in $\mathscr{C}$ and any $(t_1, \ldots, t_{n-1}) \in [0,1]^{n-1}$ the map

$$\widehat{F}_k(f_n, \ldots, f_1)(t_1, \ldots, t_{n-1})\colon \bigvee_{x \in F(u_0)} S^k \to \bigvee_{x \in F(u_n)} S^k$$

is equivariant with respect to reflections in the first two coordinates; the $\mathbb{Z}_2$-action is given by $\mathfrak{r}$ (there is also a second $\mathbb{Z}_2$-action by reflection in the second coordinate, which we will ignore). Note that since we require the box map to be a *doubly signed* refinement of $F(f_n \circ \cdots \circ f_1)$—defined using $\widehat{\Phi}$ instead of Φ from §4.1—a doubly-equivariant spatial refinement is *not* a spatial refinement of F in the usual sense. We will always denote doubly equivariant spatial refinements by $\widehat{F}_k$ to distinguish them from $\mathfrak{r}$-equivariant spatial refinements $\widetilde{F}_k$, to which they are not homotopy equivalent (even nonequivariantly).

We next record the equivariant version of Proposition 4.12:

Proposition 4.20. *Let $\mathscr{C}$ be a small category in which every sequence of composable non-identity morphisms has length at most n, and let $F\colon \mathscr{C} \to \mathscr{B}_\bullet$ be a strictly unitary 2-functor.*

(1) *If $k \geq n+1$, the refinement of Proposition 4.12 may also be constructed $\mathfrak{r}$-equivariantly, while if $k \geq n+2$, it may be constructed doubly equivariantly.*

(2) *If $k \geq n+2$, the weak equivalence from Proposition 4.12(2) may be constructed $\mathfrak{r}$-equivariantly, while if $k \geq n+3$, it may be constructed doubly equivariantly.*

Moreover, if $\bullet = \xi$, there is a ξ- and $\mathfrak{r}$-equivariant refinement of F for $k \geq n+1$, and the weak equivalence from Proposition 4.12(2) may be constructed ξ- and $\mathfrak{r}$-equivariantly for $k \geq n+2$.

Proof. The proof is essentially identical to the proof of Proposition 4.12. To make each $\widetilde{F}_k(f_n, \ldots, f_1)$ $\mathfrak{r}$- or doubly-equivariant, simply stipulate that each map $e_{f_m, \ldots, f_1}$ has image contained in $E_{\mathrm{sym}}(\{B_x \mid x \in F(v_0)\}, s_{F(f_m \circ \cdots \circ f_1)})$ or $E_{2\mathrm{sym}}(\{B_x \mid x \in F(v_0)\}, s_{F(f_m \circ \cdots \circ f_1)})$. $\qquad\square$

Proposition 4.21. *If $\widetilde{F}$ is an $\mathfrak{r}$-equivariant k-dimensional spatial refinement of $F\colon \mathscr{C} \to \mathscr{B}_\sigma$, then the following hold:*

(1) *The fixed point diagram $\widetilde{F}^{\mathbb{Z}_2}$ is a $(k-1)$-dimensional refinement of $\mathcal{F} \circ F\colon \mathscr{C} \to \mathscr{B}$.*

(2) *The orbit diagram $(\widetilde{F}/\widetilde{F}^{\mathbb{Z}_2})/\mathbb{Z}_2$ is a k-dimensional refinement of $\mathcal{F} \circ F\colon \mathscr{C} \to \mathscr{B}$.*

(3) *The quotient diagram $\widetilde{F}/\widetilde{F}^{\mathbb{Z}_2}$ is a k-dimensional refinement of $\mathcal{D} \circ F\colon \mathscr{C} \to \mathscr{B}_\xi$, with the $\mathfrak{r}$-action on $\widetilde{F}$ inducing the $\mathbb{Z}_2^+$-action on $\widetilde{F}/\widetilde{F}^{\mathbb{Z}_2}$.*

If $\widehat{F}$ is a doubly-equivariant k-dimensional spatial refinement of $F\colon \mathscr{C} \to \mathscr{B}_\sigma$, then the following hold:

(1) *The fixed point diagram $\widehat{F}^{\mathbb{Z}_2}$ is a $(k-1)$-dimensional refinement of $F\colon \mathscr{C} \to \mathscr{B}_\sigma$.*

(2) *The orbit diagram $(\widehat{F}/\widehat{F}^{\mathbb{Z}_2})/\mathbb{Z}_2$ is a k-dimensional refinement of $F\colon \mathscr{C} \to \mathscr{B}_\sigma$.*

(3) The quotient diagram $\widehat{F}/\widehat{F}^{\mathbb{Z}_2}$ is a k-dimensional refinement of $\mathcal{D} \circ F \colon \mathscr{C} \to \mathscr{B}_\xi$, with the $\mathfrak{r}$-action on $\widetilde{F}$ inducing the $\mathbb{Z}_2^-$-action on $\widetilde{F}/\widetilde{F}^{\mathbb{Z}_2}$.

If $\widetilde{F}$ is a k-dimensional $\mathfrak{r}$-equivariant spatial refinement of $F \colon \mathscr{C} \to \mathscr{B}_\xi$, the following hold:

(4) The orbit diagram $\widetilde{F}/\mathbb{Z}_2^+$ is a k-dimensional spatial refinement of $\mathcal{Q} \circ F \colon \mathscr{C} \to \mathscr{B}$.
(5) Say $F = \mathcal{D} \circ G$, and $\widetilde{\widetilde{F}}$ is ξ-invariant. Then the orbit diagram $\widetilde{F}/\mathbb{Z}_2^-$ is a $\mathfrak{r}$-equivariant k-dimensional spatial refinement of $G \colon \mathscr{C} \to \mathscr{B}_\sigma$ with the $\mathbb{Z}_2^+$-action on $\widetilde{F}$ inducing the $\mathfrak{r}$-action on $\widetilde{F}/\mathbb{Z}_2^-$.

Proof. This is an immediate consequence of the definitions. $\qquad\square$

As in §4.4, given any such equivariant spatial refinement $\widetilde{F}_k$ for $F \colon \underline{2}^n \to \mathscr{B}_\bullet$, we construct a G-equivariant CW complex $\|F\|_k$ as $\mathrm{hocolim}(\widetilde{F}_k^+)$, where $G = \mathbb{Z}_2$ if $\bullet = \sigma$ and $G = \mathbb{Z}_2^+ \times \mathbb{Z}_2^-$ if $\bullet = \xi$. By a *G-equivariant CW complex*, we mean a G-space carrying two different CW structures, so that it is a G-CW complex with respect to the first structure, and we use the second structure to define its cellular chain complex; equivalently, it is a pair of a G-CW complex and a CW complex, along with a homeomorphism connecting the two, and its cellular chain complex is defined to be cellular chain complex of the second CW complex (and so, does not automatically carry a G-action). In our construction, the fine structure from §4.4 is the first CW structure, while the standard structure is the second one. A caveat: our definition of G-equivariant CW complex is quite non-standard.

As before, for any stable functor (F, r), after fixing an equivariant spatial refinement $\widetilde{F}_k$, we let $|\Sigma^r F| = (\|F\|_k, r - k)$ denote the corresponding G-equivariant finite CW spectrum.

Similarly, if $\widehat{F}_k$ is a k-dimensional doubly-equivariant spatial refinement of $F \colon \underline{2}^n \to \mathscr{B}_\sigma$, we let $\widehat{\|F\|}_k$ denote the $\mathbb{Z}_2$-equivariant CW complex $\mathrm{hocolim}(\widehat{F}_k^+)$. For any stable functor (F, r), we let $\widehat{|\Sigma^r F|}$ denote the corresponding finite $\mathbb{Z}_2$-equivariant CW spectrum. We call $\widehat{|\Sigma^r F|}$ the *doubly-equivariant realization* of (F, r), to avoid confusion with the ordinary realization of (F, r).

Corollary 4.22. *Let $F \colon \underline{2}^n \to \mathscr{B}_\bullet$.*

(1) *If $\bullet = \sigma$, then $|F|^{\mathbb{Z}_2} = \Sigma^{-1}|\mathcal{F} \circ F|$, $|F|/|F|^{\mathbb{Z}_2} = |\mathcal{D} \circ F|$ (with the $\mathbb{Z}_2$-action on the left equaling the $\mathbb{Z}_2^+$-action on the right), and $(|F|/|F|^{\mathbb{Z}_2})/\mathbb{Z}_2 = |\mathcal{F} \circ F|$.*
(2) *If $\bullet = \xi$, then $|F|/\mathbb{Z}_2^+ = |\mathcal{Q} \circ F|$; if $F = \mathcal{D} \circ G$, then $|F|/\mathbb{Z}_2^- = |G|$ (with the $\mathbb{Z}_2^+$-action on the left equaling the $\mathbb{Z}_2$-action on the right).*
(3) *If $\bullet = \sigma$, then $\widehat{|F|}^{\mathbb{Z}_2} = \Sigma^{-1}|F|$, $\widehat{|F|}/\widehat{|F|}^{\mathbb{Z}_2} = |\mathcal{D} \circ F|$ (with the $\mathbb{Z}_2$-action on the left equaling the $\mathbb{Z}_2^-$-action on the right), and $(\widehat{|F|}/\widehat{|F|}^{\mathbb{Z}_2})/\mathbb{Z}_2 = |F|$.*

Proof. This follows from Proposition 4.20 and the properties of homotopy colimits (ho-3)—(ho-4). $\qquad\square$

Proposition 4.23. *If $F \colon \underline{2}^n \to \mathscr{B}_\xi$, then the reduced cellular complex of its realization, $\widetilde{C}_{\mathrm{cell}}(|F|)$, is isomorphic, as a $\mathbb{Z}_u$-module, to the totalization $\mathrm{Tot}(F)$ with the cells mapping to the corresponding generators. If $F \colon \underline{2}^n \to \mathscr{B}_\sigma$, then the reduced cellular complex of its doubly-equivariant realization,*

$\widetilde{C}_{\mathrm{cell}}(|\widehat{F}|)$ *is isomorphic to the totalization* $\mathrm{Tot}(\mathcal{F} \circ F)$ *with the cells mapping to the corresponding generators.*

Proof. The statement is, nonequivariantly, just Proposition 4.16. The isomorphism as $\mathbb{Z}_u$-modules follows from inspection of the proof of [LLS, Theorem 6]. The second statement is proved as in Proposition 4.16. $\qquad\square$

Lemma 4.24. *Let* $\Sigma^{r_1} F_1 \to \Sigma^{r_2} F_2$ *be an equivariant map between stable odd Burnside functors* $(F_1 \colon \underline{2}^{n_1} \to \mathscr{B}_\sigma, r_1)$ *and* $(F_2 \colon \underline{2}^{n_2} \to \mathscr{B}_\sigma, r_2)$. *Then there is an induced* $\mathbb{Z}_2$-*equivariant map of equivariant realizations*

$$|\Sigma^{r_1} F_1| \to |\Sigma^{r_2} F_2|,$$

an induced $\mathbb{Z}_2$-*equivariant map of doubly-equivariant realizations*

$$|\widehat{\Sigma^{r_1} F_1}| \to |\widehat{\Sigma^{r_2} F_2}|,$$

and an induced $\mathbb{Z}_2^+ \times \mathbb{Z}_2^-$-*equivariant map of equivariant realizations*

$$|\Sigma^{r_1} \mathcal{D} F_1| \to |\Sigma^{r_2} \mathcal{D} F_2|,$$

all well-defined up to homotopy equivalence. If the map of Burnside functors is an equivariant equivalence, then these induced maps are equivariant homotopy equivalences.

Proof. Let us sketch the arguments in the first case. The other two cases are similar.

First, note that Lemma 4.15 can be made to hold equivariantly, so that associated to a natural transformation of signed Burnside functors there is an equivariant map. So we only need to show that associated to an equivariant equivalence of Burnside functors, there is a well-defined equivariant stable homotopy equivalence of their realizations.

It will suffice to show that each of the moves in Definition 3.6 induces an equivariant stable homotopy equivalence. For the stabilization move this is clear. For the first move, assume that we have a natural transformation η, with $\mathrm{Tot}(\mathcal{D}\eta)$ a homotopy equivalence over $\mathbb{Z}_u$. By Proposition 4.16 and Lemma 4.17, the induced map between realizations is cellular and induces a non-equivariant homotopy equivalence. The induced map on fixed-point sets is induced by the underlying natural transformation $\mathcal{F}\eta$ using the identification of Corollary 4.22, which is a homotopy equivalence of chain complexes since $\mathrm{Tot}(\mathcal{D}\eta)$ is. By the proof of Lemma 4.17, and using Corollary 4.22 again, this induced map on fixed-point sets is a homotopy equivalence. By the G-Whitehead theorem, the induced map is a $\mathbb{Z}_2$-homotopy equivalence. $\qquad\square$

We record the behavior under coproducts.

Lemma 4.25. *Let* $F_1, F_2 \colon \underline{2}^n \to \mathscr{B}_\bullet$. *Then* $|F_1 \amalg F_2|$ *is equivariantly homeomorphic to* $|F_1| \vee |F_2|$. *If* $\bullet = \sigma$, *then* $|\widehat{F_1 \amalg F_2}|$ *is equivariantly homeomorphic to* $|\widehat{F_1}| \vee |\widehat{F_2}|$

Proof. The statement is an immediate consequence of (ho-2) and §3.9. $\qquad\square$

Remark 4.26. In fact, for $F \colon \underline{2}^n \to \mathscr{B}_\sigma$ and any $\ell \geq 0$, the constructions of the present section may be carried out using reflection in the first ℓ coordinates to produce a $\mathbb{Z}_2^\ell$-equivariant CW-spectrum $|F|^\ell$; we have encountered the first few cases: $|\mathcal{F}F| = |F|^0$, $|F| = |F|^1$, $|\widehat{F}| = |F|^2$. Its cellular

complex equals the totalization $\mathrm{Tot}(\mathcal{F}F)$ if ℓ is even and $\mathrm{Tot}(F)$ if ℓ is odd. Proposition 4.21, as well as its corollaries above, readily generalizes to these realizations, and the entire family is related by iterated quotients (or fixed point sets) under the various actions.

5. Khovanov homotopy types

In this section, we construct the odd Khovanov Burnside functor, and the odd Khovanov homotopy type as its realization. We also construct a reduced odd Khovanov homotopy type and the unified Khovanov homotopy type. We establish various properties such as fixed point constructions and cofibration sequences. We also construct several concordance invariants following standard procedure.

5.1. **The odd Khovanov Burnside functor.** In this section, we define a functor to the signed Burnside category associated to an oriented link diagram L with oriented crossings. After ordering the n crossings of L, we will identify the vertices of the hypercube of resolutions of L with the objects of $\underline{2}^n$, and the edges with the length one arrows of $\underline{2}^n$.

To define the odd Khovanov Burnside functor $F_o\colon \underline{2}^n \to \mathscr{B}_\sigma$, following Lemma 3.2, we need only define it on objects, length one morphisms, and across two-dimensional faces of the cube $\underline{2}^n$. On objects we set

$$F_o(u) = Kg(u).$$

For each edge $u \geqslant_1 v$ in $\underline{2}^n$, and each element $y \in F_o(v)$, write

$$\mathfrak{F}_o(\phi_{v,u}^{\mathrm{op}})(y) = \sum_{x \in F_o(u)} \epsilon_{x,y} x,$$

where $\mathfrak{F}_o$ is the odd Khovanov functor from §2.3. Note each $\epsilon_{x,y} \in \{-1, 0, 1\}$. Define

$$F_o(\phi_{u,v}) = \{(y,x) \in F_o(v) \times F_o(u) \mid \epsilon_{x,y} = \pm 1\},$$

where the sign on elements of $F_{Kh'}(\phi_{u,v})$ is given by $\epsilon_{x,y}$ of the pair, and the source and target maps are the natural ones.

We need only define the 2-morphisms across 2-dimensional faces. In fact, in contrast to the case of even Khovanov homology, where a global choice is necessary in order to define the 2-morphisms [LS14a], in odd Khovanov homology there is a unique choice of 2-morphisms compatible with the preceding data. To be more specific, for any 2-dimensional face $u \geqslant_1 v, v' \geqslant_1 w$, and any pair $(x,y) \in F_o(u) \times F_o(w)$, there is a unique bijection between

$$A_{x,y} := s^{-1}(x) \cap t^{-1}(y) \subset F_o(\phi_{v,w}) \times_{F_o(v)} F_o(\phi_{u,v})$$

and

$$A'_{x,y} := s^{-1}(x) \cap t^{-1}(y) \subset F_o(\phi_{v',w}) \times_{F_o(v')} F_o(\phi_{u,v'})$$

that preserves the signs. (That is, the signed sets $A_{x,y}, A'_{x,y}$ both have at most one element of any given sign).

The last assertion may be checked on a case-by-case basis, using [ORSz13, Figure 2]. Away from ladybug configurations (i.e., configurations of type X and Y) the involved sets both have at most one element (and $\mathfrak{F}_o$ commutes across 2d faces), so the result is automatic. For ladybug

configurations, there are sets $A_{x,y}, A'_{x,y}$ with two elements, but the elements have opposite sign, and so there is still a unique matching.

Next, we observe that the compatibility relation demanded by Lemma 3.2 is satisfied by F_o (using the unique bijections across 2-faces). For this, we must consider 3-dimensional faces $\iota\colon \underline{2}^3 \to \underline{2}^n$, and a choice of elements $x \in F_o(\iota(1,1,1)), y \in F_o(\iota(0,0,0))$ and the correspondence $A_{x,y}$. There are six distinct decompositions of the arrow $(1,1,1) \to (0,0,0)$ in $\underline{2}^3$ into a composition of nonidentity arrows, corresponding to permutations of $\{1,2,3\}$. Specifically, if e_i denotes the arrow $1 \to 0$ in the ith-factor of $\underline{2}^3$, the permutation σ corresponds to the composition $e_{\sigma(3)} \circ e_{\sigma(2)} \circ e_{\sigma(1)}$. These compositions are in turn related by 2-morphisms

$$F_{i,j}\colon F(e_i) \circ F(e_j) \to F(e_j) \circ F(e_i).$$

The compatibility relation of Lemma 3.2 boils down to the condition that the following diagram commutes:

$$
\begin{array}{ccc}
F_{e_3} \circ F_{e_1} \circ F_{e_2} & \xrightarrow{\text{Id} \times F_{12}} & F_{e_3} \circ F_{e_2} \circ F_{e_1} \\
\end{array}
$$

with maps $F_{13} \times \text{Id}$, $F_{32} \times \text{Id}$, $\text{Id} \times F_{23}$, $\text{Id} \times F_{31}$, $F_{21} \times \text{Id}$ connecting the vertices $F_{e_1} \circ F_{e_3} \circ F_{e_2}$, $F_{e_2} \circ F_{e_3} \circ F_{e_1}$, $F_{e_1} \circ F_{e_2} \circ F_{e_3}$, $F_{e_2} \circ F_{e_1} \circ F_{e_3}$.

However, it turns out that for any choice of x, y as above, there is at most one element of a given sign in $A^\sigma_{x,y} := s^{-1}(x) \cap t^{-1}(y) \subset F(e_{\sigma(3)}) \circ F(e_{\sigma(2)}) \circ F(e_{\sigma(1)})$, and therefore, the coherence check is automatic. To see this is a simple enumeration of all possible options. In more detail, following [LS14a], for 3d configurations that do not contain ladybug configurations on any of their 2d faces, each of the six sets $A^\sigma_{x,y}$ contain at most one element. For the remaining configurations, it is shown in [LS14a] that each of the six sets $A^\sigma_{x,y}$ contain at most two elements. However, since these remaining configurations contain ladybugs, these two elements must have opposite signs. (Recall that if $u \geqslant_1 v, v' \geqslant_1 w$ is a ladybug configuration and $s^{-1}(x) \cap t^{-1}(y) \subset F_o(\phi_{v,w}) \circ F_o(\phi_{u,v})$ is non-empty, then it consists of two oppositely signed points.) Therefore, each of the six sets $A^\sigma_{x,y}$ contains at most one element of each sign for the remaining configurations as well.

Definition 5.1. Define the stable signed Burnside functor associated to an oriented link diagram L with n oriented crossings and a choice of edge assignment to be $(F_o, -n_-)$, where $F_o\colon \underline{2}^n \to \mathscr{B}_\sigma$ is the functor defined above, and n_- is the number of negative crossings in L. Since the differential on the Khovanov chain complex respects the quantum grading, cf. §2, the odd Khovanov Burnside

functor splits as a coproduct of functors, one in each quantum grading:

$$F_o = \coprod_j F_o^j.$$

The total complex of the odd Khovanov Burnside functor $\Sigma^{-n_-} F_o^j$ agrees with the dual of the odd Khovanov chain complex:

$$(\mathrm{Tot}(\Sigma^{-n_-} F_o^j))^* = Kc_o^{*,j}(L).$$

Definition 5.2. We define the *odd Khovanov spectrum* $\mathcal{X}_o(L) = \bigvee_j \mathcal{X}_o^j(L)$ as a $\mathbb{Z}_2$-equivariant finite CW spectrum, where $\mathcal{X}_o^j(L)$ is a realization of the stable signed Burnside functor $\Sigma^{-n_-} F_o^j$

This odd functor recovers the even functor $F_e \colon \underline{2}^n \to \mathscr{B}$ from [LLS, LLS17] as follows.

Proposition 5.3. *The functors F_o and F_e satisfy $\mathcal{F} \circ F_o = F_e$, where $\mathcal{F} \colon \mathscr{B}_\sigma \to \mathscr{B}$ is the forgetful functor from Figure 1.*

Proof. The generators of even and odd Khovanov homology are canonically identified, cf. §2.6, so on objects we have a canonical identification $\mathcal{F}F_o(u) = F_e(u) = Kg(u)$. Similarly, since the differentials agree up to sign (for this identification), we have that $\mathcal{F}F_o(\phi_{u,v})$ is canonically identified with $F_e(\phi_{u,v})$ for $u \geqslant_1 v$. So we just need to show that the 2-morphism $\mathcal{F}F_o(\phi_{v,w}) \circ \mathcal{F}F_o(\phi_{u,v}) \to \mathcal{F}F_o(\phi_{v',w}) \circ \mathcal{F}F_o(\phi_{u,v'})$ agrees with the the 2-morphism $F_e(\phi_{v,w}) \circ F_e(\phi_{u,v}) \to F_e(\phi_{v',w}) \circ F_e(\phi_{u,v'})$ for all 2d faces $u \geqslant_1 v, v' \geqslant_1 w$.

For 2d configurations apart from ladybugs, for any $x \in Kg(u), y \in Kg(w)$, the subset $s^{-1}(x) \cap t^{-1}(y)$ in each of the correspondences contain at most one element, and so the two 2-morphisms agree. For ladybugs, one may check directly that the 2-morphism for $\mathcal{F}F_o$ agrees with that for F_e. To be more specific, the 2-morphism for $\mathcal{F}F_o$ specified by a type-X sign assignment agrees (using the above identifications of objects and 1-morphisms) with the right ladybug matching for F_e (a type-Y assignment corresponds to left ladybug matching), see Figure 2 for details. By Lemma 3.2, $\mathcal{F}F_o$ is isomorphic to F_e. $\qquad\square$

Therefore, the even Khovanov spectrum from [LS14a] is $\mathcal{X}_e(L) = \bigvee_j \mathcal{X}_e^j(L)$ with $\mathcal{X}_e^j(L) = |\Sigma^{-n_-} \mathcal{F}F_o^j|$. Using the doubly-equivariant realizations, we have a related spectrum:

Definition 5.4. We define a second even Khovanov spectrum, denoted $\mathcal{X}_e'(L) = \bigvee_j \mathcal{X}_e'^j(L)$ as a $\mathbb{Z}_2$-equivariant finite CW spectrum, where $\mathcal{X}_e'^j(L) = |\widehat{\Sigma^{-n_-} F_o^j}|$, a doubly-equivariant realization of the stable signed Burnside functor $\Sigma^{-n_-} F_o^j$.

Definition 5.5. We define the *unified Khovanov spectrum* $\mathcal{X}_u(L) = \bigvee_j \mathcal{X}_u^j(L)$ as a $\mathbb{Z}_2^+ \times \mathbb{Z}_2^-$-equivariant finite CW spectrum, where $\mathcal{X}_u^j(L)$ is a realization of the stable $\mathbb{Z}_2$-equivariant Burnside functor $\Sigma^{-n_-} \mathcal{D}F_o^j$.

Remark 5.6. Following Remark 4.26, there is in fact a family of Khovanov spaces $\mathcal{X}_\ell(L)$, for $\ell \geq 0$, whose cellular chain complexes agree with the even Khovanov chain complex $Kc_e(L)$ if ℓ is even and the odd Khovanov chain complex $Kc_o(L)$ if ℓ even is odd. (We have already encountered
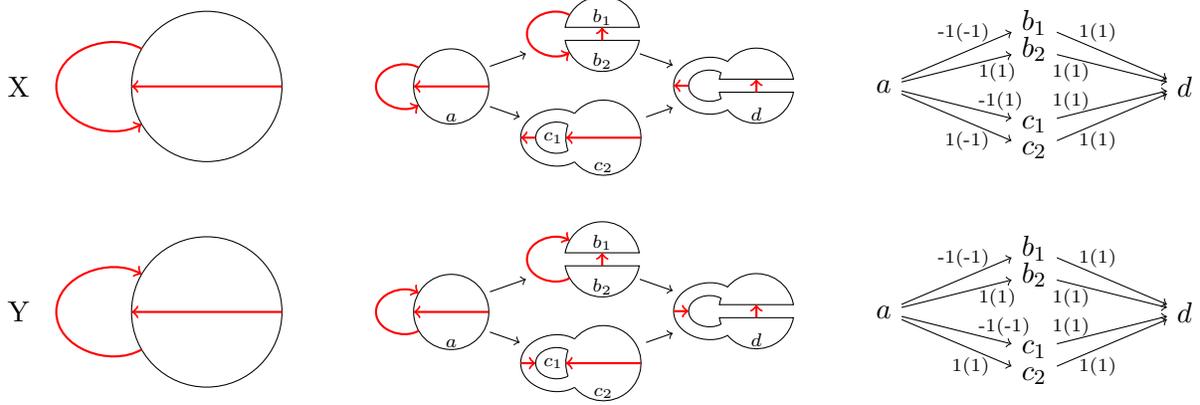
FIGURE 2. **The odd functor for the type-X assignment recovers the even functor for the right ladybug matching.** Consider the two types of ladybugs, X and Y, and name the circles appearing in the various resolutions a, b_1, b_2, c_1, c_2, d as shown (their ordering does not matter). The coefficients of the relevant portion of the functor $\mathfrak{F}_o$ (as well as those of the assignment $\mathfrak{F}'_o$ in parentheses) are shown. Since we are considering a type-X assignment, $\mathfrak{F}_o$ is chosen to differ from $\mathfrak{F}'_o$ in one edge for the X-ladybug, and is chosen to agree with $\mathfrak{F}'_o$ for the Y-ladybug. In either case, the unique sign-respecting 2-isomorphism is the matching $(a, b_1, d) \leftrightarrow (a, c_1, d), (a, b_2, d) \leftrightarrow (a, c_2, d)$, which is the right ladybug matching from [LS14a].

$\mathcal{X}_0(L) = \mathcal{X}_e(L)$, $\mathcal{X}_1(L) = \mathcal{X}_o(L)$, and $\mathcal{X}_2(L) = \mathcal{X}'_e(L)$.) There is a natural generalization of Theorem 1.6 to this family of spaces. We conjecture that $\mathcal{X}_\ell(L)$, up to homotopy equivalence, only depends on the parity of ℓ.

5.2. **Relations among the three theories.** In this section, we find relations among the three Khovanov homotopy types, in terms of geometric fixed points, geometric quotients, and cofibration sequences.

Proof of Theorem 1.6. The first statement and the first parts of statements (2) and (4) follow from Corollary 4.22. The exact sequences are a consequence of the $\mathbb{Z}/2$-actions on $\mathcal{X}'_e(L)$ and $\mathcal{X}_o(L)$, for which $\Sigma^{-1}\mathcal{X}_o(L)$ and $\Sigma^{-1}\mathcal{X}_e(L)$ are respectively the fixed point sets, and $\mathcal{X}_u(L)$ the quotients. The inclusion of the fixed-point sets are cofibrations in both cases, giving the desired exact sequences. The agreement with the exact sequences of [PS16] at the level of cohomology is a consequence of (3) and (5). So it remains to prove (3) and (5). The proofs are similar, so let us only consider (3).

Consider the Puppe sequence associated to the inclusion $\Sigma^{-1}\mathcal{X}_e(L) \hookrightarrow \mathcal{X}_o(L)$. For concreteness, assume $\mathcal{X}_o(L)$ has been constructed equivariantly using k-dimensional boxes, and all the sub-boxes of $[0,1]^k$ involved in the construction are of the form $[0,1] \times B$; that is, they extend the full length in the first coordinate. Let X denote $\mathcal{X}_o(L)$, Y denote the fixed set $\Sigma^{-1}\mathcal{X}_e(L)$, and Z denote the quotient $X/Y = \mathcal{X}_u(L)$. The Puppe sequence takes the form

$$Y \hookrightarrow X \to X \cup C(Y) \xrightarrow{P} \Sigma Y,$$

with C denoting the cone, where the last map P is quotienting by X.

The term $X \cup C(Y)$ is homotopy-equivalent to Z by quotienting by $C(Y)$:

$$Q \colon X \cup C(Y) \to X/Y = Z.$$

So the Puppe map $Z \to \Sigma Y$ is the homotopy inverse of Q, composed with P.

Meanwhile, we have the map $R \colon \mathcal{X}_u(L) = Z \to \mathcal{X}_e(L) = \Sigma Y$ given by quotienting by $\mathbb{Z}_2^+$. Recall that the $\mathbb{Z}_2^+$-action on $Z = X/Y$ is induced from the $\mathbb{Z}_2$-action on X. We wish to show that these two maps from Z to ΣY are homotopic. Since Q is a homotopy equivalence, it is enough to show that the two maps $P, R \circ Q \colon X \cup C(Y) \to \Sigma Y$ are homotopic.

Consider the quotient of X by the $\mathbb{Z}_2$-action. Since X has been constructed using boxes that stretch the full length along the first coordinate, it is not hard to see that the quotient is $C(Y)$. This produces a quotient map $S \colon X \cup C(Y) \to C(Y) \cup C(Y) = \Sigma Y$.

Both the maps P and $R \circ Q$ factor through the above map S. The first map quotients the first $C(Y)$ factor, while the second map quotients the second factor. Either is homotopic to the identity map $C(Y) \cup C(Y) \to \Sigma Y$, and so the claim follows. $\qquad\square$

5.3. **Invariance.** The main aim of this section is to prove that changes of the orientation of the crossings, as well as Reidemeister moves, result in equivariantly equivalent signed Burnside functors.

Proof of Theorem 1.7. We will now prove that the equivariant equivalence class of the odd Khovanov Burnside stable functor from Definition 5.1 is independent of the choices in its construction, namely the choice of diagram L, the orientation of the crossings, the edge assignment, and the ordering of the generators a_i at each resolution.

We first see that for a fixed diagram L, changing the other auxiliary choices results in sign reassignments (which are sometimes isomorphisms) of functors from $\underline{2}^n$ to $\mathscr{B}_\sigma$.

- **Edge assignment:** Let ϵ, ϵ' be two different edge assignments of the same type for the same oriented knot diagram L. As noted in [ORSz13, Lemma 2.2], $\epsilon\epsilon'$ is a (multiplicative) cochain in $C^1_{\mathrm{cell}}([0,1]^n, \mathbb{Z}_2)$, and hence a coboundary of a 0-cochain α on the cube of resolutions. That is, there is a map $\alpha \colon \underline{2}^n \to \{\pm 1\}$, so that for any $v \geqslant_1 w$ $\alpha(v)\alpha(w) = \epsilon(\phi^{\mathrm{op}}_{w,v})\epsilon'(\phi^{\mathrm{op}}_{w,v})$. If F_o and F'_o are the corresponding functors $\underline{2}^n \to \mathscr{B}_\sigma$, we get that F'_o is obtained from F_o by using the sign reassignment associated to α.

- **Orientations at crossings:** Recall that [ORSz13, Lemma 2.3] asserts that for oriented diagrams (L, o) and (L, o') and an edge assignment ϵ for (L, o), there exists an edge assignment of the same type ϵ' for (L, o') so that $Kc_o(L, o, \epsilon) \cong Kc_o(L, o', \epsilon')$. The isomorphism constructed in the lemma respects the Khovanov generators, and so induces an isomorphism of signed Burnside functors. To be more specific, note that the Khovanov generators $Kg(L)$ of $Kc_o(L, o)$ are independent of the orientation o (which only changes the differential). Then the choice of edge assignment ϵ' is such that the identity morphism $Kc_o(L, o, \epsilon) \to Kc_o(L, o', \epsilon')$ commutes with the differentials. Then the corresponding Burnside functors are also naturally isomorphic. (Independence of the orientations of crossings can also be proved using Reidemeister II moves twice, as in [SSSz17, Figure 4.5].)

- **Type of edge assignment:** [ORSz13, Lemma 2.4] proves that an edge assignment ϵ of a decorated link diagram (L, o) of type X can also be viewed as a type Y edge assignment for some orientation o'. That is, the type-X Burnside functor associated to (L, o, ϵ) is already the type-Y Burnside functor associated to (L, o', ϵ). (Independence of the type of edge assignment can also be achieved by Viro's trick of reflecting the knot diagram along the vertical line (which switches the X and the Y ladybug), and then using a sequence of Reidemeister moves to come back to the original diagram, cf. [LS14a, Proposition 6.5].)
- **Ordering of circles at each resolution:** Finally, we must check that reordering the circles of a resolution results in an equivariantly-equivalent signed Burnside functor. For this, let $Kg(u)$ and $Kg'(u)$ denote the Khovanov generators for two differing orderings of the circles for a fixed link diagram. These orderings are related by a bijection from $Kg(u)$ to $Kg'(u)$. It is simple to check that these bijections relate the two functors $F_o, F'_o \colon \underline{2}^n \to \mathscr{B}_\sigma$ by a sign reassignment.

Next we move on to the main issue for proving well-definedness of the stable equivalence class of the odd Khovanov Burnside functor: Reidemeister moves. For proving invariance under the Reidemeister moves, the argument mostly follows the proof of [LLS17, Theorem 1], which lifts Khovanov's invariance proof [Kho00] to the level of Burnside functors. For two diagrams differing by a Reidemeister move, Khovanov's invariance proof—see also Bar-Natan [BN02]—is built using chain maps between Khovanov complexes, which are either subcomplex inclusions or quotient complex projections that send Khovanov generators to Khovanov generators and are chain homotopy equivalences, or their chain homotopy inverses. Since these maps send Khovanov generators to Khovanov generators, it is easy to see that the argument lifts to the Burnside category level [LLS17]; we similarly sketch how in the odd case, the invariance proof from [ORSz13] lifts to the odd Burnside functor. (The astute reader will observe that for the Reidemeister III proof by Khovanov, the chain maps do not send Khovanov generators to Khovanov generators. This issue is faced in [LS14a] already. It seems possible to carry through the approach of [Kho00, BN02, ORSz13], but at the expense of considering functors from categories other than cube categories. However, we will instead follow the proof of Reidemeister III invariance of [LS14a] by considering the braid-like Reidemeister III.)

The standard way to prove Reidemeister invariance—applicable in the even, odd, and unified theory—is the following. Start with the Khovanov chain complex of one diagram, and perform a sequence of replacements to arrive at the Khovanov chain complex of the other diagram, where each replacement is either:

(c-1) *Replacing the complex with a quotient complex associated to a merge.* Namely, for a merge taking circles a_1, a_2 in L_0 to a in L_1, there is an acyclic subcomplex spanned, at L_0, by generators that do not contain a_1, and all generators at L_1; replace by the quotient by this subcomplex.

(c-2) *Replacing the complex with the subcomplex associated to a split.* Namely, for a split taking one circle a in L_0 to a_1, a_2 in L_1, there is a subcomplex spanned by all generators from L_1 which do not contain an a_1 factor, and the corresponding quotient is acyclic; replace by this subcomplex.

It is easy to check that the relevant maps—the quotient complex projection in the first case and the subcomplex inclusion in the second case—are chain homotopy equivalences in the unified theory

over $\mathbb{Z}_u$, and hence also in the odd and the even theory. (These cancellations are parametrized by *cancellation data* from [SSSz17, Definition 4.4].)

To lift this argument to the Burnside functor level, in the first case, we will replace the functor by a sub-functor, and in the second case, by a quotient functor from §3.7. (Recall, the Khovanov complex is the *dual* of the totalization of the Burnside functor, hence sub-functors correspond to quotient complexes and quotient functors correspond to subcomplexes.)

- **RI:** Consider the Reidemeister I move from a diagram $L =$ ⌣ to the diagram $L' =$ ⌣○. We have that $Kg(L') = Kg(L'_0) \amalg Kg(L'_1)$ where $L'_0 =$ ⊃○ (respectively, $L'_1 =$ ⌣) is obtained from L' by resolving the new crossing by the 0-resolution (respectively, 1-resolution). Let a be the new circle in L'_0, as shown in the picture.

  We may perform a replacement of Type (c-1) by cancelling the subcomplex of $Kc_u(L')$ spanned by all the generators in $Kg(L'_1)$ and only the generators in $Kg(L'_0)$ that do not contain a, and after that we will be left with a quotient complex that is naturally isomorphic to $Kc_u(L)$. That is, we have a quotient complex projection $Kc_u(L') \to Kc_u(L)$ that is a chain homotopy equivalence over $\mathbb{Z}_u$. This is induced from a subfunctor inclusion $F_o(L) \to F_o(L')$, that is, the dual map on the totalizations

$$(\mathrm{Tot}(\mathcal{D} \circ F_o(L')))^* \to (\mathrm{Tot}(\mathcal{D} \circ F_o(L)))^*$$

  agrees with the map on the unified Khovanov complex, where $\mathcal{D} \colon \mathscr{B}_\sigma \to \mathscr{B}_\xi$ is the doubling functor from Figure 1. Therefore, the functors $F_o(L)$ and $F_o(L')$ are equivariantly equivalent.
- **RII:** The proof for Reidemeister II move is similar, except now we have to use both types of cancellations. Say we are doing a Reidemeister II from $L =$ ⌣ to $L' =$ ⟩⟨. Once again, $Kg(L')$ decomposes as $\amalg_{(i,j)\in\{0,1\}^2} Kg(L'_{ij})$, where L'_{ij} are the partial (i,j) resolutions of L' at the new crossings. Let a be the new circle in $L'_{01} =$ ⊃○⊂.

  For the merge $L'_{01} \to L'_{11}$, we may cancel the subcomplex spanned by $Kg(L'_{11})$ and the generators in $Kg(L'_{01})$ that do not contain a. The remaining quotient complex has an acyclic subcomplex corresponding to the split $L'_{00} \to L'_{01}$, spanned by $Kg(L'_{00})$ and the remaining generators in $Kg(L'_{01})$. This produces a chain homotopy equivalence between $Kc_u(L')$ and $Kc_u(L'_{10})$ (modulo shifting the homological grading by one), and the latter is naturally identified with $Kc_u(L)$. Since these subquotient complexes come from Burnside sub-functors and Burnside quotient functors, it is automatic that the two stable Burnside functors $F_o(L) = F_o(L_{10})$ and $\Sigma^{-1} F_o(L')$ are equivariantly equivalent.
- **RIII:** The proof of Reidemeister III invariance is exactly the same as the previous proof. As discussed earlier, we deviate from the standard proofs from [Kho00, BN02, ORSz13], but instead follow the proof from [LS14a, Proposition 6.4]. Let L' be obtained from L by performing a braid-like Reidemeister III move, as in [LS14a, Figure 6.1c]. Then in the six-dimensional partial cube of resolutions of L', one can perform as sequence of cancellations—see [LS14a, Figure 6.4] and the subsequent table—of Types (c-1) and (c-2) to produce a chain homotopy equivalence between $Kc_u(L')$ and $Kc_u(L'_{000111})$ (modulo shifting gradings by three), and the latter is naturally identified to $Kc_u(L)$. The subquotient complexes come from Burnside sub-functors and Burnside

quotient functors, and once again it follows that the two stable Burnside functors $F_o(L) = F_o(L_{000111})$ and $\Sigma^{-3} F_o(L')$ are equivariantly equivalent.

We leave it to the reader to convince themselves that the above equivalences automatically respect the decomposition of the Burnside functors according to quantum gradings.   □

Proof of Theorem 1.1. Recall that $\mathcal{X}_o(L) = |\Sigma^{-n_-} F_o| = (\|F_o\|_k, -n_- - k)$. By Lemma 4.17, $|\Sigma^{n_-} F_o|$ depends, up to (nonequivariant) stable homotopy equivalence, only on the stable equivalence class of $\Sigma^{n_-} F_o$. Then by Theorem 1.7, the stable homotopy class of $|\Sigma^{n_-} F_o|$ is an invariant of L. Proposition 4.16 identifies the cellular chain complex of $|\Sigma^{n_-} F_o|$ as the totalization of $\Sigma^{-n_-} F_o$, which is the dual of the Khovanov complex (see discussion after Definition 5.1), so Theorem 1.1 follows (nonequivariantly). To see that $\mathcal{X}_o(L)$ is well-defined up to equivariant stable homotopy equivalence, we use Lemma 4.24 in place of Lemma 4.17.   □

Proof of Theorem 1.4. As with the proof of Theorem 1.1, we see that $\mathcal{X}'_e(L)$, up to equivariant stable homotopy equivalence, depends only on the equivariant equivalence class of $\Sigma^{n_-} F_o$, by Lemma 4.24. Theorem 1.7 establishes that the equivariant equivalence class of $\Sigma^{n_-} F_o$ is a link invariant, and the theorem follows.   □

Proof of Theorem 1.5. The well-definedness follows as in Theorems 1.1 and 1.4 from Lemma 4.24. For the CW description, we use Proposition 4.23, which establishes that the equality in Theorem 1.5 is an isomorphism of $\mathbb{Z}_u$-modules. The statement about the $\mathbb{Z}_2$-actions on the reduced cellular chain complex follows from the construction.   □

5.4. Reduced odd Khovanov homotopy type.

We briefly address the reduced theory.

Given a (generic) point p on a link diagram L, there is a natural sub-functor of $F_o^j(L)$ generated by only those Khovanov generators that do not contain the circle c_p containing p. Let $\widetilde{F}_{o,+}^{j-1}(L, p)$ denote this subfunctor and $\widetilde{F}_{o,-}^{j+1}(L, p)$ denote the corresponding quotient functor.

Next we show that the two reduced functors $\widetilde{F}_{o,+}^j$ and $\widetilde{F}_{o,-}^j$ are canonically identified. Arranging for convenience that the ordering of circles at each resolution has that c_p—the circle containing p—is always last, we have a canonical bijection between $\widetilde{F}_{o,-}$ and $\widetilde{F}_{o,+}$. This bijection is compatible with the 1-morphisms of the even Burnside functor; however, we must also check that these respect the sign map. To be specific, for $u \geqslant_1 v$, the bijection $\widetilde{F}_{o,-}(\phi_{u,v}) \to \widetilde{F}_{o,+}(\phi_{u,v})$ preserves all signs cf. §2.3, as the reader may check. We will refer to either functor as $\widetilde{F}_o$.

We would expect, based on what happens for odd Khovanov chain complex, that the unreduced functor F_o^j should be stably equivalent to two copies of the reduced functor, $\widetilde{F}_o^{j-1} \amalg \widetilde{F}_o^{j+1}$. However, the chain level splitting from [ORSz13] does not generalize. Indeed, any such stable equivalence cannot be a equivariant equivalence, cf. Definition 3.6, since by Proposition 5.3, $\mathcal{F} F_o = F_e$ (and similarly, $\mathcal{F} \widetilde{F}_o = \widetilde{F}_e$, where $\widetilde{F}_e$ is the reduced even Burnside functor), and the even Burnside functor (and indeed, the even Khovanov chain complex) does not split as two copies of its reduced version.

Definition 5.7. We define the *reduced odd Khovanov spectrum* $\widetilde{\mathcal{X}}_o(L, p) = \bigvee_j \widetilde{\mathcal{X}}_o^{\,j}(L, p)$ as a $\mathbb{Z}_2$-equivariant finite CW spectrum, where $\widetilde{\mathcal{X}}_o^{\,j}(L, p)$ is a realization of the stable signed Burnside functor $\Sigma^{-n_-} \widetilde{F}_o^j$.

Definition 5.8. We define the *reduced unified Khovanov spectrum* $\widetilde{\mathcal{X}}_u(L, p) = \bigvee_j \widetilde{\mathcal{X}}_u^j(L, p)$ as a $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant finite CW spectrum, where $\widetilde{\mathcal{X}}_u^j(L, p)$ is a realization of the stable signed Burnside functor $\Sigma^{-n_-} \mathcal{D} \widetilde{F}_o^j$.

Proof of Theorem 1.2. Well-definedness of $\widetilde{\mathcal{X}}_o(L)$, up to equivariant stable homotopy equivalence, will follow from showing that $\Sigma^{-n_-} \widetilde{F}_o$ is well-defined up to equivariant equivalence, depending only on the isotopy class of (L, p). Isotopy invariance is immediate for Reidemeister moves away from the basepoint (using the maps induced by Reidemeister moves on $\Sigma^{-n_-} F_o$, and observing that they preserve $\Sigma^{-n_-} \widetilde{F}_o$). As observed in [Kho03], any two diagrams for isotopic pointed links can be related by Reidemeister moves not crossing the basepoint and isotopies in S^2, from which well-definedness follows. The cofibration sequence is a consequence of Lemma 4.15, using the cofibration sequence of Burnside functors:

$$\widetilde{F}_{o,+}^{j-1}(L, p) \to F_o^j(L) \to \widetilde{F}_{o,-}^{j+1}(L, p).$$

Finally, the description of the reduced cellular cochain complex is a consequence of Proposition 4.16, as in the proof of Theorem 1.1. $\qquad\square$

Proposition 5.9. *The (stable) homotopy type of the reduced unified Khovanov spectrum $\widetilde{\mathcal{X}}_o(L, p) = \bigvee_j \widetilde{\mathcal{X}}_o^j(L, p)$ from Definition 5.8 is independent of the choices in its construction and is an invariant of the isotopy class of the pointed link corresponding to (L, p). Its reduced cellular cochain complex agrees with the reduced unified Khovanov complex $\widetilde{Kc}_u(L)$,*

$$\widetilde{C}_{\mathrm{cell}}^i(\widetilde{\mathcal{X}}_u^j(L, p)) = \widetilde{Kc}_u^{\,i,j}(L),$$

with the cells mapping to the distinguished generators of $\widetilde{Kc}_u(L)$. There is a cofibration sequence

$$\widetilde{\mathcal{X}}_u^{j-1}(L, p) \to \mathcal{X}_u^j(L) \to \widetilde{\mathcal{X}}_u^{j+1}(L, p).$$

Proof. The proof of Theorem 1.2 goes through mostly unchanged. The only new observation necessary is that the double of a cofibration sequence of Burnside functors is again a cofibration sequence. $\qquad\square$

5.5. **Cobordism maps.** For every smooth link cobordism $L \to L'$ embedded in $\mathbb{R}^3 \times [0, 1]$, there is an induced map on the even Khovanov complex $Kc(L) \to Kc(L')$ [Jac04, BN05, Kho06], well-defined up to chain homotopy and an overall sign. (The dependence on the overall sign can be removed, see [CMW].)

[LS14b] lifted these maps to the even Khovanov homotopy types, $\mathcal{X}_e(L') \to \mathcal{X}_e(L)$, so that the induced map on the cellular cochain complex is the previous map, but did not check well-definedness. It is fairly easy to check that the map $\mathcal{X}_e(L') \to \mathcal{X}_e(L)$ defined in [LS14b] comes from a map of the even Burnside functors $F_e(L') \to F_e(L)$, so that the dual of the map on their totalizations is the map $Kc(L) \to Kc(L')$.

In this section, we will further lift these to maps of the odd Burnside functor $F_o(L') \to F_o(L)$, so that the even Burnside functor map is obtained by applying the forgetful functor $\mathcal{F} \colon \mathscr{B}_\sigma \to \mathscr{B}$. In particular, we will get maps on the odd Khovanov homotopy type, $\mathcal{X}_o(L') \to \mathcal{X}_o(L)$ and the odd Khovanov complex, $Kc_o(L) \to Kc_o(L')$. We will not check the well-definedness of any of these maps.

The standard way to define these maps is by decomposing the cobordism as movie, which is a sequence of knot diagrams so that each one is obtained from the previous one by a planar isotopy, Reidemeister move, or a Morse critical point, which can be a birth, death, or a saddle. In §5.3, we have already constructed maps of odd Burnside functors corresponding to the Reidemeister moves (which were also equivariant equivalences). So we only need to construct maps associated to the three Morse singularities.

First we consider the cup and cap cobordisms. Let L a link diagram, and $L' = L \amalg U$, the disjoint union of L and an unknot, introducing no new crossings. The elementary cobordism from L to L' is called a *cup* or a *birth*, while that from L' to L is a *cap* or a *death*. We construct natural transformations

$$\Phi_o^\cup \colon F_o(L') \to F_o(L)$$
$$\Phi_o^\cap \colon F_o(L) \to F_o(L'),$$

decreasing quantum grading by 1, lifting the natural transformations

$$\Phi_e^\cup \colon F_e(L') \to F_e(L)$$
$$\Phi_e^\cap \colon F_e(L) \to F_e(L')$$

for the even Burnside functors from [LS14b].

In each resolution of L' there is a component corresponding to U. We can write $Kg(L') = Kg(L)_+ \amalg Kg(L)_-$ where $Kg(L)_-$ (respectively, $Kg(L)_+$) is the subset of generators in $Kg(L')$ which contain U (respectively, do not contain U); either is canonically identified with $Kg(L)$ by ordering the circles at each resolution so that U is last. Let $F_o(L)_-$ (respectively, $F_o(L)_+$) be the subfunctor of $F_o(L')$ generated by $Kg(L)_-$ (respectively, $Kg(L)_+$); either is isomorphic to $F_o(L)$, modulo a quantum grading shift of ± 1. Then $F_o(L') = F_o(L)_- \amalg F_o(L)_+$, and so there is a subfunctor inclusion $F_o(L)_- \to F_o(L')$ and quotient functor projection $F_o(L') \to F_o(L)_+$.

We then set the cobordism maps according to:

$$\Phi_o^\cup \colon F_o(L') \to F_o(L)_+ \cong F_o(L)$$
$$\Phi_o^\cap \colon F_o(L) \cong F_o(L)_- \to F_o(L').$$

Next, we handle the saddle case. Let L_0, L_1 be n-crossing link diagrams before and after the saddle, as in [LS14b, Figure 3.2], and let $F_e(L_0), F_e(L_1) \colon \underline{2}^n \to \mathscr{B}$ be the two even Burnside functors (we have implicitly identified the crossings in L_0 with the crossings in L_1). Associated to the saddle cobordism, [LS14b] constructs a natural transformation $\Phi_e^s \colon F_e(L_1) \to F_e(L_0)$ as follows. There is a $(n+1)$-crossing diagram L so that L_i is the i-resolution of L at the new crossing, for $i = 0, 1$. Then $\Phi_e^s \colon \underline{2}^{n+1} \to \mathscr{B}$ is simply defined to be $F_e(L) \colon \underline{2}^{n+1} \to \mathscr{B}$—the even Burnside functor associated to L. One easily checks that the natural transformation increases the quantum grading by 1.

The generalization to the signed Burnside functor version is immediate, and we obtain a natural transformation $\Phi_o^s\colon F_o^s(L_1) \to F_o^s(L_0)$.

Lemma 5.10. *Associated to a (movie presentation of a) link cobordism $L \to L'$, the map on the odd Burnside functors $\Phi_o\colon F_o^{j+\chi(S)}(L') \to F_o^j(L)$ lifts the map on the even Burnside functors $\Phi_e\colon F_e^{j+\chi(S)}(L') \to F_e^j(L)$ that is (implicitly) constructed in* [LS14b]*:*

$$\mathcal{F}\Phi_o = \Phi_e.$$

In particular, the induced map on the $\mathbb{Z}_2$ chain complex, $Kc^{*,j}(L;\mathbb{Z}_2) \to Kc^{*,j+\chi(S)}(L';\mathbb{Z}_2)$, agrees with the Khovanov map (up to chain homotopy).

Proof. This is immediate from the definitions for each of the maps—the Reideister invariance maps, as well as the cups, caps, and the saddles. $\square$

Let $\mathcal{A}_2$ denote the mod-2 Steenrod algebra. Let $\mathcal{A}^\sigma$ denote the free product of two copies of $\mathcal{A}_2$. It acts on $Kh(L;\mathbb{Z}_2)$ as follows: the first (respectively, second) copy acts by viewing $Kh(L;\mathbb{Z}_2)$ as the mod-2 cohomology of $\mathcal{X}_e(L)$ (respectively, $\mathcal{X}_o(L)$).

Proof of Theorem 1.3. We follow the proof of [LS14b, Corollary 5]. Let $S\colon L \to L'$ be the link cobordism and $Kh_S\colon Kh(L;\mathbb{Z}_2) \to Kh(L';\mathbb{Z}_2)$ the induced map. By Lemma 5.10, Kh_S comes from a map of Burnside functors $\Phi_o^S\colon F_o(L') \to F_o(L)$.

The even and odd realizations give two actions of $\mathcal{A}_2$ on the mod-2 Khovanov homology, and, in particular, by naturality of Steenrod operations on either the odd or even spatial realization, we have a commutative diagram

$$
\begin{array}{ccc}
Kh^{i,j}(L;\mathbb{F}) & \xrightarrow{\ \ \ \alpha\ \ \ } & Kh^{i+n,j}(L;\mathbb{F}) \\
{\scriptstyle Kh_S}\Big\downarrow & & \Big\downarrow{\scriptstyle Kh_S} \\
Kh^{i,j+\chi(S)}(L';\mathbb{F}) & \xrightarrow{\ \ \ \alpha\ \ \ } & Kh^{i+n,j+\chi(S)}(L';\mathbb{F})
\end{array}
$$

for α a stable cohomology operation of degree n coming from either copy of $\mathcal{A}_2$. It is then clear that the diagram then also commutes for α a linear combination or composition of elements of the two copies of $\mathcal{A}_2$. The theorem follows directly from these diagrams. $\square$

5.6. Concordance invariants.

Theorem 1.3 allows one to define knot concordance invariants, once again borrowing arguments directly from [LS14b]. In this section, we work only with knots, not links.

For a knot diagram K and field $\mathbb{F}$, there is a spectral sequence with E^2-page Khovanov homology $Kh^{i,j}(K;\mathbb{F})$ converging to $\mathbb{F}^2$ coming from a descending filtration $\mathcal{F}$ of the Khovanov chain complex $Kc(K)$ so that

$$Kc^{*,j}(K;\mathbb{F}) = \mathcal{F}_j/\mathcal{F}_{j+2}.$$

This was defined by Lee [Lee05] for fields not of characteristic 2, and for all fields by Bar-Natan [BN05], see also [Tur06, Nao06]; from now on, fix our field as $\mathbb{F} = \mathbb{F}_2$, the field with two elements, and we work their variant.

The Rasmussen s invariant of K from [Ras10], cf. [LS14b], is defined by

$$s^{\mathbb{F}}(K) = \max\{q \in 2\mathbb{Z} + 1 \mid H^*(\mathcal{F}_q) \to H^*(\mathcal{F}_{-\infty}) \cong \mathbb{F}^2 \text{ surjective }\} + 1,$$
$$= \max\{q \in 2\mathbb{Z} + 1 \mid H^*(\mathcal{F}_q) \to H^*(\mathcal{F}_{-\infty}) \cong \mathbb{F}^2 \text{ nonzero }\} - 1,$$

Definition 5.11. Fix $\alpha \in \mathcal{A}^\sigma$ of grading $n > 0$. Call $q \in 2\mathbb{Z} + 1$ α-*half full* if there exist elements $\widetilde{a} \in Kh^{-n,q}(K; \mathbb{F})$, $\widehat{a} \in Kh^{0,q}(K; \mathbb{F})$, $a \in H^0(\mathcal{F}_q; \mathbb{F})$, $\bar{a} \in H^0(\mathcal{F}_{-\infty}; \mathbb{F})$ such that

(1) the map $\alpha \colon Kh^{-n,q}(K; \mathbb{F}) \to Kh^{0,q}(K, \mathbb{F})$ from Theorem 1.3 sends $\widetilde{a}$ to $\widehat{a}$.
(2) The map $H^0(\mathcal{F}_q; \mathbb{F}) \to Kh^{0,q}(K; \mathbb{F}) = H^0(\mathcal{F}_q/\mathcal{F}_{q+2}; \mathbb{F})$ sends a to $\widehat{a}$.
(3) The map $H^0(\mathcal{F}_q; \mathbb{F}) \to H^0(\mathcal{F}_{-\infty}; \mathbb{F})$ sends a to $\bar{a}$.
(4) $\bar{a} \in H^0(\mathcal{F}_{-\infty}; \mathbb{F}) = \mathbb{F} \oplus \mathbb{F}$ is nonzero.

Call q α-*full* if there exists tuples $(\widetilde{a}, \widehat{a}, a, \bar{a})$ and $(\widetilde{b}, \widehat{b}, b, \bar{b})$ as above, with properties (1)–(3), and so that $\bar{a}, \bar{b}$ is a basis of $H^0(\mathcal{F}_{-\infty}; \mathbb{F})$.

Definition 5.12. For a knot K, define $r_+^\alpha(K) = \max\{q \in 2\mathbb{Z} + 1 \mid q \text{ is } \alpha\text{-half-full}\} + 1$, and $s_+^\alpha(K) = \max\{q \in 2\mathbb{Z} + 1 \mid q \text{ is } \alpha\text{-full}\} + 3$. If $m(K)$ is the mirror of K, let $r_-^\alpha(K) = -r_+^\alpha(m(K))$ and $s_-^\alpha(K) = -s_+^\alpha(m(K))$.

Theorem 5.13. *Let* $\alpha \in \mathcal{A}^\sigma$ *and* S *a connected, embedded cobordism in* $\mathbb{R}^3 \times [0, 1]$ *from* K *to* K' *of genus* g. *Then*

$$|r_\pm^\alpha(K) - r_\pm^\alpha(K')| \leq 2g$$
$$|s_\pm^\alpha(K) - s_\pm^\alpha(K')| \leq 2g.$$

In particular, $|r_\pm^\alpha(K)|/2, |s_\pm^\alpha(K)|/2$ *are concordance invariants and lower bounds for the slice genus* $g_4(K)$.

Proof. This follows from Theorem 1.3, arguing as in [LS14b, Theorem 1]. $\square$

5.7. **Questions.** We conclude with some structural questions about the odd Khovanov space:

(q-1) In §5.6 we constructed concordance invariants using the action of the mod-2 Steenrod algebra on $\widetilde{H}^*(\mathcal{X}_o(L); \mathbb{Z}_2)$. It is natural to ask for concordance invariants defined from homology using different coefficient fields. Indeed, [LS14b] defines such invariants using stable cohomology operations with any coefficient field. For this, perhaps one needs an analogue of the Lee spectral sequence for odd Khovanov homology.

(q-2) Ozsváth-Rasmussen-Szabó [ORSz13] showed that $Kh_o^{*,j}(L) = \widetilde{Kh}_o^{*,j-1}(L) \oplus \widetilde{Kh}_o^{*,j+1}(L)$ for any link L. Is it the case that $\mathcal{X}_o^j(L) \simeq \widetilde{\mathcal{X}}_o^{j-1}(L) \vee \widetilde{\mathcal{X}}_o^{j+1}(L)$? More specifically, is there a stable equivalence between the signed Burnside functors $F_o^j(L)$ and $\widetilde{F}_o^{j-1}(L) \amalg \widetilde{F}_o^{j+1}(L)$? (Such an equivalence has to be non-equivariant.)

(q-3) So far, calculations of the odd Khovanov homotopy type are limited. Is it always a wedge sum of Moore spaces? Do there exist links L for which $\mathcal{X}_o(L)$ is not a wedge sum of smash products of Moore spaces?

(q-4) In [PS16] there are short exact sequences:

$$Kc_e(L) \to Kc_u(L) \to Kc_o(L) \qquad \text{and} \qquad Kc_o(L) \to Kc_u(L) \to Kc_e(L)$$

At the level of cohomology, these exact sequences are induced from the cofibration sequences in Theorem 1.6. However, the maps in that proposition were not cellular for the coarse CW structure. That leads to the question: are there CW cofibration sequences

$$\mathcal{X}_o(L) \to \mathcal{X}_u(L) \to \mathcal{X}_e(L) \qquad \text{and} \qquad \mathcal{X}_e(L) \to \mathcal{X}_u(L) \to \mathcal{X}_o(L)$$

(with respect to the coarse CW structure) inducing the maps of [PS16] on cellular chain complexes?

(q-5) One of the applications of the technology of [LLS] was to show

$$\mathcal{X}_e(m(L)) \simeq \mathcal{X}_e(L)^{\vee}$$

where $m(L)$ is the mirror of L, and $\vee$ denotes the Spanier-Whitehead dual. We conjecture, similarly, that $\mathcal{X}_o(m(L)) \simeq \mathcal{X}_o(L)^{\vee}$. The proof of the statement in even Khovanov homology involved the TQFT structure of even Khovanov homology, and does not immediately generalize to odd Khovanov homology.

(q-6) It would be desirable to understand the behavior of the odd Khovanov spectra for disjoint unions and connected sums. Is it possible to express the (equivariant) homotopy type $\mathcal{X}_o(L_1 \amalg L_2)$ in terms of the (equivariant) odd Khovanov spectra of L_1, L_2? (In the even theory, it is merely the smash product.) Is it possible to express the odd and unified (unreduced) Khovanov spectra of $L_1 \# L_2$ in terms of the spectra of the component links? For even Khovanov homotopy, this was dealt with as [LLS, Theorem 8]: $\mathcal{X}_e(L_1 \# L_2)$ is the derived tensor product of $\mathcal{X}_e(L_i)$ over the even Khovanov spectrum of the unknot.

(q-7) Is the old even Khovanov spectrum $\mathcal{X}_e(L)$ stable homotopy equivalent to the new even Khovanov spectrum $\mathcal{X}_e'(L)$? More generally, does the Khovanov spectrum $\mathcal{X}_\ell(L)$ from Remark 5.6 depend only on the parity of ℓ?

References

[Bén67]   Jean Bénabou, *Introduction to bicategories*, Reports of the Midwest Category Seminar, Springer, Berlin, 1967, pp. 1–77. MR 0220789 (36 #3841)

[BN02]   Dror Bar-Natan, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. **2** (2002), 337–370. MR 1917056

[BN05]   _____, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. **9** (2005), 1443–1499. MR 2174270

[CJS95]   R. L. Cohen, J. D. S. Jones, and G. B. Segal, *Floer's infinite-dimensional Morse theory and homotopy theory*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 297–325. MR 1362832

[CMW]   David Clark, Scott Morrison, and Kevin Walker, *Fixing the functoriality of Khovanov homology*, arXiv:0701339.

[GM95]   J. P. C. Greenlees and J. P. May, *Equivariant stable homotopy theory*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 277–323. MR 1361893

[HKK16]   Po Hu, Daniel Kriz, and Igor Kriz, *Field theories, stable homotopy theory, and Khovanov homology*, Topology Proc. **48** (2016), 327–360. MR 3465966

[Jac04]   Magnus Jacobsson, *An invariant of link cobordisms from Khovanov homology*, Algebr. Geom. Topol. **4** (2004), 1211–1251. MR 2113903

[Kho00]   Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426. MR 1740682

[Kho02]   _____, *A functor-valued invariant of tangles*, Algebr. Geom. Topol. **2** (2002), 665–741. MR 1928174

[Kho03]   _____, *Patterns in knot cohomology. I*, Experiment. Math. **12** (2003), no. 3, 365–374. MR 2034399 (2004m:57022)

[Kho06]   _____, *An invariant of tangle cobordisms*, Trans. Amer. Math. Soc. **358** (2006), no. 1, 315–327. MR 2171235

[KM93]   P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces. I*, Topology **32** (1993), no. 4, 773–826. MR 1241873

[KM11]   _____, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci. (2011), no. 113, 97–208. MR 2805599

[KR08a]   Mikhail Khovanov and Lev Rozansky, *Matrix factorizations and link homology*, Fund. Math. **199** (2008), no. 1, 1–91. MR 2391017

[KR08b]   _____, *Matrix factorizations and link homology. II*, Geom. Topol. **12** (2008), no. 3, 1387–1425. MR 2421131

[Lee05]   Eun Soo Lee, *An endomorphism of the Khovanov invariant*, Adv. Math. **197** (2005), no. 2, 554–586. MR 2173845

[LLS]   Tyler Lawson, Robert Lipshitz, and Sucharit Sarkar, *Khovanov homotopy type, Burnside category, and products*, arXiv:1505.00213.

[LLS17]   _____, *The cube and the Burnside category*, Categorification in geometry, topology, and physics, Contemp. Math., vol. 684, Amer. Math. Soc., Providence, RI, 2017, pp. 63–85. MR 3611723

[LS14a]   Robert Lipshitz and Sucharit Sarkar, *A Khovanov stable homotopy type*, J. Amer. Math. Soc. **27** (2014), no. 4, 983–1042. MR 3230817

[LS14b]   _____, *A refinement of Rasmussen's S-invariant*, Duke Math. J. **163** (2014), no. 5, 923–952. MR 3189434

[LS14c]   _____, *A Steenrod square on Khovanov homology*, J. Topol. **7** (2014), no. 3, 817–848. MR 3252965

[Nao06]   Gad Naot, *The universal Khovanov link homology theory*, Algebr. Geom. Topol. **6** (2006), 1863–1892. MR 2263052

[ORSz13]   Peter S. Ozsváth, Jacob Rasmussen, and Zoltán Szabó, *Odd Khovanov homology*, Algebr. Geom. Topol. **13** (2013), no. 3, 1465–1488. MR 3071132

[OSz04]   Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158. MR 2113019

[OSz05]   _____, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33. MR 2141852

[PS16]   Krzysztof K. Putyra and Alexander N. Shumakovitch, *Knot invariants arising from homological operations on Khovanov homology*, J. Knot Theory Ramifications **25** (2016), no. 3, 1640012, 18. MR 3475079

[Put14]   Krzysztof K. Putyra, *A 2-category of chronological cobordisms and odd Khovanov homology*, Knots in Poland III. Part III, Banach Center Publ., vol. 103, Polish Acad. Sci. Inst. Math., Warsaw, 2014, pp. 291–355. MR 3363817

[Ras10]   Jacob Rasmussen, *Khovanov homology and the slice genus*, Invent. Math. **182** (2010), no. 2, 419–447. MR 2729272

[See]   Cotton Seed, *Computations of the Lipshitz-Sarkar Steenrod square on Khovanov homology*, arXiv:1210.1882.

[SS06]   Paul Seidel and Ivan Smith, *A link invariant from the symplectic geometry of nilpotent slices*, Duke Math. J. **134** (2006), no. 3, 453–514. MR 2254624

[SSSz17]   Sucharit Sarkar, Cotton Seed, and Zoltán Szabó, *A perturbation of the geometric spectral sequence in Khovanov homology*, Quantum Topol. **8** (2017), no. 3, 571–628. MR 3692911

[Tur06]   Paul R. Turner, *Calculating Bar-Natan's characteristic two Khovanov homology*, J. Knot Theory Ramifications **15** (2006), no. 10, 1335–1356. MR 2286127

[Vog73]   Rainer M. Vogt, *Homotopy limits and colimits*, Math. Z. **134** (1973), 11–52. MR 0331376

E-mail address: `sucharit@math.ucla.edu`

Department of Mathematics, University of California, Los Angeles, CA 90095

E-mail address: `cscaduto@scgp.stonybrook.edu`

Simons Center for Geometry and Physics, Stony Brook, NY 11794

E-mail address: `mstoff@mit.edu`

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02142