ON LOCALLY CONSTRUCTIBLE MANIFOLDS

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On Locally Constructible Manifolds

by Bruno Benedetti

To Giulietta Signanini
for teaching me addition

Contents

Chapter 0

Introduction

Ambjørn, Boulatov, Durhuus, Gross, Jonsson, and other physicists have worked to develop a three-dimensional analogue of the simplicial quantum gravity theory, as provided for two dimensions by Regge [122]. (See Ambjørn et al. [5] [7], Loll [98] or Regge–Williams [123] for surveys.) The discretized version of quantum gravity considers simplicial complexes instead of smooth manifolds; the metric properties are artificially introduced by assigning length a to any edge. (This approach is due to Weingarten [142] and known as *Theory of Dynamical Triangulations.*) A crucial path integral over metrics, the "partition function for gravity", is then defined via a weighted sum over all triangulated manifolds of fixed topology. In three dimensions, the whole model is convergent only if the number of triangulated 3-spheres with N facets grows not faster than C^N, for some constant C. But does this hold? How many simplicial spheres are there with N facets, for N large?

Without the restriction to local constructibility this crucial question still represents a major open problem, which was also put into the spotlight by Gromov [61, pp. 156-157]. Its 2D analogue, however, was answered long time ago by Tutte [137] [138], who proved that there are asymptotically fewer than $\left(\frac{16}{3\sqrt{3}}\right)^N$ combinatorial types of triangulated 2-spheres. By Steinitz' theorem, cf. [153, Lect. 4], this quantity equivalently counts the maximal planar maps on $n \geq 4$ vertices, which have $N = 2n - 4$ faces, and also the combinatorial types of 3-dimensional simplicial polytopes with N facets.

Why are 2-spheres "not so many"? Every combinatorial type of simplicial 2-sphere can be generated as follows (Figure 0.1): First for some even $N \geq 4$ build a tree of N triangles (which combinatorially is the same thing as a triangulation of an $(N + 2)$-gon without interior vertices), and then glue edges according to a complete matching of the boundary edges.

According to Jordan's theorem, a necessary condition in order to obtain a 2-sphere is that such a matching is *planar*. (See Figure 0.1 below.) Planar matchings and triangulations of $(N + 2)$-gons are both enumerated by a Catalan number C_{N+2}, and since the Catalan numbers satisfy an exponential bound $C_N = \frac{1}{N+1}\binom{2N}{N} < 4^N$, we get an exponential upper bound for the number of triangulations.

Figure 0.1: LEFT: How to get an octahedron from a tree of eight triangles (i.e., a triangulated decagon). RIGHT: How **not** to get an octahedron: No matter how you complete the matching, the obtained 2-complex will not be a 2-sphere, otherwise the dashed path would form a non-separating loop inside it (a contradiction with Jordan's theorem).

Neither this simple argument nor Tutte's precise count can be easily extended to higher dimensions. While it is still true that there are only exponentially-many "trees of N d-simplices", the matchings that can be used to glue d-spheres are not planar any more.

An observation by Durhuus [49] [50, p. 184] led to a new approach. If our goal is to produce a 2-*sphere* from a tree of polygons, there is no loss of generality in considering only *local gluings*, i.e. pairwise identifications of edges that are *adjacent*. Adjacency is meant as a dynamic requirement: After two edges have been glued together, new pairs of edges become adjacent and may thus be identified. For example, a dice can be constructed via local gluings as shown in Figure 0.2.

The intermediate steps in the gluing process might fail to be polytopal complexes. For example, after the red and the green identifications in Figure 0.2 are performed, what we get is a regular CW complex homeomorphic to
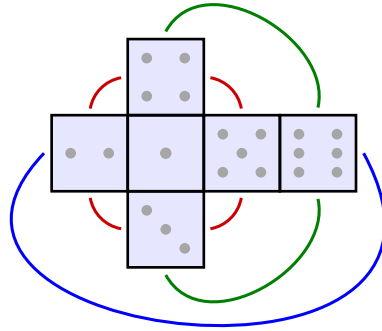
Figure 0.2: A local construction for a dice: Perform red, green and blue identifications in this order.

a ball, but not a polytopal complex. Ignore this difficulty for the moment. Note, however, that the boundary of any tree of polygons is a 1-sphere, and each local gluing (except the last one) does not change the homeomorphism type of the boundary. Therefore, the only closed 2-manifolds that we can produce via local gluings are 2-spheres. Conversely, once we start to perform local gluings in the boundary of a given a tree of polygons, no matter which sequence we choose we will never get stuck. From this it follows that every 2-sphere can be obtained via local gluings from some (actually, *any*) "spanning tree of polygons".

How much of this generalizes to 3-spheres? In 1995, Durhuus and Jonsson [50] introduced the notion of "locally constructible" (LC) 3-manifolds, to describe the manifolds obtainable from a tree of polytopes by identifying pairs of *adjacent* polygons in the boundary. (Of course a boundary triangle has to be identified with another triangle, a square with a square, and so on.) "Adjacent" means here "sharing at least an edge", and represents (as before) a dynamic requirement.

Durhuus and Jonsson [50, Theorem 1] found an exponential upper bound on the number of combinatorially distinct simplicial LC 3-manifolds with N facets. Based also on computer simulations by Ambjørn–Varsted [8] (see also Hamber–Williams [70] and others [2] [4] [6] [33] [38]) they conjectured that *the class of 3-spheres and the class of LC 3-manifolds coincide*.

In fact, they were able to show [50, Theorem 2] one of the two inclusions: *all LC 3-manifolds are spheres*. The idea is a more complicated version of the analogous statement for 2-manifolds (explained before): The boundary of every tree of polytopes is a 2-sphere, and each local gluing either preserves the topology of the boundary, or kills one of its connected components, or pinches the boundary in a vertex, or disconnects the boundary at some pinch

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point. However, each connected component of the boundary stays simply connected (cf. Theorem 1.6.6), so that the closed 3-manifolds that we may produce via local constructions are all simply connected and homeomorphic to the 3-sphere.

A positive solution of the Durhuus–Jonsson conjecture would have implied that there are are at most C^N simplicial 3-spheres with N facets (for a suitable constant C) — which would have been the desired missing link to implement discrete quantum gravity in three dimensions. This drew further attention to the subject and at the same time raised deeper and deeper questions: Are (LC) simplicial 4-spheres exponentially many? Is it possible that all simply connected 3-manifolds are LC? Can we tackle these problems using combinatorial group theory? Compare the following (adapted) quotes by Ambjørn et al. [5, pp. 295–296]

> There is still no asymptotic estimate of the number of non-isomorphic triangulations with a given number of simplices for $d > 2$. In [50] it is proved that the number of triangulations of S^3 is exponentially bounded if a plausible technical assumption holds.

> Progress has been made on related problems of counting so-called ball coverings as well as the counting of possible curvature assignments to a given manifold [14]. Computer simulations support these analytic results [6] [33] and indicate that the number of non-isomorphic triangulations of S^4 is exponentially bounded as a function of the number of 4-dimensional simplices.

by Durhuus–Jonsson themselves [50, p. 191]

> it should be noted that proving the local constructibility of all simply connected simplicial 3-manifolds is a far more ambitious project than proving this for manifolds with the topology of S^3. By Corollary 1.6.7, such a result would imply the Poincaré conjecture.

and by Boulatov [30, p. 21], who gave an incomplete proof that 3-spheres are exponentially many [29] [30]:

> Combinatorial group theory gives a natural mathematical framework and sets up a standard language for physical problems related to lattice models of 3-dimensional quantum gravity. All the formal group constructions with relators and generators have a natural geometrical realization in terms of 2-dimensional complexes (or fake surfaces, in a less formal parlance). And vice versa, geometrical constructions can be formalized in the group theory terms. It would be interesting to find physical models which could be formulated and solved

entirely in terms of abelian presentations. It might be a mathematically adequate way to make physically meaningful approximations.

In 2002, the sensational work by Perelman [117] [118], who managed to prove the Poincaré conjecture, nurtured the hopes in a positive answer to Durhuus–Jonsson's question.

We will show here that the conjecture of Durhuus and Jonsson has in fact a *negative* answer: There are both simplicial and non-simplicial 3-spheres that are not LC. We will also give elementary topological obstructions to local constructibility, using tools from combinatorial group theory, as Boulatov had foreseen.

With this, however, we will not resolve the question whether there are fewer than C^N simplicial 3-spheres on N facets, for some constant C. Via Heegaard splittings, we will link this question to the following concrete geometric problem (cf. Section 2.4): Given a specific triangulation of a handlebody H with N facets, can you complete it with linearly many tetrahedra to a triangulation of a 3-sphere?

0.1 Main results

By LC d-manifolds we mean those obtained from a tree of d-polytopes by repeatedly identifying two adjacent boundary facets. (We assume $d \geq 2$, as we find it vacuous to talk about local constructibility when $d = 0$ or $d = 1$.) One of our first results is the following extension and sharpening of Durhuus–Jonsson's one [50, Theorem 1].

Main Theorem 1 (Theorem 2.5.1). *For fixed $d \geq 2$, the number of combinatorially distinct simplicial LC d-manifolds with N facets grows no faster than $2^{d^2 \cdot N}$.*

Durhuus and Jonsson discussed only the case when $d = 3$ and in addition the produced complexes are simplicial spheres. We will give a proof for Main Theorem 1 in Chapter 2; an analogous upper bound, with the same type of proof, holds for LC non-simplicial d-manifolds if the d-polytopes have a bounded number of facets.

On the contrary, the other main result of Durhuus–Jonsson [50], i.e. "all LC d-manifolds are spheres for $d \leq 3$", does not extend to higher dimensions:

Main Theorem 2 (Corollary 5.2.7). *Any product of LC manifolds is an LC manifold. In particular, some LC 4-manifolds are not 4-spheres.*

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This depends on some facts on product complexes that will be proven in Section 3.3. Most of all, though, Main Theorem 2 relies on the following characterization of LC manifolds, which relates the locally constructible notion defined by physicists to concepts that originally arose in topological combinatorics:

Main Theorem 3 (Theorem 5.2.6). *A closed d-manifold ($d \geq 2$) is LC if and only if the manifold minus a facet can be collapsed down to a complex of dimension $d-2$. Furthermore, there are the following inclusion relations between families of d-spheres ($d \geq 3$):*

$$\{\text{shellable}\} \subseteq \{\text{constructible}\} \subsetneq \{\text{LC}\} \subsetneq \{\text{all } d\text{-spheres}\}.$$

The inclusions all hold with equality for $d = 2$: All 2-spheres are shellable (see Newman [112]). It is not known whether a non-shellable constructible 3-sphere exists. The fact that for each $d \geq 3$ not all d-spheres are locally constructible answers the Durhuus–Jonsson conjecture negatively in all dimensions.

In 1988, Kalai [81] constructed for each $d \geq 4$ a family of more than exponentially many simplicial d-spheres on n vertices; Lee [90] later showed that all of Kalai's spheres are shellable. Combining this with Main Theorem 1 and Main Theorem 3, we obtain the following asymptotic result:

Corollary. *For fixed $d \geq 4$, the number of shellable simplicial d-spheres grows more than exponentially with respect to the number n of vertices, but only exponentially with respect to the number N of facets.*

In general, the asymptotic counts of combinatorial types of spheres according to the number n of vertices and according to the number N of facets are equivalent only for $d = 2$: In fact, by the Lower resp. Upper Bound Theorem for d-spheres (see Kalai [80] resp. Stanley [133]), there are sharp inequalities $l(n) \leq N \leq u(n)$, for some functions $l = \Theta(n)$ and $u = \Theta(n^{\lfloor \frac{d+1}{2} \rfloor})$. (For example, in the case of 3-spheres one has $l(n) = 3n-10$ and $u(n) = \frac{1}{2}n(n-3)$. Note that when $d \geq 3$, the exponent $\lfloor \frac{d+1}{2} \rfloor$ is bigger than one.)

Inspired by finiteness theorems by Cheeger–Grove–Petersen–Wu [39] [62] [63], in 1996 Bartocci et al. [14] focused on d-manifolds of "fluctuating topology" (not necessarily spheres) but "bounded geometry" (curvature and diameter bounded from above, and volume bounded from below). In [14], they obtained an exponential upper bound for the number of simplicial d-manifolds with bounded Grove–Petersen constant (cf. [62, Lemma 3.3]). This constant is the smallest integer C such that, for any ϵ-net (i.e. a

family of open balls of radius ϵ that cover the manifold and would become pairwise disjoint if we halved their radius), the number of radius-ϵ-balls that intersect a given one is at most C. The combinatorial interpretation of this technical condition is unclear. However, Bartocci et al. [14, p. 7] suggested that a possible "translation" for $d = 2$ could be to consider triangulated orientable surfaces with bounded vertex degree.

(In a related paper, Ambjørn et al. studied manifolds with bounded average curvature [3, p. 5]. The combinatorial counterpart seems to be the class of d-manifolds with $\sum_F \deg(F) \leq C \cdot f_{d-2}(M)$, where the sum ranges over all $(d-2)$-faces F of M and $\deg(F)$ counts the number of $(d-1)$-faces containing F. For $d = 2$ the surfaces in this class might have vertices with a high degree: "Bounded vertex degree" is a stronger requirement than "bounded average vertex degree".)

In Chapter 2, we show that focusing on (average) vertex degrees might be misleading: When $d = 2$, the right strategy consists in bounding the genus (which is stronger than bounding the average vertex degree), while bounding the vertex degree is not enough.

Main Theorem 4 (Corollaries 2.3.2 & 5.5.3, Remark 2.3.3). *Simplicial orientable 2-manifolds are more than exponentially many.*
Simplicial orientable 2-manifolds with bounded vertex degree, or bounded average vertex degree, are still more than exponentially many.
However, simplicial orientable 2-manifolds with bounded genus are exponentially many.

In other words, for each $d \geq 2$ the class of d-manifolds is so numerous (measured with respect to the number of facets) that an integral like the partition function for gravity diverges on it. However, for $d = 2$ the topological genus yields a good cut-off: If we integrate only on 2-manifolds with genus bounded by g, and then let g grow, the partition function is not ill-defined. (Obviously every surface has genus bounded by some g, but there is no g such that all surfaces have genus $\leq g$.)

Let us state clearly that this result and its interpretation are not new. The consistency of discrete quantum gravity for $d = 2$ is well-known to all quantum gravity experts, see for example [5] [7]. That said, our proof of Main Theorem 4 is not based on simulations or Monte-Carlo methods; it is entirely combinatorial, and it generalizes to d-manifolds via k-local constructibility, a concept related to a computer science paper [54] by Eğecioğlu and Gonzalez (see Section 5.5). As a matter of fact,

(1) for fixed k and for each $d \geq 3$ there exists a d-manifold (and actually even a d-sphere) that is not k-LC (Corollary 5.5.6);
(2) every d-manifold is k-LC for a suitable $k > 0$;

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(3) for fixed k and d, the number of k-LC d-manifolds with N facets grows exponentially in N.

So the partition function for gravity restricted to k-LC d-manifolds would be well-defined for any fixed k, but unless $d = 2$ there is no k large enough to include all spheres in the k-LC class.

In order to show that not all spheres are neither LC nor k-LC, we study in detail 3-spheres with a "knotted triangle"; these are obtained by adding a cone over the boundary of a ball with a knotted spanning edge. This is an old trick in combinatorial topology, dating back to Furch's 1924 paper [57, p. 73] and rediscovered by Bing in 1959 [21, p. 110]; we will explain it in Chapter 4, providing also some background notions in knot theory.

Spheres with a knotted triangle cannot be boundaries of polytopes. Lickorish [93] showed in 1991 that

> a 3-sphere with a knotted triangle is not shellable if the knot is
> at least 3-complicated.

Here "at least 3-complicated" refers to the technical requirement that the fundamental group of the complement of the knot has no presentation with less than four generators. A concatenation of three or more trefoil knots satisfies this condition. In 2000, Hachimori and Ziegler [65] [69] demonstrated that Lickorish's technical requirement is not necessary for his result:

> a 3-sphere with any knotted triangle is not constructible.

We re-justify Lickorish's technical assumption, showing that this is exactly what is needed if we are to reach a stronger conclusion, namely, a topological obstruction to local constructibility. Thus, the following result is established in order to prove that the last inclusion of the hierarchy in Main Theorem 3 is strict.

Main Theorem 5 (Theorem 5.3.3 and Cor. 5.5.5). *A 3-sphere with a knotted triangle is not LC if the knot is at least 3-complicated.*
More generally, the $(d-3)$-rd suspension of a 3-sphere with a triangular knot in its 1-skeleton is a d-sphere that

– *cannot be LC if the knot is at least $3 \cdot 2^{d-3}$-complicated, and*
– *cannot be k-LC, if the knot is at least $(3 \cdot 2^{d-3} + k)$-complicated.*

The requirement about knot complexity is now necessary, as non-constructible spheres with a single trefoil knot can still be LC (see Theorem 5.3.7). Also, in order to derive Main Theorem 5 we had to strengthen and generalize Lickorish's result: See Theorems 3.5.1 and 5.5.4.

We point out that the presence of some knot in the 1-skeleton can be realized as a *local* property, so the number of knotted spheres (resp. knotted balls) with N facets has the same asymptotic growth as the global number of spheres (resp. balls). So, in some sense, most of the 3-spheres are knotted.

The combinatorial topology of d-balls and that of d-spheres are of course closely related. In Chapter 6, we adapted our methods to manifolds with boundary:

Main Theorem 6 (Theorems 6.1.9 & 6.0.1, Lemmas 1.6.3 & 5.1.1).
A d-manifold with boundary (d ≥ 2) is LC if and only if after the removal of a facet it collapses down to the union of the boundary with a complex of dimension at most d − 2. Furthermore, there are the following inclusion relations between families of d-manifolds with boundary (d ≥ 3):

$$\{\text{shellable}\} \subsetneq \{\text{constructible}\} \subsetneq \{\text{LC}\} \subsetneq \left\{ \begin{array}{c} \text{all simply connected} \\ d\text{-manifolds with boundary} \end{array} \right\}.$$

In particular, for each d ≥ 3 we have the following hierarchy for d-balls:

$$\{\text{shellable}\} \subsetneq \{\text{constr.}\} \subsetneq \{\text{LC}\} \subsetneq \left\{ \begin{array}{c} \text{collapsible onto a} \\ (d-2)\text{-complex} \end{array} \right\} \subsetneq \{\text{all } d\text{-balls}\}.$$

Again, the 2-dimensional case is much simpler and had been completely solved quite some time ago: All simply connected 2-manifolds with boundary are 2-balls (or 2-spheres, if the boundary is empty), and all 2-balls and 2-spheres are shellable [113].

When $d \leq 3$, collapsibility onto a $(d-2)$-complex is equivalent to collapsibility. Thus Main Theorem 6 settles the question by Hachimori [66, pp. 54, 66] of whether all constructible 3-balls are collapsible. Furthermore, we show in Corollary 6.3.7 that some collapsible 3-balls do not collapse onto their boundary minus a facet, a property that comes up in classical studies in combinatorial topology (see e.g. Chillingworth [40] or Lickorish [95]). In particular, a result of Chillingworth can be rephrased as "if for any geometric simplicial complex Δ the support (union) $|\Delta|$ is a convex 3-dimensional polytope, then Δ is necessarily an LC 3-ball"; see Theorem 6.3.10. Hence, any geometric subdivision of the 3-simplex is LC.

0.2   Where to find what

We divided the material according to the topics it relates to, and not according to the chronological order of discovery. So the first chapter already contains new results, whereas the last chapter still contains elementary definitions. In case you find this too chaotic, we hope these guidelines will help you.

Chapter 1 collects most of the basic definitions, as well as the (easy and not so easy) results that directly follow from them. For example, in Chapter 1 we prove that local constructibility is maintained under taking cones or barycentric subdivisions. In a subsection called "Operations on complexes", we also explain the meaning of expressions like "coning off the boundary", "suspending" or "taking links", all of which occur frequently in the combinatorial topology literature. Don't miss the definition of local constructibility in Section 1.6.

Chapter 2 contains all the asymptotic enumeration results, with the exception of the count of bounded genus surfaces, which will be presented in Section 5.5. Indeed, it is in Chapter 2 that we demonstrate why surfaces are more than exponentially many, while LC d-manifolds are only exponentially many for each $d \geq 2$.

Chapter 3 collects all we know about collapses. We present this classical notion from a new perspective, focusing on how many dimensions down one can get via collapsing sequences. (Such integer will be called "collapse depth".) These results may seem a little abstruse and plethoric, but they are all needed to prove that neither all d-spheres are LC (for each $d \geq 3$), nor all LC d-manifolds are spheres (for each $d \geq 4$).

Chapter 4 briefly explains what knots are, and why they might show up inside a finely triangulated 3-ball (or 3-sphere); the main focus is on knots as obstructions to collapsibility and shellability.

Chapter 5 contains all our main results on LC spheres. All the notions introduced and discussed in the previous chapters converge into Main Theorem 3 and our hierarchy for 3-spheres (Theorem 5.3.12). If instead what you are looking for is a result on LC balls, or more generally on manifolds with boundary, look into Chapter 6.

The authorship of theorems, propositions etc. is usually displayed within the claim, like for example **Theorem 6.3.10 (Chillingworth [40]).** Some results have no explicit authorship, but are announced as "well known" in the text preceding them. The remaining results with no author displayed are to be understood as new, in the sense that either they appeared in the preprint [18], possibly in slightly more specific formulations, or they appear here for the first time, as far as we know. The paper [18] is joint work with Günter Ziegler, who is also the present thesis' advisor. Finally, subsections 5.4.1 and 5.4.2 are joint work with Frank Lutz.

0.3 Acknowledgements

Chapter 1

Getting started

1.1 Polytopal complexes

A *polytope P* is the convex hull of a finite set of points in some $\mathbb{R}^k$. A *face* of P is any set of the form $F = P \cap \{\mathbf{x} \in \mathbb{R}^k : \mathbf{c} \cdot \mathbf{x} = c_0\}$, provided $\mathbf{c} \cdot \mathbf{x} \leq c_0$ is a linear inequality satisfied by all points of P. The *dimension* of a face is the dimension of its affine hull. Taking $\mathbf{c} = \mathbf{0}$ and $c_0 = 0$ in the definition above, we see that P is a face of itself; all other faces of P all called *proper*.

A *polytopal complex* is a finite, nonempty collection C of polytopes (called the *faces* of C) in some Euclidean space $\mathbb{R}^k$, such that:

1. if σ is in C then all the faces of the polytope σ are elements of C;
2. the intersection of any two polytopes σ and τ of C is a face of both σ and τ.

A polytopal complex C is called *simplicial complex* if all of its facets are simplices.

Given a polytopal complex C, the *face poset* of C is the finite set of all polytopes in C, ordered by inclusion. Two polytopal complexes are *(combinatorially) equivalent* if the respective face posets are isomorphic.

Conventionally, the inclusion-maximal faces of a d-complex are called *facets*, and the inclusion-maximal proper subfaces of the facets are called *ridges*. The *k-faces* are the faces of dimension k; the 0-faces are called *ver*-

tices, and the 1-faces *edges*[1]. The *dimension* of C is the largest dimension of a polytope of C; *d-complex* is just a shortening for d-dimensional polytopal complex. The k-skeleton of a d-complex C ($k \leq d$) is the k-complex of all the polytopes of C that have dimension at most k.
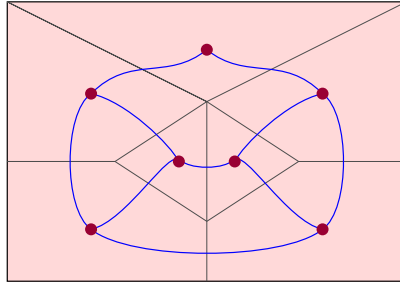
Figure 1.1: The dual graph.

Pure complexes are complexes where all facets have the same dimension. The *dual graph* of a pure d-complex C is a graph whose vertices are the facets of C; two vertices of the graph are connected by an edge if and only if the corresponding facets of C have a common ridge. (See Fig. 1.1.)

A pure d-complex is *strongly connected* if its dual graph is connected[2].

The *underlying space* $|C|$ of a polytopal complex C is the union of all its faces. Conversely, if C is a simplicial complex, C is called *triangulation* of $|C|$ (and of any topological space homeomorphic to $|C|$). A *d-sphere* is a d-complex whose underlying space is homeomorphic to $\{\mathbf{x} \in \mathbb{R}^{d+1} : |\mathbf{x}| = 1\}$. Similarly, a *d-ball* is a complex whose underlying space is homeomorphic to $\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \leq 1\}$.

With a little abuse of notation, we will call *d-manifold* (resp. *d-manifold with boundary*) any d-dimensional polytopal complex whose underlying space is homeomorphic to a topological manifold (resp. to a topological manifold with boundary). All d-manifolds are pure and strongly connected[3]. Fur-

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[1]In quantum gravity literature, the edges are usually called *links*; we will refrain from this notation, since the word "*link*" has a different meaning in combinatorial topology, namely, a "localization" of a complex at a given face. For the same reason, unlike most knot theory studies, we will not use the word "link" to denote a disjoint union of knots.

[2]Some authors prefer to write "connected in codimension one" instead of "strongly connected", especially in view of the connection with Commutative Algebra established via Stanley-Reisner rings. See Stanley [134].

[3]In the combinatorial topology literature, a "strongly connected d-complex such that every ridge lies in exactly two facets" is often called a "pseudomanifold". We will refrain from this notation since we have a different use in mind for the word "pseudomanifold": something that might be disconnected, too. (See Section 1.6.)

thermore, every ridge in a manifold lies in at most two facets; the boundary consists precisely of those ridges that lie in one facet only.

A *tree of d-polytopes* is a *d*-manifold with boundary whose dual graph is a tree. Every tree of *d*-polytopes is a *d*-ball, but some *d*-balls are not trees of polytopes (for example, balls with interior vertices). A *stacked d-sphere* is any simplicial sphere which is combinatorially equivalent to the boundary of a tree of $(d+1)$-simplices.

Let P be a *d*-dimensional polytope. For each $k \in \{0, \ldots, d\}$ and for each *k*-dimensional face F_i of P, denote by b_i the barycenter of F_i. For example, the barycenter of an edge is its midpoint, while the barycenter of a vertex is the vertex itself.
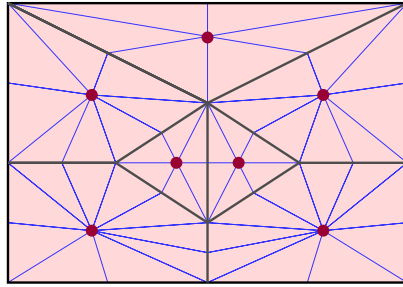
Figure 1.2: Barycentric subdivision (in blue).

Definition 1.1.1 (Barycentric subdivision)**.** The *barycentric subdivision* $sd(P)$ of a polytope P is the simplicial complex described as follows:

–   the vertices of $sd(P)$ are the barycenters of all the faces of P;
–   the facets of $sd(P)$ are the convex hulls of $(d+1)$ barycenters $b_0, \ldots, b_d$ whose corresponding faces F_i form a *flag* $F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_d = P$.

The *barycentric subdivision* $sd(C)$ of a polytopal complex C is the simplicial complex obtained by subdividing each polytope of C barycentrically.

The underlying spaces of C and $sd(C)$ are the same, but $sd(C)$ is *simplicial* even when C is not. Bayer [15] showed that if P and Q are polytopes with combinatorially equivalent barycentric subdivisions, then P is combinatorially equivalent either to Q or to Q^*, where Q^* is the *dual polytope* of Q (also known as "polar polytope": see Ziegler [153, Sect. 2.3]). Surprisingly, this holds no more for regular CW complexes (see Bayer [15, p. 7]).

1.2 PL manifolds

Let C be a simplicial complex, and σ a face of C. The *closed star* of σ is the subcomplex of all faces containing σ, together with their faces. The *link* of a face σ is the subcomplex $\mathrm{link}_C \, \sigma$ of C consisting of the simplices that are disjoint from σ but contained in a face that contains σ. For example, the link of any vertex in a 2-sphere is a 1-sphere. The link of any edge in a 2-sphere is a disjoint union of two points (in other words, a 0-sphere).

Remark 1.2.1. The definition of "link" above creates some difficulties in the non-simplicial case. Following the notation of Courdurier [43], for polytopal complexes we will distinguish between

  (i) the *link* of σ, i.e. the subcomplex of all the faces disjoint from σ but contained in a face that contains σ, and
  (ii) the *spherical link* of σ, i.e. any polytopal complex whose face poset is isomorphic to the upper ideal of σ in the face poset of C.

In simplicial complexes, the two notions above coincide; in addition, the facets of the link are in 1–1 correspondence with the facets of the closed star.

Definition 1.2.2 (PL)**.** A d-sphere is called *PL* if it is piecewise-linearly homeomorphic to the boundary of a $(d+1)$-simplex; a d-ball is *PL* if it is piecewise-linearly homeomorphic to a d-simplex. *PL manifolds* (with or without boundary) are usually defined as follows:

  – all simplicial 1-spheres and 1-balls are PL;
  – a simplicial d-manifold with boundary is PL if and only if the links of its vertices are either PL $(d-1)$-spheres, or PL $(d-1)$-balls;
  – a non-simplicial d-manifold with boundary is called PL if its barycentric subdivision is PL.

The definition is non-ambiguous for all $d \ne 4$ [26, p. 5] [104, pp. 9-11]. When $d = 4$, non-ambiguity is an open problem: a priori, there might be a 4-sphere S that is not piecewise-linearly homeomorphic to the boundary of a 5-simplex, so that all vertex links in S are piecewise-linearly homeomorphic to the boundary of a 4-simplex. To be on the safe side, we will call a simplicial 4-sphere "PL as manifold" when its links are PL 3-spheres and "PL" if it is also piecewise linearly homeomorphic to the boundary of a 5-simplex.

Are all spheres PL? This classical question (see e.g. [45]) was solved negatively in 1975 by Edwards [51], who showed that the double suspension of the Mazur "homology 3-sphere" is a non-PL 5-sphere. Cannon [35]

later generalized Edwards' result proving that the double suspension of any "homology d-sphere" is a non-PL $(d+2)$-sphere; more recently Björner and Lutz [26] [27] found a triangulation of a non-PL simplicial 5-sphere with 18 vertices and 261 facets. Every triangulated d-sphere with less than $d+6$ vertices is PL by [13].

While non-PL d-spheres exist for each $d \geq 5$, it is known that all d-spheres are PL for each $d \leq 3$. Whether all 4-spheres are PL is still an open question [104, pp. 9–11]. However, all 4-manifolds are "PL as manifolds" [34, p. 10] in virtue of the Poincaré conjecture, which was recently proven by Perelman [117] [118] [88] [36] [107] [108].

Given two disjoint simplices α and β, the join $\alpha * \beta$ is a simplex whose vertices are the vertices of α plus the vertices of β. By convention, $\emptyset * \beta$ is β itself. The *join* of two simplicial complexes A and B is defined as $A * B := \{\alpha * \beta : \alpha \in A, \ \beta \in B\}$. We quote from Lickorish [94, p. 380] adapting the notation:

> Piecewise linear topology is always dominated by the idea of a join. Suppose a $(d+1)$-simplex C is regarded as the join of two of its disjoint faces A and B. Then $\dim A + \dim B = d$ and
>
> $$\partial C = \partial(A * B) = (\partial A * B) \cup (A * \partial B).$$
>
> Assuming the pair (A, B) is ordered, this gives, up to isomorphism, $d+1$ ways of expressing the standard triangulation of the d-sphere (as the boundary of a $(d+1)$-simplex) as the union of two triangulations of d-balls glued along their boundaries. Suppose that A is an r-simplex in a triangulated PL d-manifold K and that $\text{link}_K A = \partial B$ for some $(d-r)$-simplex $B \notin K$. Note that B is a *new* simplex not seen in K. Ignore the fact that this condition may seem unlikely. The ***bistellar move*** $\chi(A, B)$ consists of changing K to $\tilde{K}$ by removing $A * \partial B$ and inserting $\partial A * B$.
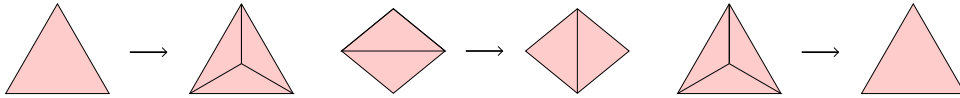
Figure 1.3: The three types of bistellar move in dimension two. Note that the move $\chi(A, B)$ has $\chi(B, A)$ as an inverse.

Bistellar moves are also known as *bistellar flips* or *Pachner moves*. Each two-dimensional Pachner move can be viewed as a switch from the top view to the bottom view of a given tetrahedron. Analogously, the four three-dimensional Pachner moves correspond to top view/bottom view switches in the visualization of a 4-simplex (cf. Eppstein [19]).

Pachner moves are named after Udo Pachner, who showed that these $d + 1$ operations suffice for moving from one triangulated manifold with boundary to any PL-homeomorphic triangulation of it [115] [116] (see also [37]). In other words, Pachner moves are *local* and *ergodic*:

– a simplicial d-sphere is PL if and only if it can be obtained from the boundary of the $(d+1)$-simplex via a finite sequence of Pachner moves;
– a simplicial d-ball is PL if and only if it can be obtained from the d-simplex via a finite sequence of Pachner moves.

If we start with a PL d-sphere S, a helter-skelter sequence of Pachner moves might not result in the boundary of the $(d+1)$-simplex. (For example, a Pachner move followed by its inverse leaves S unchanged.) However, suppose that *no* sequence of m moves transforms S into the boundary of a $(d + 1)$-simplex: If m is sufficiently large, can we conclude that S is *not* PL? The answer is

- "yes" for $d = 3$ and m greater than $6 \cdot 10^6 \cdot N^2 \cdot 2^{5 \cdot 10^4 \cdot N^2}$, by the work of Mijatovic [103] (see also King [86]);
- unknown for $d = 4$ (even if the expected answer is negative: see Nabutovski–Ben Av [111]);
- negative for $d \geq 5$: A deep result of Novikov [140] (see also Stillwell [136]) states that PL d-spheres are not algorithmically recognizable for any $d \geq 5$.

In the terminology of quantum gravity papers (e.g. [32]), this translates into saying that Pachner moves are *ergodic* for all d, but *finitely ergodic* only for $d \leq 3$.

Pachner moves are specific for *simplicial* PL manifolds. However, there is an analogous set of moves for d-dimensional *cubical* PL manifolds, corresponding to switching from the top view to the bottom view of a $(d+1)$-cube (cf. Bern et al. [19]).

1.3   Shellability and constructibility

Definition 1.3.1 (Shellability [153, p. 233])**.** Let C be a pure d-dimensional polytopal complex, $d \geq 1$. A *shelling* of C is a linear ordering $F_1, \ldots, F_s$ of the facets of C, so that for each $i \in \{2, \ldots, s\}$ the polytopal complex $F_i \cap \bigcup_{j=1}^{i-1} F_j$ is pure $(d-1)$-dimensional and yields a beginning segment for a shelling of the boundary ∂F_i of F_i.

A pure polytopal d-complex C is *shellable* if it has a shelling, or if it is 0-dimensional.

Definition 1.3.1 can be simplified for simplicial or cubical complexes, because d-simplices and d-cubes are "extendably shellable" [46, p. 37] [153, pp. 235–236]. If the intersection of F_j with the previous facets is shellable, it yields automatically a beginning segment for a shelling of ∂F_j.

Also, any pure $(d-1)$-dimensional subcomplex of the boundary of a d-simplex is necessarily connected, strongly connected and shellable. (On the contrary, a pure subcomplex of the d-cube need not be connected: Cubes contain pairs of disjoint facets.) So, for pure simplicial complexes, shellability can be characterized as follows:

– every simplex is shellable;
– a d-dimensional pure simplicial complex C, different from a simplex, is shellable if and only if it can be written as $C = C_1 \cup C_2$, where C_1 is a shellable simplicial d-complex, C_2 is a d-simplex, and $C_1 \cap C_2$ is a pure $(d-1)$-complex.

For a short history of the shellability notion, see Ziegler [154]. One of the most celebrated result is certainly Bruggesser and Mani's theorem (see [153, Lect. 8]), which says that the boundary of any $(d+1)$-polytope is a shellable d-sphere.

Constructibility is a weakening of shellability, defined by:

– every simplex is constructible;
– a d-dimensional pure simplicial complex C (different than a simplex) is constructible if and only if it can be written as $C = C_1 \cup C_2$, where C_1 and C_2 are constructible simplicial d-complexes, and $C_1 \cap C_2$ is a constructible simplicial $(d-1)$-complex.

This notion was introduced in 1972 by Hochster [75], in connection with notions from commutative algebra ("constructible complexes are Cohen–Macaulay"), but it had been implicitly used long before by combinatorial topologists [101, p. 103]. For example, Zeeman's work [152] contains a proof of the fact that any constructible polytopal d-complex C such that each ridge of C belongs to at most two facets is either a PL d-ball or a PL d-sphere. Also, Lickorish's two-page paper from 1970 [92] shows that some 3-balls do not have embedded 2-discs, thus proving that not all 3-balls are constructible.

In 1960 Curtis and Zeeman [45] conjectured the existence of non-PL 5-spheres; the conjecture was proven by Edwards in 1975 [51]. Since non-PL spheres are a subclass of non-constructible spheres, from Edwards' work it follows that for all $d \geq 5$ some d-spheres are not constructible. Recently Hachimori [65] extended this claim to $d \geq 3$: In fact, arbitrary 3-spheres may contain a knot, while constructible 3-spheres may not [69].

The following examples show that shellability is strictly stronger than constructibility:

Example 1.3.2 (Ziegler's 3-ball [154]). Let B_1 be the 3-ball on the vertices $1, \ldots, 8$ given by the seven facets

$$\{1,3,4,7\} \quad \{1,4,5,7\} \quad \{2,3,4,8\} \quad \{2,3,6,8\} \quad \{3,4,7,8\}$$
$$\{3,6,7,8\} \quad \{4,5,7,8\}.$$

Let B be the 3-ball on ten vertices (labelled by $0, \ldots, 9$) given by the twenty-one facets

$$\{1,3,4,7\} \quad \{1,4,5,7\} \quad \{2,3,4,8\} \quad \{2,3,6,8\} \quad \{3,4,7,8\}$$
$$\{3,6,7,8\} \quad \{4,5,7,8\} \quad \{0,1,2,3\} \quad \{0,1,2,5\} \quad \{0,2,3,7\}$$
$$\{0,2,5,6\} \quad \{0,2,6,7\} \quad \{1,2,3,4\} \quad \{1,2,4,9\} \quad \{1,2,5,6\}$$
$$\{1,2,6,9\} \quad \{1,4,5,8\} \quad \{1,4,8,9\} \quad \{1,5,6,9\} \quad \{1,5,8,9\}$$
$$\{2,3,6,7\}.$$

We can see B_1 as a subcomplex of B. Let B_2 be the closure of $B - B_1$, that is, the smallest subcomplex of B that contains all faces of B not in B_1. It is known [154] that:

 – both B_1 and B_2 are shellable 3-balls;
 – the intersection $B_1 \cap B_2$ is a 2-ball, hence B is constructible;
 – B is not shellable.

Example 1.3.3 (Lutz's 3-ball [99]). Let B be the 3-ball on nine vertices (labelled by $0, \ldots, 8$) given by the eighteen facets

$$\{0,1,2,3\} \quad \{0,1,2,4\} \quad \{0,1,4,5\} \quad \{0,1,5,7\} \quad \{0,1,6,8\}$$
$$\{0,1,7,8\} \quad \{0,2,3,4\} \quad \{0,6,7,8\} \quad \{1,2,3,6\} \quad \{1,2,4,5\}$$
$$\{1,2,5,8\} \quad \{1,2,6,8\} \quad \{1,5,7,8\} \quad \{2,3,4,7\} \quad \{2,3,6,7\}$$
$$\{2,4,6,7\} \quad \{2,4,6,8\} \quad \{4,6,7,8\}.$$

This B is constructible, but not shellable. It is in fact a "vertex-minimal" example of a non-shellable ball: In fact, all 3-balls with less than nine vertices are shellable [99].

Example 1.3.4 (Rudin's 3-ball). The following 14 points lie on the boundary of a tetrahedron in $\mathbb{R}^3$.

$$X_1 = (0,0,0) \qquad X_2 = (0.5, 0.866, 0)$$
$$X_3 = (1,0,0) \qquad X_4 = (0.5, 0.289, 0.75)$$
$$Y_1 = (0.2245, 0.019941, 0.05175) \quad Y_2 = (0.4655, 0.696616, 0.1425)$$
$$Y_3 = (0.7755, 0.059754, 0) \qquad Y_4 = (0.5345, 0.378689, 0.55575)$$
$$U_1 = (0.5, 0, 0) \qquad U_2 = (0.5, 0.5775, 0.375)$$

Rudin's 3-ball is the simplicial ball whose vertices are the previous 14 points. This ball is non-shellable [126] but constructible [121]; see also Wotzlaw [149].

A very recent paper in Communications in Algebra [114] ends with the following problem: Are there many examples of constructible simplicial complexes that are not shellable?

The following simple observation (cf. Hachimori [65, Lemma 1]) will enable the reader to answer the question positively. If we glue together two simplicial 3-balls B_1 and B_2 alongside a triangle in their boundary, then $B_1 \cup B_2$ is constructible if and only if both the B_i's are constructible, while $B_1 \cup B_2$ is shellable only if the B_i's are both shellable. So *every* shellable 3-ball with $N - 18$ facets, when glued onto Lutz's 3-ball alongside a boundary triangle, yields a constructible non-shellable 3-ball with N facets. (In Chapter 2 we will see that there are exponentially many shellable 3-balls with $N - 18$ facets, for N large. So counting with respect to the number N of facets, there are exponentially many examples of constructible non-shellable 3-balls.)

Surprisingly, it is still not known whether a constructible non-shellable 3-sphere exists; see Kamei [82] for a survey of the attempts done so far. Examples of non-shellable spheres were obtained with different methods by Armentrout [9], Hachimori [66], Lickorish [84] [93] and Vince [139]. The first three examples are not constructible. Chronologically, the first example of a non-shellable ball dates back to 1924 and is due to Furch [57, p. 73]. Recently Hachimori proved that Furch's ball is not constructible [65]; we will prove an even stronger statement in Example 6.3.4.

Shellability and constructibility are not topological properties, for a shellable complex can be homeomorphic to a non-shellable one. However, when a simplicial complex is shellable (resp. constructible), its barycentric subdivision is also shellable (resp. constructible). Provan and Billera [121, Theorem 3.3.1] showed that the barycentric subdivision of every shellable simplicial complex is even vertex decomposable (see Paragraph 1.4 for the definition; see also Björner and Wachs [28, p. 3967]).

Bruggesser and Mani [31, p. 200] proved that every d-ball or d-sphere becomes shellable, and thus also constructible, after performing sufficiently many barycentric subdivisions. However, for each $d \geq 2$, there are constructible d-complexes that remain non-shellable even after performing arbitrarily many barycentric subdivisions [67, p. 2310]. Also, there is no integer r such that the r-th subdivision of every 3-sphere is shellable [60]. An analogous result holds for d-spheres or d-balls as well: See Kearton–Lickorish's revision [84] of the work by Goodrick [60].

1.4 Vertex-decomposability

Let C be a simplicial complex and let v be one of its vertices. The *deletion* $\mathrm{del}_C\, v$ is the subcomplex of C formed by all the faces of C that do not contain the vertex v. A pure simplicial d-complex C is *vertex decomposable* if either $d = 0$, or C is a simplex, or there is a vertex v of C such that

(1) $\mathrm{link}_C\, v$ is $(d-1)$-dimensional and vertex decomposable, and
(2) $\mathrm{del}_C\, v$ is d-dimensional and vertex decomposable.

The notion of vertex decomposability was introduced by Provan and Billera in their proof of 1980 [121] that vertex decomposable simplicial complexes satisfy the famous Hirsch conjecture from linear programming, which states that the diameter of the dual graph of a pure simplicial d-complex with n vertices is bounded above by $n - (d+1)$. (At present, the conjecture is still open: see e.g. [153, Chapter 3].) All vertex decomposable complexes are shellable [121]: this follows (by induction on the dimension and the number of facets) from the following well known fact:

Lemma 1.4.1. *Let v be a vertex of a simplicial complex C. If $\mathrm{link}_C\, v$ and $\mathrm{del}_C\, v$ are both shellable, then C is shellable too.*

Whether an analogous Lemma holds true for polytopal complexes as well, is still an open question; the expected answer is a negative one, which is why Ehrenborg and Hachimori wrote [52, p. 475] that

> shellability and constructibility naturally extend to polytopal complexes, whereas vertex decomposability only applies to simplicial complexes.

Provan and Billera [121, Theorem 3.3.1] (see also [28, p. 3967]) showed that the barycentric subdivision of every shellable (simplicial) complex is vertex decomposable. This means that every sphere or ball becomes vertex decomposable, after performing sufficiently many barycentric subdivisions. Provan and Billera also showed that all simplicial 2-balls and 2-spheres are vertex decomposable. For $d \geq 3$, Klee and Kleinschmidt [87] proved that all simplicial d-balls (resp. d-spheres) with at most $d+3$ (resp. $d+4$) vertices are vertex-decomposable.

However, not all shellable 3-spheres are vertex decomposable: Lockeberg [97] [87] (see also [66, p. 26] for the correction of a typo) gave an example of a simplicial 4-polytope with a non-vertex-decomposable boundary. Since the boundary of a polytope can be shelled starting at the star of any vertex [153, Corollary 8.13], the deletion of any vertex from the boundary of Lockeberg's polytope yields a simplicial shellable 3-ball which is not vertex-

decomposable. A vertex-minimal example of non-vertex-decomposable shellable 3-ball is given in [100].

Further obstructions to vertex decomposability arise from knot theory: See the work of Hachimori–Ziegler [69], or wait for Section 4.4.

1.5 Regular CW complexes

Björner [23] showed how to extend the notion of shellability to finite pure regular CW complexes; Hachimori and Shimokawa [68] did the same with constructibility. Let us introduce some notation (cf. Hatcher [72, p. 519]).

A *d-disk* $\bar{e}^d_\alpha$ is a topological space homeomorphic to the closed unit ball in $\mathbb{R}^d$. The part of $\bar{e}^d_\alpha$ homeomorphic to the unit sphere in $\mathbb{R}^d$ under the previous homeomorphism is called *boundary* of the disk; its removal from $\bar{e}^d_\alpha$ yields an *open d-disk*, denoted e^d_α.

Definition 1.5.1 (CW complex). A (finite)[4] *CW complex* is a space constructed via the following procedure:

(1) start with a set X^0 of n points, the so-called 0-*cells*;
(2) recursively, form the *d-skeleton* X^d by *attaching* open *d*-disks e^d_α (called *d-cells*) onto X^{d-1}, via maps

$$\varphi_\alpha \ : \ S^{d-1} \longrightarrow X^{d-1};$$

the word "attaching" means that X^d is the quotient space of the disjoint union $X^{d-1} \sqcup_\alpha \bar{e}^d_\alpha$ under the identifications $x \equiv \varphi_\alpha(x)$, for each x in the boundary of $\bar{e}^d_\alpha$;
(3) stop the inductive process at a finite stage, setting $X = X^d$ for some d (called the *dimension* of X).

A CW complex is *regular* if the attaching maps for the cells are injective (see e.g. Björner [24]). A regular CW-complex is *simplicial* if for every proper face F, the interval $[0, F]$ in the face poset of the complex is *boolean* (i.e. isomorphic to the poset $B_k := \left(2^{[k]}, \subseteq\right)$ of all subsets of a k-element set, for some k).

Every polytopal complex is a regular CW-complex; every simplicial complex (and in particular, any triangulated manifold) is a simplicial regular CW-complex.

The k-dimensional cells of a regular CW complex C are called *k-faces*; the inclusion-maximal faces are called *facets*, and the inclusion-maximal proper subfaces of the facets are called *ridges*. Conventionally, the 0-faces

---

[4]All complexes that we consider are finite, therefore also finite-dimensional.

are called *vertices*, and the 1-faces *edges*. The *dimension* of C is the largest dimension of a facet; *pure* complexes are complexes where all facets have the same dimension.

Let C be a pure regular CW-complex of dimension $d \geq 1$. A *shelling* of C is a linear ordering $F_1, \ldots, F_s$ of the facets of C, such that:

(1) the boundary ∂F_1 of F_1 is shellable;
(2) for each $i \in \{2, \ldots, s\}$, the CW complex $F_i \cap \bigcup_{j=1}^{i-1} F_j$ is pure, regular, $(d-1)$-dimensional, and it is also a beginning segment for a shelling of the boundary ∂F_i of F_i.

A pure CW complex C is *shellable* if either C is 0-dimensional, or C has a shelling.

A d-dimensional pure regular CW complex is *constructible* if:

– either $d = 0$, or
– it consists of only one facet whose boundary is constructible, or
– it splits into the union of two d-dimensional constructible subcomplexes C_1 and C_2, such that the intersection $C_1 \cap C_2$ is a $(d-1)$-dimensional constructible CW complex.

All shellable CW complexes are constructible [68]. Furthermore, if C is a constructible regular CW complex and if each ridge of C lies in at most two facets of C, then by Zeeman's work [152] C is homeomorphic either to a PL ball or to a PL sphere. A partial converse of this theorem is given by Newman's result [113], according to which every regular CW complex homeomorphic to a 2-ball or to a 2-sphere must be shellable and in particular constructible. Newman's claim is best possible, since 3-balls (and 3-spheres) might be non-shellable and non-constructible.

1.6   Local constructibility

Definition 1.6.1 (Pseudomanifold)**.** By a *d-pseudomanifold* we mean a finite regular CW-complex P that is d-dimensional, pure, and such that each $(d-1)$-dimensional cell belongs to at most two d-cells. A d-pseudomanifold is *simplicial* if it is simplicial as CW complex, that is, if all its facets are d-simplices. The *boundary* of the pseudomanifold P, denoted ∂P, is the smallest subcomplex of P (possibly empty) containing all the $(d-1)$-cells of P that belong to exactly one d-cell of P.

According to our definition, a pseudomanifold need not be a polytopal complex; it might be disconnected; and its boundary might not be a pseu-

domanifold (compare Lemma 1.6.4). Every d-manifold with boundary is also a d-pseudomanifold.

Definition 1.6.2 (Locally constructible pseudomanifold). For $d \geq 2$, let C be a pure d-dimensional polytopal complex with N facets. A *local construction* for C is a sequence $T_1, T_2, \ldots, T_N, T_{N+1}, \ldots, T_k$ ($k \geq N$) such that T_i is a d-pseudomanifold for each i and

(1) T_1 is a d-dimensional polytope;
(2) if $i \leq N - 1$, then T_{i+1} is obtained from T_i by gluing a new d-polytope to T_i alongside one of the $(d-1)$-cells in ∂T_i;
(3) if $i \geq N$, then T_{i+1} is obtained from T_i by identifying a pair σ, τ of combinatorially equivalent $(d-1)$-cells in the boundary ∂T_i, provided the intersection of σ and τ contains at least a $(d-2)$-cell;
(4) $T_k = C$.

We say that C is *locally constructible*, or *LC*, if a local construction for C exists. With a little abuse of notation, we will call each T_i an *LC pseudomanifold*. We also say that C is locally constructed *along T*, if T is the dual graph of T_N, and thus a spanning tree of the dual graph of C.

The identifications described in item (3) above (called *local gluings*) are operations that are not closed with respect to the class of polytopal complexes. Local constructions where all steps are polytopal complexes produce only a very limited class of pseudomanifolds, consisting of d-balls with no interior $(d-3)$-faces. (When in an LC step the identified boundary facets intersect in *exactly* a $(d-2)$-cell, no $(d-3)$-face is sunk into the interior, and the topology stays the same; compare Lemma 6.3.1.)

However, since by definition the local construction in the end must arrive at a pseudomanifold C that *is* a polytopal complex, each intermediate step T_i must satisfy severe restrictions: for each $t \leq d$,

– distinct t-polytopes that are not in the boundary of T_i share at most one $(t-1)$-face;
– distinct t-polytopes in the boundary of T_i that share more than one $(t-1)$-face will need to be identified by the time the construction of C is completed.

Moreover,

– if σ, τ are the two $(d-1)$-cells glued together in the step from T_i to T_{i+1}, σ and τ cannot belong to the same d-polytope of T_i; nor can they belong to two d-polytopes that are already adjacent in T_i.

For example, in each step of the local construction of a *simplicial* 3-sphere, no two tetrahedra share more than one triangle. Moreover, any two distinct

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interior triangles either are disjoint, or they share a vertex, or they share an edge; but they cannot share two edges, nor three; and they also cannot share one edge and the opposite vertex. If we glue together two boundary triangles that belong to adjacent tetrahedra, no matter what we do afterwards, we will not end up with a simplicial complex any more. So,

> *a locally constructible 3-sphere is a combinatorial 3-sphere obtained from a tree of polytopes T_N by repeatedly identifying two adjacent polygons in the boundary.*

LC pseudomanifolds were introduced in 1995 by two physicists, Durhuus and Jonsson [50], in connection with enumerative results (cf. Chapter 2). To get acquainted with this class of complexes, we present a few preliminary results.

Lemma 1.6.3 (Durhuus-Jonsson). *Every LC pseudomanifold is simply connected and strongly connected.*

Proof. A tree of polytopes satisfies both simply connectedness and strongly connectedness; any local gluing maintains these properties. $\square$

(In spite of their name, simply connectedness and strongly connectedness are independent properties: A wedge of 2-balls yields a simply-, not strongly-connected 2-complex, while a triangulated annulus is a strongly-, not simply-connected 2-complex.)

Lemma 1.6.4. *The boundary of an LC d-pseudomanifold is a (not necessarily LC) $(d-1)$-pseudomanifold.*

Proof. Let P be an LC d-pseudomanifold. If P is a tree of d-polytopes then ∂P is a (stacked) $(d-1)$-sphere, and the claim is obvious. Any LC gluing does not increase the number of $(d-2)$-faces per $(d-1)$-face: Therefore, every pseudomanifold in the local construction of P has a boundary which is a regular CW $(d-1)$-complex, and every $(d-2)$-face of such boundary belongs to at most two boundary facets.

A *fake cube*, which is a $3 \times 3 \times 3$ pile of cubes with the central cube missing, is an LC 3-manifold (one can show this either directly or via Lemma 5.1.1, cf. Figure 5.1) whose boundary is homeomorphic to the disjoint union of two 2-spheres. Being disconnected, the boundary of the fake cube cannot be LC: Compare Lemma 1.6.3. $\square$

In order to reach the conclusion of Lemma 1.6.4, the LC assumption is essential: in fact, two triangles sharing a vertex yield an easy example of a 2-pseudomanifold whose boundary is not a 1-pseudomanifold.

Lemma 1.6.5. *The barycentric subdivision of an LC d-pseudomanifold is an LC simplicial pseudomanifold.*

Proof. We prove the theorem only in case $d = 3$, the general case being analogous.

The barycentric subdivision of a tree of polytopes is locally constructible: this can be shown either directly, or via Lemma 5.1.1, since the barycentric subdivision of a shellable complex is shellable. Thus it suffices to show that a sequence of local gluings that produces a pseudomanifold C from a tree of polytopes T_N corresponds to a (longer) sequence of local gluings that produces $sd(C)$ from $sd(T_N)$.

Consider a single local gluing $\sigma' \equiv \sigma''$ of two boundary m-gons sharing an edge e in ∂T_i. Let σ be the interior m-gon of T_{i+1} generated by the gluing. Since the barycentric subdivision of σ is strongly connected, the facets of $sd(\sigma)$ can be labeled $1, 2, \ldots, 2m$, so that:

- the facet labeled by 1 contains a "portion" of e;
- each facet labeled by $k > 1$ is adjacent to some facet labeled j with $j < k$.

This induces a corresponding labeling $1', 2', \ldots, (2m)'$ (resp. $1'', 2'', \ldots, (2m)''$) of the facets of $sd(\sigma)'$ (resp. $sd(\sigma)''$). Now glue together the two copies k' and k'' of the facet k, in the labeling order. All these gluings are *local* by definition. Eventually, they produce $sd(T_{i+1})$ from $sd(T_i)$. □

In case $d = 3$, the topology of LC 3-pseudomanifolds (and of their boundaries) is controlled by the following result.
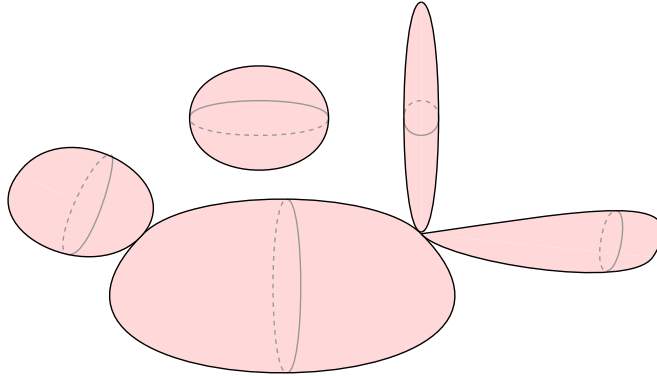
Figure 1.4: The boundary of an LC 3-pseudomanifold is a disjoint union of 'cacti of 2-spheres'. (Every LC 3-pseudomanifold is simply- and strongly-connected, so obviously the 3-pseudomanifold "inside" the pink surface cannot be LC. Its *complement* inside S^3 can be LC.)

Theorem 1.6.6 (Durhuus–Jonsson [50]). *Every LC 3-pseudomanifold P is homeomorphic to a 3-sphere with a finite number of "cacti of 3-balls" removed. (A cactus of 3-balls is a tree-like connected structure in which any two 3-balls share at most one point.) Thus the boundary ∂P is a finite disjoint union of cacti of 2-spheres. In particular, each connected component of ∂P is a simply-connected 2-pseudomanifold.*

(Durhuus and Jonsson proved the result above only in the simplicial case. That said, if P is an LC polytopal complex, by Lemma 1.6.5 the barycentric subdivision $sd(P)$ is an LC simplicial complex with the same topology of P.)

Corollary 1.6.7 (Durhuus-Jonsson). *Every 3-dimensional LC pseudo-manifold without boundary is a sphere.*

We will see in Theorem 5.2.7 that for $d > 3$ other topological types such as products of spheres are possible.

1.7   Operations on complexes

Let X and Y be (finite) regular CW complexes. The *product* $X \times Y$ is a regular CW complex with cells the products $e_\alpha^i \times e_\beta^j$, where e_α^i ranges over the cells of X and e_β^j ranges over the cells of Y (cf. Hatcher [72, p. 8]). If A is a subcomplex of X, the *quotient* space X/A also inherits a natural CW complex structure from X (cf. Hatcher [72, p. 8]): The cells of X/A are the cells of X not in A, plus one new 0-cell, the image of A in X/A. If $\varphi_\alpha : S^{d-1} \longrightarrow X^{d-1}$ is the attaching map of a cell e_α^d of X, then the attaching map for the corresponding cell in X/A is the composition

$$S^{d-1} \longrightarrow X^{d-1} \longrightarrow X^{d-1}/A^{d-1}.$$

If $\mathbb{I}$ is the interval $[0,1]$, the *cone* over X is defined as the quotient $(X \times \mathbb{I})/(X \times \{0\})$; the *apex* of the cone is the 0-cell given by the image of $\{X \times 0\}$ in $(X \times \mathbb{I})/(X \times \{0\})$. The union alongside X of two copies of the cone over X is called *suspension* of X. For example, if X is a d-sphere, the suspension of X is a $(d+1)$-sphere.

If σ is a cell of a regular CW complex C, the *link* of σ is the complex of the cells disjoint from σ but contained in a cell that contains σ; the *spherical link* of σ is instead any regular CW complex whose face poset is isomorphic to the upper ideal of σ in the face poset of C. As we noticed in Remark 1.2.1, the two notions coincide for simplicial complexes, but are distinct for polytopal complexes and also for simplicial regular CW complexes.

All *spherical* links in a shellable polytopal complex are shellable (see Björner [22, p. 170]). Also, spherical links in a constructible complexes are constructible (see e.g. [66, p. 23]). It is an open question whether the same results hold true for links as well: Courdurier [43] recently showed that the closed star of a vertex in a shellable polytopal complex is shellable; however, shellable complexes might have nonshellable boundaries [115, Theorem 2].

If the closed star of a vertex v in C coincides with the join of v with its link, then the link and the spherical link are the same. In particular, the (spherical) link of v inside the cone $v * C$ is just C. This leads to the following (known) Lemma:

Lemma 1.7.1. *Let C be a regular CW complex and v a new vertex. Then C is shellable (resp. constructible) if and only if $v * C$ is shellable (resp. constructible).*

We will show in Figure 5.2 an example of an LC simplicial complex with a vertex v whose (spherical) link is not LC. In other words, the LC property, differently from shellability, constructibility and vertex decomposability, is not inherited by spherical links. Nevertheless, the analogous result to Lemma 1.7.1 still holds:

Proposition 1.7.2. *Let C be a d-pseudomanifold and v a new vertex. Then C is LC if and only if $v * C$ is LC.*

Proof. The implication "if C is LC, then $v * C$ is LC" is straightforward.

For the converse, assume T_i and T_{i+1} are intermediate steps in the local construction of $v * C$, so that passing from T_i to T_{i+1} we glue together two adjacent $(d-1)$-faces σ', σ'' of ∂T_i. Let F be any $(d-2)$-face of T_i. If F does not contain v, then F is in the boundary of $v * C$, so $F \in \partial T_{i+1}$. Therefore, F cannot belong to the intersection of σ' and σ'', because all $(d-2)$-faces of $\sigma' \cap \sigma''$ are sunk into the interior of T_{i+1}.

So, every $(d-2)$-face in the intersection $\sigma' \cap \sigma''$ must contain the vertex v. This implies that $\sigma' = v * S'$ and $\sigma'' = v * S''$, with S' and S'' *distinct* $(d-2)$-faces. Certainly S' and S'' share at least a codimension-one face, otherwise σ' and σ' would not be adjacent. Thus, from a local construction of $v * C$ we can read off a local construction of C. $\square$

Even if the links in an LC pseudomanifold need not be LC, they all have to be strongly connected:

Proposition 1.7.3. *In a simplicial LC d-pseudomanifold, all links are strongly connected.*

The spherical *link of any* $(d-2)$*-face in an LC d-pseudomanifold is a* 1*-ball if the face lies on the boundary, and a* 1*-sphere otherwise.*

Proof. We fix a local construction $T_1, \ldots, T_k$ for P and proceed by induction on the number of local gluings. If P is a tree of simplices, the link of a k-face is a $(d-k-1)$-ball. By contradiction, suppose that the link of each k-face of T_i is strongly connected, but the link of some k-face F inside T_{i+1} is not strongly connected. The only way this could happen is if F is obtained identifying two k-faces F', F'' of ∂T_i, and the strongly connected links of F' and F'' have merged "wrongly".

Now, for any triple of t-complexes A, B, C, suppose that B and C are both strongly connected and $A = B \cup C$: Then A is strongly connected if and only if $B \cap C$ is $(t-1)$-dimensional. In particular, since

$$\text{link}_{T_{i+1}} F = \text{link}_{T_i} F' \cup \text{link}_{T_i} F'',$$

the strongly connectedness of the complexes on the right and the non-strongly-connectedness of the one on the left imply that the intersection $\text{link}_{T_i} F' \cap \text{link}_{T_i} F''$ cannot be $(d-k-2)$-dimensional. In the following, we will obtain a contradiction by pinpointing a $(d-k-2)$-face contained in the complex $\text{link}_{T_i} F' \cap \text{link}_{T_i} F''$.

Let σ' and σ'' be the boundary facets sharing a $(d-2)$-face r that have been identified in the step $T_i \rightsquigarrow T_{i+1}$. Up to relabeling, F' is contained in σ' and F'' is contained in σ'', but neither F' nor F'' are completely contained in r, so that $\dim(F \cap r) = \dim F - 1$.

Define $\rho := \text{link}_r(F \cap r)$; clearly ρ lies in $\text{link}_{T_i} F' \cap \text{link}_{T_i} F''$. On the other hand,

$$\dim \rho = \dim r - \dim(F \cap r) - 1 = \dim r - (\dim F - 1) - 1 = (d-2) - k,$$

a contradiction.

As far as the second part of the claim is concerned, the spherical link of a $(d-2)$-face of a d-pseudomanifold is a 1-pseudomanifold. Strongly connected 1-pseudomanifolds can only be 1-spheres or 1-balls. The spherical link of a face is a sphere if and only if such face lies in the interior of the pseudomanifold. □

The previous proof can be adapted to show that all spherical links in an LC pseudomanifolds are strongly connected. On the contrary, in the second part of the claim of Proposition 1.7.3, the word "spherical links" cannot be replaced by "links": In fact, the link of F might be a 1-sphere even if F lies on the boundary. For example, let S be any 2-sphere, and let

$T_1, T_2, \ldots, T_N, \ldots, T_k$ be any local construction of S. Then T_{k-1} is a regular CW complex homeomorphic to a 2-ball, with exactly two vertices on the boundary which are connected by a double edge. The link of any vertex in T_{k-1} is a 1-sphere.

We leave it to the reader to generalize Proposition 1.7.3 to the non-simplicial case.

A very common operation that transforms a d-ball in a d-sphere consists in "coning off the boundary". When $d = 2$ the name is self-explanatory: given a disk, we look at the boundary of a cone that has the given disk as basis. This yields a 2-sphere. In higher dimensions, if B is a polytopal d-ball (resp. a simplicial d-ball) and v is a new vertex, then $B \cup (v * \partial B)$ is a polytopal d-sphere (resp. a simplicial d-sphere). The following results are known:

Lemma 1.7.4. *Let B be a d-ball. Let $S_B := B \cup (v * \partial B)$.*

 (i) *If B and ∂B are both shellable (resp. constructible), then S_B is also shellable (resp. constructible).*
 (ii) *If S_B is shellable (resp. constructible), then ∂B is shellable (resp. constructible).*
(iii) *The converses of both the previous implications are false. In addition, if B is shellable, S_B might be nonshellable.*

Proof. Any shelling of ∂B translates into a shelling of $v * \partial B$, which placed after any shelling of B yields a shelling for S_B. (Similarly for constructibility: if ∂B is constructible, then $v * \partial B$ is also constructible.) A counterexample for the converse of the implication in (i) was given by Kamei [82]: He found constructible nonshellable 3-balls B with S_B shellable (and also non-constructible 3-balls B with S_B shellable).

The item (ii) follows from the fact that $\mathrm{link}_{S_B} v = \partial B$. By Lemma 1.7.1, the link of v in S_B coincides with the spherical link of v in S_B. The spherical link of any face in a shellable (resp. constructible) complex is shellable (resp. constructible). A counterexample for the converse of (ii) is given by any non-constructible 3-sphere S, because every 3-sphere can be obtained coning off the boundary of some 3-ball: In fact, for each vertex v of S, if $C = \mathrm{del}_{S_B}(v)$ one has $S = S_C$. Yet ∂C is a 2-sphere and thus shellable.

Shellable balls B such that S_B is non-shellable can be obtained by the work of Pachner [115, Theorem 2, p. 79], who proved that any 3-sphere is combinatorially equivalent to the boundary of some shellable 4-ball. In particular, if B is a shellable ball with nonshellable boundary, by (ii) S_B cannot be shellable. $\square$

Remark 1.7.5. An alternative way to show the strictness of the implication (i) comes from knot theory. Hachimori and Ziegler proved the existence of shellable simplicial 3-sphere S with a quadrilateral knot (cf. Theorem 4.4.3) and the non-existence of constructible 3-balls with a knotted spanning arc of two edges (cf. Theorem 4.4.1). Yet the deletion from S of one of the four knot vertices yields a ball with a knotted spanning arc of two edges.

Remark 1.7.6. If B is an LC d-ball and ∂B is an LC $(d-1)$-sphere, then S_B is also LC. In fact, by Proposition 1.7.2 the d-sphere S_B is the union of two LC d-balls (B and $v * \partial B$) that intersect in a $(d-1)$-sphere (namely, ∂B). We will see in Lemma 5.1.1 that this suffices to prove that S_B is LC. (See also Remark 6.2.2 for a more general result.) However, B might be non-LC even if S_B is LC: To see this, choose as B the collapsible ball C_2 that we will construct in Theorem 6.3.6 (cf. Proposition 3.4.2).

Chapter 2

Asymptotic enumeration of manifolds

In Weingarten's discrete approach [142] to the physical theory of quantum gravity (cf. Regge [122] [123]), the partition function for gravity is rendered by a weighted sum over all orientable d-manifolds. The model converges if the number of triangulated d-manifolds with N facets grows not faster than C^N, for some constant C. However, this is false for each $d \geq 2$: As we explain in Corollary 2.3.2, orientable simplicial 2-manifolds with N facets are at least $\left(\frac{N}{20}\right)!$.

To bypass this obstacle, one cuts off artificially the class of manifolds over which one is integrating. When $d = 2$, restricting the *topology* does the trick. In fact, as we will see in Corollaries 2.3.2 and 5.5.3,

- – simplicial orientable 2-manifolds are more than exponentially many (both in N and n), but
- – 2-spheres are exponentially many (both in N and n),
- – 2-dimensional tori are exponentially many (both in N and n), and
- – simplicial orientable 2-manifolds with genus bounded by a constant are exponentially many (both in N and n).

Note that the key assumption to obtain an exponential bound is neither "bounded average curvature" nor "bounded vertex degree", as previous studies on the subject seemed to suggest [3, p. 5] [14, p. 7], but rather "bounded genus". Indeed, we will see in Remark 2.3.3 that surfaces with bounded curvature but unbounded genus are still more than exponentially many.

When $d \geq 3$, however, the strategy of fixing the topology encounters

deep problems. As mentioned by Gromov [61, pp. 156-157] in his list of crucial open problems in modern geometry, we still do not know whether simplicial 3-spheres with N facets are exponentially many, or more.

In 1995 two quantum gravity physicists, Durhuus and Jonsson [50], introduced the class of LC 3-spheres (see Definition 1.6.2) proving an exponential upper bound for its cardinality. We show here that their bound extends from spheres to manifolds (with or without boundary), from dimension $d \in \{2,3\}$ to any dimension $d \geq 2$, from simplicial complexes to polytopal complexes of "bounded facet complexity", and from "LC" to broader classes (such as k-LC manifolds: cf. Definition 2.6.4).

Main Theorem 7 (Corollary 2.5.2, Theorem 2.6.5, Theorem 2.6.3).
Let k, d, A be nonnegative integers, with $A > d > 1$. There are exponentially many simplicial k-LC d-manifolds (with or without boundary) with N facets. The same holds for k-LC d-manifolds (with or without boundary, simplicial or not) provided each facet is a d-polytope with at most A faces.

Thus one has the following prospect:

▷ The asymptotics of 3-spheres is unknown. If $v(n)$ is the number of 3-spheres with n vertices, by the work of Pfeifle and Ziegler [120]

$$\Omega(n^{\frac{5}{4}}) \ \leq \ \log v(n) \ \leq O(n^2 \log n);$$

on the other hand, if $f(N)$ is the number of 3-spheres with N facets, one has
$$\Omega(N) \ \leq \ \log f(N) \ \leq O(N \log N).$$

(The lower bound $\Omega(N)$ follows for example from the count of stacked 3-spheres, cf. Corollary 2.1.4, while the upper bound $O(N \log N)$ may be derived from Corollary 2.1.5.)

▷ If $p(n,d)$ (resp. $p(N,d)$) counts the number of d-spheres with n vertices (resp. with N facets) that are combinatorially equivalent to the boundary of some $(d+1)$-polytope, one has

$$\log p(n,d) = \Theta(n \log n),$$

by the work of Shemer [129], Goodman–Pollack [59] and Alon [1]; at the same time,

$$\left(1 + \frac{1}{d}\right)^N \ \leq \ p(N,d) \leq 2^{d^2 \cdot N},$$

by Corollary 2.1.4 and by Theorem 2.5.1, together with the fact that all boundaries of polytopes are shellable [153, Lect. 8] and all shellable spheres are LC (cf. Lemma 5.1.1).

▷ Simplicial 3-manifolds are more than exponentially many, both in n and N:
$$\log m(N) = \Theta(N \log N),$$
where $m(N)$ counts simplicial 3-manifolds with N tetrahedra (see Corollary 2.1.5 resp. Corollary 2.3.5 for an upper resp. lower bound);

▷ however, if $h(N)$ (resp. $h_k(N)$) is the number of LC (resp. k-LC) simplicial 3-manifolds on N tetrahedra, then by Theorem 2.6.5
$$\log h(N) = \Theta(N) \quad \text{and} \quad \log h_k(N) = \Theta(N) \quad \text{for fixed } k.$$

Most of the previous bounds generalize to higher dimensions. For $d \geq 4$, we know by the work of Kalai [81] and Lee [90] that shellable d-spheres are more than exponentially many in n (cf. Pfeifle [119]); at the same time, we will show in Lemma 5.1.1 that shellable d-spheres are LC; via Main Theorem 7, this implies that shellable d-spheres are exponentially many in N.

From these conclusions it is clear that counting with respect to vertices or facets is not the same. That said, since $n < N$ what is more than exponential in N is also more than exponential in n.

2.1 Few trees of simplices

We will here establish that there are less than $C_d(N) := \frac{1}{(d-1)N+1}\binom{dN}{N}$ trees of N d-simplices. The idea is that there are less trees of d-simplices than planted plane d-ary trees, which are counted by order d Fuss–Catalan numbers. Also, we will see that this exponential upper bound for trees of simplices is essentially sharp. This will be crucial in determining upper and lower bounds for d-manifolds.

Lemma 2.1.1. *Every tree of N d-simplices has $(d-1)N+2$ boundary facets of dimension $d-1$ and $N-1$ interior faces of dimension $d-1$.*
It has $\frac{d}{2}((d-1)N+2)$ faces of dimension $d-2$, all of them lying in the boundary.

Proof. By induction: A d-simplex has $d+1$ boundary facets and $\binom{d+1}{2}$ (boundary) ridges. Whenever we attach a d-simplex alongside a boundary facet onto a tree of d-simplices,

– we create one interior $(d-1)$-face,
– we add $(d-1)$ boundary facets and
– we add d ridges, which all lie on the boundary.

□

By *rooted* tree of simplices we mean a tree of simplices B together with a distinguished facet δ of ∂B, whose vertices have been labeled $1, 2, \ldots, d$. Rooted trees of d-simplices are in bijection with "planted plane d-ary trees", that is, plane rooted trees such that every non-leaf vertex has exactly d (left-to-right-ordered) sons; cf. [102].

Proposition 2.1.2. *There is a bijection between rooted trees of N d-simplices and planted plane d-ary trees with N non-leaf vertices, which in turn are counted by the Fuss–Catalan numbers $C_d(N) = \frac{1}{(d-1)N+1} \binom{dN}{N}$. Thus, the number of combinatorially-distinct trees of N d-simplices satisfies*

$$\frac{1}{(d-1)N+2} \frac{1}{d!} C_d(N) \quad \leq \quad \# \{ \text{ trees of } N \ d\text{-simplices } \} \quad \leq \quad C_d(N).$$

Proof. Given a rooted tree of d-simplices with a distinguished facet δ in its boundary, there is a unique extension of the labeling of the vertices of δ to a labeling of all the vertices by labels $1, 2, \ldots, d+1$, such that no two adjacent vertices get the same label. Thus each d-simplex receives all $d+1$ labels exactly once.

Now, label each $(d-1)$-face by the unique label that none of its vertices has. With this we get an edge-labeled rooted d-ary tree whose non-leaf vertices correspond to the N d-simplices; the root corresponds to the d-simplex that contains δ, and the labeled edges correspond to all the $(d-1)$-faces other than δ. We get a plane tree by ordering the down-edges at each non-leaf vertex left to right according to the label of the corresponding $(d-1)$-face.

The whole process can be reversed. Given an arbitrary rooted planted plane d-ary tree T, we $(d+1)$-color it as follows: First we label its root by 1, and then recursively we label by $1, \ldots, i-1, i+1, \ldots, d+1$, in this order, the left-to-right sons of each non-leaf node labeled by i. Next, we take a d-simplex Σ_R (where R stands for "root"), we label its vertices by $1, \ldots, d+1$, and we introduce new simplices Σ_v in bijection with the non-leaves v of the tree T as follows: If the i-labeled son w of a non-leaf v is itself a non-leaf,

(1) we stack the facet of Σ_v opposite to the vertex of Σ_v labeled by i,
(2) we call Σ_w the newly introduced d-simplex, and
(3) we label by i the newly introduced vertex (i.e. the vertex of Σ_w that is not in Σ_v).

This way from a rooted planted plane d-ary tree we obtain a rooted tree of d-simplices, the "distinguished facet" being the $(d-1)$-face spanned by the d vertices of Σ_R that are labeled by $1, \ldots, d$.

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There are exactly $C_d(N) = \frac{1}{(d-1)N+1}\binom{dN}{N}$ planted plane d-ary trees with N interior vertices (see e.g. Aval [11]; the integers $C_2(N)$ are the "Catalan numbers", which appear in many combinatorial problems, see e.g. Stanley [135, Ex. 6.19] [132]). Any tree of N d-simplices has exactly $(d-1)N+2$ boundary facets, so it can be rooted in exactly $((d-1)N+2)\,d!$ ways, which however need not be inequivalent. This explains the first inequality claimed in the lemma. Finally, combinatorially-inequivalent trees of d-simplices also yield inequivalent rooted trees, whence the second inequality follows. $\qquad\square$

Corollary 2.1.3. *The number of trees of N d-simplices, for N large, is bounded by*

$$\binom{dN}{N} \; \sim \; \left(d\cdot\left(\tfrac{d}{d-1}\right)^{d-1}\right)^N \; < \; (de)^N.$$

Corollary 2.1.4. *The number of stacked d-spheres on N facets for N large is approximately*

$$\binom{(N-2)\frac{d+1}{d}}{\frac{N-2}{d}} \; \approx \; \left(\frac{d+1}{d}\sqrt[d]{d+1}\right)^N.$$

Proof. By Lemma 2.1.1, any stacked d-sphere on N facets is the boundary of a tree of $\frac{N-2}{d}$ simplices of dimension $d+1$. The conclusion follows by Corollary 2.1.3. $\qquad\square$

Corollary 2.1.5. *The number of simplicial d-manifolds with boundary with N facets, for N large, is bounded by*

$$(de)^N \cdot ((d-1)N+2)!!,$$

which is smaller than $(dN)^{dN}$.

Proof. Any triangulated d-manifold (with boundary) with N facets can be obtained from a tree of N d-simplices by pairwise identifying boundary facets. (Just look at the tree of simplices determined by a spanning tree of the dual graph of the manifold.) By Lemma 2.1.1 there are $(d-1)N+2$ boundary facets in a tree of d-simplices; the conclusion follows then via Corollary 2.1.3, because the number of perfect matchings of a set of $2k$ objects is

$$(2k)!! = (2k)\cdot(2k-2)\cdot\ldots\cdot 4\cdot 2 = k!\,\cdot\,2^k.$$

$\qquad\square$

We point out that the previous bound is far from being sharp, because not every matching results in a manifold, but most of all because each manifold is overcounted several times: In fact, every manifold can be constructed out of *any* of its "spanning trees of simplices".

2.2 Few 2-spheres

The fact that 2-spheres are exponentially many was discovered in the Sixties by Tutte [137] [138] (see also Richmond–Wormald [124], Bender [16] [17], Gao [58] and Di Francesco [55]), who gave the asymptotic estimate

$$\frac{3}{16\sqrt{6\pi n^5}} \left(\frac{256}{27}\right)^{n-2}$$

for combinatorially distinct rooted simplicial 3-polytopes with n vertices. This implies that the number of simplicial 2-spheres with N triangles is approximately

$$\left(\frac{16}{3\sqrt{3}}\right)^N.$$

Out of these $\frac{16}{3\sqrt{3}}^N \approx 3.08^N$ 2-spheres, roughly $\frac{3\sqrt{3}}{2}^N \approx 2.6^N$ are stacked, by Corollary 2.1.4.

An elementary exponential upper bound (independently from Tutte's work) can be found with various approaches:

▷ As we mentioned in the Introduction, every simplicial 2-sphere with N triangles can be generated from an $(N+2)$-gon (triangulated without interior vertices!) by identifying the boundary edges pairwise, according to a complete matching. A necessary condition in order to obtain a 2-sphere from a tree of N triangles is that this matching be *planar*. Planar matchings and triangulations of $(N+2)$-gons are both enumerated by a Catalan number C_{N+2}, and since the Catalan numbers satisfy a polynomial bound $C_N = \frac{1}{N+1}\binom{2N}{N} < 4^N$, we get an exponential upper bound for the number of triangulations.

▷ A variation of the previous approach uses convexity rather than planarity: By Steinitz' theorem (cf. [153, Lect. 4]), all 2-spheres are "polytopal", i.e. combinatorially equivalent to the boundary of some convex 3-polytope. Yet 3-polytopes (and $(d+1)$-polytopes in general) are not so many: compare Goodman–Pollack [59], Alon [1], and Corollary 5.1.3. (Note that this approach works for all 2-spheres, simplicial or not.)

▷ Durhuus [49] [50, p. 184] observed that all simplicial 2-spheres are locally constructible. Also this approach leads to an exponential upper bound, since LC spheres are exponentially many (cf. Theorem 2.5.1).

The last approach generalizes from spheres to surfaces of bounded genus: see Section 5.5.

2.3 Many surfaces and many handlebodies

In this paragraph we show that there are many orientable 2-manifolds with N facets. The idea is to take out $2g$ holes from a finely triangulated sphere of "size" $\Theta(g)$, and then to complete it to a genus-g-surface by attaching g handles — which can be done in $g!$ inequivalent ways.

We introduce some notation first, referring the reader to Stillwell [136, pp. 252-253], Johannson [78] or Scharlemann [127] for details. A *handlebody* is the tubular neighbourhood of some finite graph G inside $\mathbb{S}^3$ (or inside $\mathbb{R}^3$). All triangulated handlebodies are simplicial orientable compact 3-manifolds with boundary.
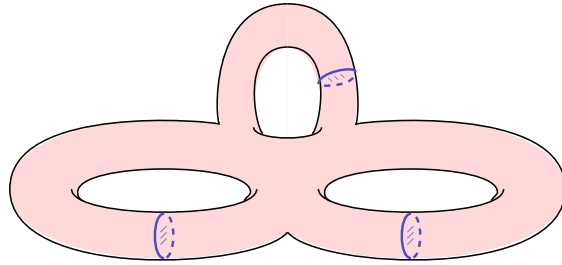
Figure 2.1: A genus-3-handlebody with three disjoint meridian discs (shaded).

Every handlebody H has a family of g disjoint 1-spheres in its boundary (called *meridians*) that bound disjoint discs $B_1, \ldots, B_g$ inside H, so that the manifold obtained cutting H open along $B_1, \ldots, B_g$ is a 3-ball. The meridians of H can be always be completed to a homology basis for the surface ∂H; one way to do this is to add the 'parallels', i.e. the meridians of the 'complement handlebody' inside $\mathbb{S}^3$ (cf. Section 2.4 or [47, Theorem 1, p. 358]).

We show now how to construct many triangulated handlebodies with the same number of facets:

Theorem 2.3.1. *For any $g \geq 1$, there are at least $g!$ combinatorially distinct triangulated orientable surfaces with genus g and $20g$ triangles. Furthermore, each of these surfaces can be realized in $\mathbb{R}^3$ as boundary of a triangulated handlebody of genus g, with exactly $11g - 1$ tetrahedra and with g triangular meridians in its 1-skeleton.*

Proof. Take a horizontal $1 \times 4g$ strip of squares and triangulate the first resp. the last $2g$ squares by inserting backslash resp. slash diagonals. Cutting away the last triangle we obtain a 2-ball B with $8g - 1$ triangles.

Set $a_j := \begin{cases} 4j - 2 & \text{if } j \in \{1, \ldots, g\} \\ 4j - 1 & \text{if } j \in \{g + 1, \ldots, 2g\} \end{cases}$; obviously the a_i-th triangle is disjoint from the a_j-th, unless $i = j$. Relabel these $2g$ disjoint triangles by $1, \ldots, g, 1', \ldots, g'$, in this order.

Given a new vertex v, we form the 2-sphere $S_B := B \cup v * \partial B$ and remove from S_B the interiors of the triangles $1, \ldots, g, 1', \ldots, g'$. The resulting "2-sphere with holes" can be completed to a closed 2-manifold by attaching g handles. First we need to fix a bijection $\pi : \{1, \ldots, g\} \longrightarrow \{1', \ldots, g'\}$. In the triangle i, let x_i resp. u_i be the leftmost resp. the upper vertex; symmetrically, in the triangle i' let $x_{i'}$ resp. $u_{i'}$ be the rightmost resp. the upper vertex. For each $i \in \{1, \ldots, g\}$, we attach a non-twisted triangular prism onto the holes i and $\pi(i)$, so that x_i resp. u_i gets connected via an edge to $x_{i'}$ resp. $u_{i'}$. Each prism can be triangulated with six facets by subdividing each lateral rectangle into two; as a result, we obtain a simplicial closed 2-manifold $M_g(\pi)$.

The number of triangles of $M_g(\pi)$ equals the number of triangles of S_B, minus $2g$ (the holes), plus $6g$ (the handles). Being the boundary of a tree of $8g - 1$ tetrahedra, S_B has $2(8g - 1) + 2$ facets: Therefore, $M_g(\pi)$ has genus g and $20g$ facets. A system of homotopy generators for $M_g(\pi)$ is given by the following $2g$ 1-spheres ($i = 1, \ldots, g$):

(A) the triangle $[x_i, y_i], [y_i, u_i], [u_i, x_i]$ (which is the boundary of the triangle i in B);

(B) the triangle $[u_i, v], [v, u_{\pi(i)}], [u_i, u_{\pi(i)}]$.

The 1-spheres of type (A) are pairwise disjoint, while any two spheres of type (B) intersect at v. (Every sphere of type (A) intersects exactly one sphere of type (B), in exactly one point.) Since S_B is boundary of a tree of $8g - 1$ tetrahedra, if we triangulate the interior of each prism using three tetrahedra we can view $M_g(\pi)$ as the boundary of a handlebody with exactly $11g - 1$ tetrahedra. By construction, the 1-spheres of type (A) form a system of meridians for such handlebody.

The conclusion follows by noticing that any two different permutations π and ρ give rise to two combinatorially different surfaces $M_g(\pi)$ and $M_g(\rho)$. $\qquad \square$

Corollary 2.3.2. *Simplicial 2-manifolds are more than exponentially many with respect to the number of facets (and thus also with respect to the number of vertices).*

Proof. By Theorem 2.3.1, when N is a multiple of 20 there are $(\frac{N}{20})!$ combinatorially distinct simplicial 2-manifolds with N facets (and genus $\frac{N}{20}$). For each real number R, the ratio of R^N to $(\frac{N}{20})!$ tends to zero for N large. $\qquad \square$

Remark 2.3.3. A variant of the main idea in Theorem 2.3.1 consists in starting with a tree of $6g$ tetrahedra in which no vertex has degree larger than six. Again, one can locate $2g$ disjoint boundary triangles and remove them: this way one gets a sphere with $2g$ holes that can be completed to a genus-g-surface (by attaching g handles) in $g!$ different ways. The news is that no vertex of the genus-g-surface has degree larger than eight.

Thus, there are more than exponentially many simplicial 2-manifolds with N facets, even if the degree of each vertex is at most eight.

Remark 2.3.4. We point out that all of the surfaces constructed in Theorem 2.3.1 and in Remark 2.3.3 have relatively high genus: Compare Corollary 5.5.3.

Corollary 2.3.5. *There are more than exponentially many combinatorial types of simplicial* 3-*manifolds with boundary with N tetrahedra.*

Proof. By Theorem 2.3.1, when N is congruent to 10 modulo 11 there are $\frac{N+1}{11}!$ triangulated handlebodies with N facets. $\qquad\square$

On the other hand, we have seen in Corollary 2.1.5 that there are less than $(dN)^{dN}$ simplicial d-manifolds with boundary, for N large. Therefore, the number $m(N)$ of simplicial 3-manifolds with boundary with N facets satisfies
$$\log m(N) = \Theta(N \log N).$$

Given an arbitrary surface embedded in $\mathbb{R}^3$, is it possible to 'fill it up' with linearly many tetrahedra (with respect to the number of facets of the surface)?

Theorem 2.3.6. *Let H be a handlebody of genus g. Suppose that a triangulation $\mathfrak{T}$ of ∂H contains in its 1-skeleton a family of meridians $\gamma_1, \ldots, \gamma_g$ of H. Then $\mathfrak{T}$ can be extended to a triangulation $\tilde{\mathfrak{T}}$ of H, so that*

$$f_3(\tilde{\mathfrak{T}}) \ = \ 24\, f_2(\mathfrak{T}) \ + \ 24 \sum_{i=1}^{g} f_1(\gamma_i).$$

Moreover, $\tilde{\mathfrak{T}}$ determines $\mathfrak{T}$ uniquely.

Proof. For each $i = 1, \ldots, g$, we triangulate the meridian disk B_i spanned by the polygonal curve γ_i by inserting an interior vertex and coning. This produces $f_1(\gamma_i)$ new triangles.

The manifold obtained cutting H open along the meridian disks is a 3-ball, by definition of "meridian". We can triangulate it by taking a point v inside H (but disjoint from all the meridian disks) and coning. The result is a CW complex homeomorphic to H; however, it is not a simplicial complex

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(unless $g = 0$), because there are pairs of distinct tetrahedra sharing all four vertices (namely, all tetrahedra of the type $v * \sigma$, with σ in some B_i).

To obtain a simplicial complex, the simplest way is to take the barycentric subdivision, which increases the number of facets by a factor of twenty-four. (Different subdivision methods may lead to better constants.)

The uniqueness part can be proven by analysing the vertex degrees: The vertex with the highest degree has to be v. Once we know v, from its link we can easily recover the starting surface and the meridian disks. (Thus from $\tilde{\mathfrak{T}}$ one can recover not only $\mathfrak{T}$, but also the meridians chosen.) $\qquad\square$

Note that $\sum_{i=1}^{g} f_1(\gamma_i)$ is certainly smaller or equal than $\frac{3}{2} f_2(\mathfrak{T})$, which is the total number of edges of $\mathfrak{T}$. Therefore, $f_3(\tilde{\mathfrak{T}}) \le 60 \cdot f_2(\mathfrak{T})$.

Corollary 2.3.7. *Let H be a handlebody. Any triangulation of ∂H with N triangles that contains a family of meridians of H in its 1-skeleton can be uniquely extended to a triangulation of H with $O(N)$ tetrahedra.*

2.4 Many 3-spheres?

There is a well known relation between handlebodies and 3-spheres, given by Heegaard splittings. Again, we refer to Stillwell [136, pp. 252-253], Johannson [78] or Scharlemann [127] for details and proofs.

Theorem 2.4.1 (Heegaard [73]). *For each $g \in \mathbb{N}$, there exists a decomposition of a 3-sphere into two solid handlebodies H_1 and H_2 that are identified along a surface $M_g = H_1 \cap H_2$ of genus g.*

The previous decomposition is determined up to isotopy by the genus (see Waldhausen [141]), and thus it is usually called *the Heegaard splitting of genus g*.

Conversely, given any orientable 2-manifold M_g of genus g, any embedding of M_g in $\mathbb{S}^3$ separates $\mathbb{S}^3$ into two distinct genus-g handlebodies, the "inside" $\mathbb{I}$ and the "outside" $\mathbb{O}$. Following [47], let us call *handle loop* a 1-sphere in the 1-skeleton of M_g that has trivial first homology in $\mathbb{I}$ and nontrivial first homology in $\mathbb{O}$, and *tunnel loop* a 1-sphere in M_g that has trivial first homology in $\mathbb{O}$ and nontrivial first homology in $\mathbb{I}$. (By definition, sets of tunnel loops and handle loops are always disjoint; however, some loops in M_g might be neither tunnel nor handles, having nontrivial homology in $\mathbb{I}$ as well as in $\mathbb{O}$, cf. [47, Figure 2].) Dey, Li and Sun [47, Theorem 1, p. 358] showed that every orientable genus-g-surface, once embedded in $\mathbb{S}^3$, admits a homology basis that consists of g handle loops (which themselves

form a basis of $H_1(\mathbb{O})$) and g tunnel loops (which themselves form a basis of $H_1(\mathbb{I})$).

This "handles & tunnels homology basis" depends of course on the embedding chosen. However, it is possible to define a homology basis for a surface in a way that does not refer to an embedding:

Definition 2.4.2 (Lazarus et al. [89]). A *canonical polygonal schema* for an orientable 2-manifold of genus g is a $(4g)$-gon with successive edges labeled by

$$\overrightarrow{x_1}, \overrightarrow{y_1}, \overleftarrow{x_1}, \overleftarrow{y_1}, \ldots, \overrightarrow{x_g}, \overrightarrow{y_g}, \overleftarrow{x_g}, \overleftarrow{y_g},$$

where corresponding edges $\overrightarrow{x_i}$ and $\overleftarrow{x_i}$ (resp. $\overrightarrow{y_i}$ and $\overleftarrow{y_i}$) are oriented in different directions, such that gluing together corresponding edges recovers the original manifold.

The "edges" of the polygon correspond to "loops" inside the manifold; it is easy to see that all these loops have a common point.

The definition above is purely topological and does not refer to a specific triangulation. We say that a *triangulated* 2-manifold *M has a canonical polygonal schema in its* 1-*skeleton* if M has a subcomplex homeomorphic to a bouquet of $2g$ 1-spheres, so that cutting M open along this subcomplex one gets a 2-ball that is a triangulation of a canonical polygonal schema. (See Figure 2.2 below).
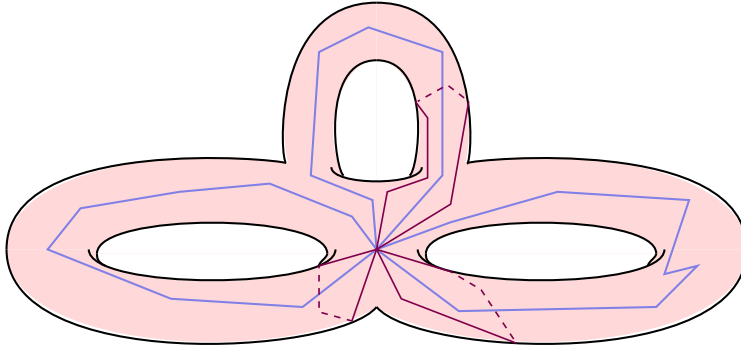
Figure 2.2: A triangulated 2-manifold M with a canonical polygonal schema in its 1-skeleton. (The rest of the triangulation is not drawn.) Cutting M open alongside this bouquet of $2g$ loops recovers the original "polygon" (= a triangulated 2-ball).

Not all 2-manifolds have a canonical polygonal schema in their 1-skeleton [89]; however, it is possible to induce one by cleverly subdividing:

Lemma 2.4.3 (Lazarus et al. [89, Theorem 1], see also [42], [120]).
Any triangulation of a closed 2-manifold of genus g with a total of $f = f_0 + f_1 + f_2$ cells can be refined to a triangulation with $O(fg)$ vertices that contains a canonical polygonal schema in its 1-skeleton. Furthermore, each meridian uses $O(f)$ vertices and edges.

Theorem 2.4.4 (essentially Pfeifle–Ziegler [120, p. 835]). *Let M be a closed 2-manifold of genus g with N facets that contains a canonical polygonal schema in its 1-skeleton. There exists a 3-sphere S_M with $O(N)$ facets such that the barycentric subdivision of M is a subcomplex of S_M. Moreover, M determines S_M uniquely.*

Proof. As in the proof of Theorem 2.3.6, the idea is to fill in the meridian disks of each handlebody first; the remaining two 3-balls are then triangulated by coning. Uniqueness is shown as in Theorem 2.3.6. $\qquad\square$

(The original proof of Pfeifle and Ziegler is slightly more complicated because it was carried out with the intent to triangulate S_M with as few additional vertices as possible. This goal was achieved by doubling all the meridian disks of M, cf. [120, p. 835].)

When $g = \frac{(q-1)(q-4)}{4}$ and q is a prime number congruent to 1 modulo 4, a genus-g surface admits a special triangulation (called *Heffter triangulation*) that uses only $O(\sqrt{g})$ vertices and $O(g)$ facets [120, Prop. 1]. Using Lemma 2.4.3 and Theorem 2.4.4, one can extend the Heffter triangulation $\mathfrak{T}_g$ of a surface M_g to a triangulation $\tilde{\mathfrak{T}}_g$ of a 3-sphere with $O(g^2)$ facets that admits a Heegard splitting along M_g. "Saving" as many vertices as possible, Pfeifle and Ziegler managed to realize $\tilde{\mathfrak{T}}_g$ with only $O(g^2)$ vertices.

From here, they were able to conclude that 3-spheres are more than exponentially many with respect to the number of vertices. In fact, they "thickened" the starting surface M_g a bit, by pulling the handlebodies apart; the interior of this thickening was subdivided into a stack of prisms over M_g, which in turn were subdivided into $O(g^2\sqrt{g})$ octahedra using only $O(g^2)$ vertices. Triangulating each octahedron independently, they achieved the following result:

Theorem 2.4.5 (Pfeifle–Ziegler [120, Theorem 1]). *There are at least $2^{\Omega(n\sqrt[4]{n})}$ simplicial 3-spheres on n vertices.*

Now, in Theorem 2.4.4 we have seen that any triangulation of a genus-g-surface with a canonical polygonal schema in its 1-skeleton can be completed to a triangulation of a 3-sphere with linearly many facets. On the other hand, we had produced in Theorem 2.3.1 many genus-g-surfaces $M_g(\pi)$ with

a homology basis – which is "almost" a canonical polygonal schema, in some sense – in their 1-skeleton. This motivates the following concrete problem:

Question 1. *Can each triangulated surface $M_g(\pi)$ be completed to a triangulation of a 3-sphere using $O(g)$ tetrahedra?*

A "yes" answer to the previous question would instantly imply that there are more than exponentially many 3-spheres with N facets. On the contrary, a "no" answer would also be interesting in connection with the natural follow-up question, "For which bijections π can the surface $M_G(\pi)$ be completed with $O(g)$ tetrahedra to a triangulation of $\mathbb{S}^3$?"

For the moment, we know that it is possible to fill the *inside* of the surface with linearly many tetrahedra, by Theorem 2.3.6, and that the outside can be triangulated with *quadratically* many tetrahedra, basically according to Lemma 2.4.3 and Theorem 2.4.4.

Another interesting problem that naturally arises from Theorem 2.4.4 is the following:

Question 2. *Are there more than exponentially many orientable surfaces with a canonical polygonal schema in their 1-skeleton?*

(Orientable surfaces with a canonical polygonal schema in their 1-skeleton are at least exponentially many. To see this, apply Lemma 2.4.3 to surfaces with bounded genera.)

Again, a "yes" answer[1] would imply that 3-spheres with N facets are more than exponentially many. If instead the numerical simulations by Ambjørn–Varsted [8] and Hamber–Williams [70] are accurate and indeed there are only exponentially many 3-spheres, we could use Theorem 2.4.4 to conclude that *most* of the orientable simplicial surfaces do not have any canonical polygonal schema in their 1-skeleton.

2.5 Few LC simplicial d-manifolds

For fixed $d \geq 2$ and a suitable constant C that depends on d, there are less than C^N combinatorial types of simplicial LC d-spheres with N facets. Our proof for this fact is a d-dimensional version of the main theorem of Durhuus & Jonsson [50] and allows us to determine an explicit constant C, for any d.

We know from Corollary 2.1.3 that there are exponentially many trees of N d-simplices. The idea is now to count the number of "LC matchings" according to ridges in the tree of simplices.

---

[1]After the submission of the present thesis, we established that Question 2 has a negative answer. The details will be found elsewhere.

Theorem 2.5.1. *Fix $d \geq 2$. The number of combinatorially distinct simplicial LC d-manifolds (with boundary) with N facets, for N large, is not larger than*

$$\left(d \cdot \left(\tfrac{d}{d-1} \right)^{d-1} \cdot 2^{\frac{2d^2-d}{3}} \right)^N.$$

Proof. Let S be a simplicial LC d-manifold with N facet. Let us fix a tree of N d-simplices B inside S. We adopt the word "couple" to denote a pair of facets in the boundary of B that are glued to one another during the local construction of S.

Let us set $D := \frac{1}{2}(2 + N(d-1))$. Note that the number of ridges in a d-manifold is $\frac{(d+1)N}{2}$, so $(d+1)N$ is even: therefore $(d-1)N$ is also even, which implies that D is an integer. By Lemma 2.1.1, the boundary of the tree of N d-simplices contains $2D$ facets, so each perfect matching is just a set of D pairwise disjoint couples. We are going to partition every perfect matching into "rounds". The first round will contain couples that are adjacent in the boundary of the tree of simplices. Recursively, the $(i + 1)$-th round will consist of all pairs of facets that *become* adjacent only after a pair of facets are glued together in the i-th round.

Selecting a pair of adjacent facets is the same as choosing the ridge between them; and by Lemma 2.1.1, the boundary contains dD ridges. Thus the first round of identifications consists in choosing n_1 ridges out of dD, where n_1 is some positive integer. After each identification, at most $d - 1$ new ridges are created; so, after this first round of identifications, there are at most $(d - 1)n_1$ new pairs of adjacent facets.

In the second round, we identify $2n_2$ of these newly adjacent facets: as before, it is a matter of choosing n_2 ridges, out of the at most $(d-1)n_1$ just created ones. Once this is done, at most $(d - 1)n_2$ ridges are created. And so on.

We proceed this way until all the $2D$ facets in the boundary of B have been matched (after f steps, say). Clearly $n_1 + \ldots + n_f = D$, and since the n_i's are positive integers, $f \leq D$ must hold. This means there are at most

$$\sum_{f=1}^{D} \quad \sum_{\substack{n_1, \ldots, n_f \\ n_i \geq 1, \sum n_i = D \\ n_{i+1} \leq (d-1)n_i}} \binom{dD}{n_1}\binom{(d - 1)n_1}{n_2}\binom{(d - 1)n_2}{n_3} \cdots \binom{(d - 1)n_{f-1}}{n_f}$$

possible perfect matchings of $(d - 1)$-simplices in the boundary of a tree of N d-simplices.

We sharpen this bound by observing that not all ridges may be chosen in the first round of identifications. For example, we should exclude those

ridges that belong to just two d-simplices of B. An easy double-counting argument reveals that in a tree of d-simplices, the number of ridges belonging to at least 3 d-simplices is smaller or equal than $\frac{N}{3} \binom{d+1}{2}$. So in the upper bound above we may replace the first factor $\binom{dD}{n_1}$ with the smaller factor $\binom{\frac{N}{3} \binom{d+1}{2}}{n_1}$.

To bound the sum from above, we use $\binom{a}{b} \leq 2^a$ and $n_1 + \cdots + n_{f-1} < < n_1 + \cdots + n_f = D$, while ignoring the conditions $n_{i+1} \leq (d-1)n_i$. Thus we obtain the upper bound

$$2^{\frac{N}{3}\binom{d+1}{2}+\frac{N}{2}(d-1)^2+(d-1)} \cdot \sum_{f=1}^{D} \binom{D-1}{f-1} \;=\; 2^{\frac{N}{3}(2d^2-d)+(d-1)}.$$

Thus the number of ways to fold a tree of N d-simplices into a manifold via a local construction sequence is smaller than $2^{\frac{2d^2-d}{3}N}$. Combining this with Proposition 2.1.2, we conclude the proof for the case of simplicial d-manifolds. We leave the adaption of the proof for simplicial d-manifolds with boundary (or simplicial LC d-pseudomanifolds) to the reader. $\qquad\square$

The upper bound of Theorem 2.5.1 can be simplified in many ways. For example, for $d \geq 16$ it is smaller than $\sqrt[3]{4}^{d^2}$. From Theorem 2.5.1 we obtain explicit upper bounds:

- there are less than 216^N simplicial LC 3-spheres with N facets,
- there are less than 6117^N simplicial LC 4-spheres with N facets,

and so on. We point out that these upper bounds are not sharp, as we overcounted both on the combinatorial side and on the algebraic side. When $d = 3$, however, our constant is smaller than what follows from Durhuus–Jonsson's original argument:

– we improved the matchings-bound from 384^N to 32^N;
– for the count of trees of N tetrahedra we obtain an essentially sharp bound of 6.75^N. (The value implicit in the Durhuus–Jonsson argument [50, p. 184] is larger since one has to take into account that different trees of tetrahedra can have the same unlabeled dual graph.)

Corollary 2.5.2. *For any fixed $d \geq 2$, there are exponential lower and upper bounds for the number of simplicial LC d-manifolds on N facets.*

Proof. Juxtapose Corollary 2.1.4 and Theorem 2.5.1. $\qquad\square$

2.6 Beyond the LC class

In the proof of Theorem 2.5.1, the final exponential bound is the output of three factors:

(1) the total number of $(d-1)$-faces to match is linear in N;
(2) there is a restriction (namely, the adjacency condition) on the couples of $(d-1)$-faces that may be identified in the first round. As a result, the number of *admissible couples* is just *linear* in N (while the number of all pairs of boundary facets is *quadratic* in N);
(3) the number of new admissible couples created after every single gluing is bounded by a constant (specifically, the constant $d-1$).

In the following, we weaken the LC notion maintaining the properties (1), (2) and (3). As a result, we will still have exponential upper bounds.

2.6.1 LC manifolds with bounded facet complexity

If we try to consider polytopal complexes instead of simplicial complexes, how do we bound the number of trees of polytopes?

Definition 2.6.1. Let $A > d > 1$ be integers. A d-manifold M on N facets has *facet complexity bounded by A* if every d-polytope of M has at most A facets.

Proposition 2.6.2. *There are exponentially many trees of N d-polytopes of bounded facet complexity.*

Proof. Enumerate from 1 to C_A the different combinatorial types of d-polytopes with at most A facets. By looking at its dual graph, a tree of N polytopes of complexity bounded by A can be represented by a tree of N "coloured" nodes, where each colour is just an integer in $\{1, \ldots, C_A\}$. Since C_A is a constant, there are only exponentially many coloured unlabeled trees on N nodes.

Already for $d = 2$, however, different trees of triangles may have the same dual graph. Thus the previous representation is not unique. Now, let R_A be the number of different ways in which one can glue together two d-polytopes with at most A facets: This "rotational factor" R_A is finite and (like C_A) should be regarded as a constant. Given a tree T on N vertices, there are at most $(R_A)^N$ different trees of d-simplices whose dual graph is T. Multiplying this exponential factor with the one obtained from counting the coloured trees on N nodes, one gets the desired exponential upper bound. $\square$

Suppose that every boundary facet of a tree of d-polytopes for M has at most B ridges. The existence of A implies the existence of such a constant B: in fact, $B \leq A - 1$, because if F is a facet of a polytope P, the number of ridges of F equals the number of facets of P that are adjacent to F, which is certainly not bigger than the number of facets of P different from F. Note that in the simplicial case $A = d + 1$ and $B = d$, while in the cubical case $A = 2d$ and $B = 2d - 2$.

Let us check that the conditions (1), (2) and (3) are fulfilled:

(1)   The total number of facets to match is at most $A + (N-1)(A-2)$. (The bound is sharp if every d-face has *exactly* A subfaces of codimension one.) Any matching (and a fortiori any partial matching) will consist of at most $D_A := \frac{A + (N-1)(A-2)}{2}$ pairs.

(2)   In the first round, we are allowed to identify only *adjacent* $(d-1)$-faces that belong to different, non-adjacent d-polytopes. Pairs of adjacent $(d-1)$-faces are counted by $(d-2)$-faces, namely, the boundary ridges. Since each d-polytope has at most $\frac{AB}{2}$ boundary ridges, the global number of boundary ridges in a tree of N d-polytopes is at most $\frac{AB}{2} + (N-1)(\frac{AB}{2} - d)$, which is linear in N. (In the cubical case, we may replace the summand d by $2d - 2$.)

(3)   The constant B can be used to (over)count how many new adjacencies we produce after a single identification: At most $B - 1$.

Theorem 2.6.3. *Let $A, d \in \mathbb{N}$, with $A > d > 1$. There are at most exponentially many LC d-manifolds with N d-polytopes, provided each d-polytope has at most A facets.*

Proof. In view of Proposition 2.6.2, we only need to find an exponential upper bound for the number of matchings. Analogously to the proof of Theorem 2.5.1, such a bound is given by

$$\sum_{f=1}^{D_A} \sum_{\substack{n_1, \ldots, n_f \\ n_i \geq 1,\ \sum n_i = D_A \\ n_{i+1} \leq (B-1)n_i}} \binom{\frac{AB}{2}N - dN + d}{n_1} \binom{(B-1)n_1}{n_2} \cdots \binom{(B-1)n_{f-1}}{n_f}$$

where $B \leq A - 1$ is the maximal number of "ridges-per-polytope", and $D_A := \frac{A + (N-1)(A-2)}{2}$. To check that this bound is exponential in N, apply everywhere the inequality $\binom{a}{b} < 2^a$. $\qquad \square$

2.6.2 Simplicial k-LC manifolds

Definition 2.6.4. Let $k, d \in \mathbb{N}$, with $d \geq 2$. Let M be a simplicial d-manifold on N facets. M is k-*LC* if and only if M can be obtained from a tree of N d-simplices by gluing together $2k$ boundary facets pairwise (preserving the orientation) and then by repeatedly identifying two *adjacent* boundary facets.

When $k = 0$ we recover the classical LC notion: "0-LC" is just another word for "LC". As k gets larger, however, more and more manifolds are captured in the k-LC class. Eventually, every d-manifold is k-LC for some k. (In fact, when $k = f_{d-1}(M) - f_d(M) - 2$, all of the boundary facets but two are matched in the first phase: Since the two $(d-1)$-faces left have to be adjacent, their gluing is a local one.)

As in the proof of Theorem 2.5.1, we divide the sequence of LC gluings into "rounds". The novelty is that there is now a round zero in which we are allowed to match a constant number of faces wildly. A tree of N d-simplices has $(d-1)N + 2$ boundary facets of dimension $d - 1$. There are $\binom{(d-1)N+2}{2k}$ ways to choose $2k$ facets out of them and $(2k)!! = 2k \cdot (2k-2) \cdot \ldots \cdot 2 = 2^k \cdot k!$ ways to match these $2k$ facets. This leads to a factor

$$2^k \cdot k! \cdot \binom{(d-1)N + 2}{2k} \leq 2^k \cdot k! \cdot 2^{(d-1)N+2},$$

which is exponential in N because k and d are to be regarded as constants. Each wild matching produces at most d new adjacencies: Thus when round zero is over at most dk new adjacencies have been formed. We proceed then via local gluings, so conditions (1), (2) and (3) are met.

Theorem 2.6.5. *Let $k, d \in \mathbb{N}$, with $d \geq 2$. There are at most $2^{d^2 \cdot N} \cdot k! \cdot (de)^N$ combinatorial types of k-LC simplicial d-manifolds with N facets.*

Proof. The number of distinct trees of d-simplices is bounded by $(de)^N$ according to Corollary 2.1.3. As far as the matchings are concerned, reasoning as in the proof of Theorem 2.5.1 we obtain the exponential upper bound

$$2^{(d-1)N+k+2} \cdot k! \sum_{f=1}^{D} \sum_{\substack{n_1, \ldots, n_f \\ \sum n_i = D}} \binom{dD + dk}{n_1} \binom{(d-1)n_1}{n_2} \cdots \binom{(d-1)n_{f-1}}{n_f},$$

where $D = \frac{(d-1)N+2}{2}$. Applying the inequality $\binom{a}{b} < 2^a$ everywhere, we obtain the upper bound

$$k! \cdot 2^{(2d+1)(d-1)\frac{N}{2}} \cdot 2^{(k+2)(d+1)},$$

whence we conclude using that $k + 2 \leq f_{d-1}(M) - f_d(M) = \frac{N}{2}$. $\hfill\square$

We leave it to the reader to adapt the previous Theorem to the case of k-LC manifolds with bounded facet complexity.

2.6.3 Distance-m-to-LC spheres

Definition 2.6.6. The *distance* of two facets in a strongly connected pure regular CW complex is their distance as vertices of the dual graph, that is, the minimal length of a dual path that connects them. If m is a positive integer, a simplicial d-pseudomanifold is *distance-m-to-LC* if it can be obtained from a tree of d-simplices by repeatedly identifying two boundary facets (preserving orientation) that are at distance at most $m + 1$ from one another.

When $m = 0$, distance-m-to-LC spheres are just LC spheres. As m gets larger, the class of distance-m-to-LC manifolds gets larger too. Eventually, every simplicial d-sphere is distance-m-to-LC for some m.

We proceed now to check whether condition (1), (2) and (3) hold. Let us assume $d \geq 3$. (The case $d = 2$ is not so interesting, since we already know that 2-spheres are "not so many"). Fix a facet σ of the boundary of a tree of d-simplices B.

– The number $c_{m+1}(\sigma)$ of boundary facets at distance $m + 1$ from σ is bounded above by the number of non-self-intersecting walks in the dual graph of ∂B, starting at σ and having length $(m + 1)$. Since we have d choices of where to go with our first step and $d - 1$ choices in each subsequent step, we obtain that

$$c_{m+1}(\sigma) \leq d \cdot (d - 1)^m.$$

– The number $c_{\leq m+1}(\sigma)$ of boundary facets at distance *at most* $m + 1$ from σ is thus bounded above by

$$\sum_{i=1}^{m+1} d \cdot (d-1)^{i-1} = d \cdot \left(\frac{(d-1)^{m+1} - 1}{(d-1) - 1} - 1 \right) <$$
$$< \frac{d}{d-2}(d-1)^{m+1} \leq 3\,(d-1)^{m+1}.$$

– The number f_{m+1} of pairs of facets at distance $m + 1$ from one another can be estimated by recursion. The initial value is $f_1 = \frac{d}{2}((d-1)N+2)$, since the pairs of adjacent $(d-1)$-faces in ∂B are in bijection with the ridges of ∂B. Further, $f_2 = ((d-1)N+2)\binom{d}{2} = (d-1)f_1$: In fact, every

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pair $\{\sigma, \tau\}$ of $(d-1)$-simplices at distance two is uniquely determined by the $(d-1)$-simplex in between them, once you choose two $(d-2)$-faces of it (the ones adjacent to σ and τ). Finally, $f_{m+1} \le (d-1)^2 f_{m-1}$, because every path of length $m+1$ has a central subpath of length $m-1$ which can be re-extended laterally to a path of length $(m+1)$ in $(d-1)^2$ possible ways. Recursively,

$$f_{m+1} \le (d-1)^m f_1 = \frac{d}{2}((d-1)N + 2) \cdot (d-1)^m.$$

– In particular, the number $f_{\le m+1}$ of (unordered) pairs of facets at distance $\le m+1$ from one another is $\frac{d}{2} \cdot ((d-1)N + 2) \cdot \left(\frac{(d-1)^{m+1}-1}{(d-1)-1} - 1 \right)$, which is at most $\frac{3}{2}((d-1)N + 2)(d-1)^{m+1}$.

Thus conditions (1) and (2) are satisfied, since the admissible couples (meaning in this case the pairs of facets at distance at most $m+1$) are at most

$$F_{m+1} := \frac{3}{2}((d-1)N + 2)(d-1)^{m+1}.$$

To check that condition (3) holds as well, fix two facets σ and σ' at distance at most $m+1$ from one another, and define $g_{m+1}(\sigma, \sigma')$ as the number of new admissible couples created by an identification $\sigma \equiv \sigma'$. Our goal is to show that $g_{m+1}(\sigma, \sigma')$ is bounded by a linear function in N that does not depend on σ or σ'.

Each new couple created by the identification $\sigma \equiv \sigma'$ consist of two boundary facets μ, μ' whose distance have become smaller or equal than $m+1$ after we glued σ and σ' together. Thus (up to relabeling) μ was at distance j from σ for some $j \in \{1, \ldots, m+1\}$, while μ' was at distance at most $m-j+2$ from σ'. (For example, when $m=0$, μ had to be adjacent to σ and μ' had to be adjacent to σ'.) So, μ belongs to a class of cardinality $c_j(\sigma) \le d(d-1)^{j-1}$, while μ' is in a class of cardinality $c_{\le m-j+2}(\sigma) \le 3(d-1)^{m-j+2}$. Since j can be arbitrarily chosen in $\{1, \ldots, m+1\}$, we obtain

$$g_{m+1}(\sigma, \sigma') \le \sum_{j=1}^{m+1} 3d(d-1)^{j-1}(d-1)^{m-j+2} \le$$

$$\le 3d \sum_{j=1}^{m+1} (d-1)^{m+1} = 3(m+1)d(d-1)^{m+1}.$$

Thus $g_{m+1}(\sigma, \sigma')$ is bounded above by a constant

$$G_{m+1} := 3(m+1)d(d-1)^{m+1},$$

and condition (3) is safe. So, we can announce the following result:

Theorem 2.6.7. *For every $m \in \mathbb{N}$, there are exponentially many simplicial distant-m-to-LC d-spheres with N facets.*

Proof. Trees of d-simplices are exponentially many in N. To bound the number of matchings, define F_{m+1} and G_{m+1} as above and proceed as in the proof of Theorem 2.5.1 to obtain the upper bound

$$\sum_{f=1}^{N+1} \sum_{\substack{n_1, \ldots, n_f \\ n_i \geq 1, \sum n_i = D \\ n_{i+1} \leq G_{m+1} n_i}} \binom{F_{m+1}}{n_1}\binom{G_{m+1}\, n_1}{n_2}\binom{G_{m+1}\, n_2}{n_3}\cdots\binom{G_{m+1}\, n_{f-1}}{n_f},$$

which is exponential in N. $\qquad\qquad\square$

2.6.4   Mogami's "weakly LC spheres"

In 1995 Mogami [106] proposed a weaker version of local constructibility. Here is his idea: During a local construction of a 3-sphere, one is also allowed to identify two (or even more) edges that share a common vertex v in the boundary, provided (1) no edge participates in this process twice and (2) the identified edges do not belong to the same 3-face, nor to 2-faces that are already adjacent.[2] We will call this identification a *Mogami move on the vertex v*.

Any two boundary facets that share a vertex v can be identified performing a Mogami move on the vertex v, followed by an LC move. Thus, all the simplicial manifolds obtained from a tree of tetrahedra by repeatedly identifying two *non-disjoint* boundary triangles (preserving orientation) are "weakly LC" in the sense of Mogami.

Mogami moves preserve simply connectedness. However, Mogami moves are not internal to the world of pseudomanifolds: If we identify two adjacent edges in the boundary of a finely triangulated 3-ball, we obtain a CW complex in which one boundary edge is shared by *four* distinct boundary triangles.

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[2]Condition (2) is not explicitly mentioned in Mogami's paper. However, identifying two edges that belong to the same polytope would destroy the regularity of the CW complex, whereas we are interested in local constructions that eventually yield a polytopal complex. On the other hand, identifying edges that belong to already adjacent faces σ' and σ'' is a waste of time: Later σ' and σ'' must be glued together if we want to reach a polytopal complex, so why not performing the local gluing $\sigma' \equiv \sigma''$ in the first place?

Mogami showed that "weakly-LC" simplicial 3-manifolds on N facets are exponentially many in N. His proof [106] is slightly different from the proof of Theorem 2.5.1 and we do not present it here. However, from [106] it is not clear how much Mogami moves help in expanding the classical LC notion: Are there non-weakly LC 3-spheres? Are there weakly-LC 3-spheres that are not LC? As far as we know, both questions are still open. However, in Figure 5.3 we will give an example of a 3-pseudomanifold that is weakly-LC but not LC.

Chapter 3

Collapses

If we remove the interior of a d-face from a d-manifold M, does the resulting complex collapse onto a $(d-1)$-complex? Does it collapse onto a $(d-2)$-complex? Does it collapse onto a $(d-t)$-complex, for some $t \leq d-1$? These questions will have great consequence in the rest of this book.

Let us first recall the meaning of the word 'collapse'. Let C be a d-dimensional polytopal complex. An *elementary collapse* is the simultaneous removal from C of a pair of faces (σ, Σ) with the following prerogatives:

– $\dim \Sigma = \dim \sigma + 1$;
– σ is a proper face of Σ;
– σ is not a proper face of any other face of C.

The three conditions above (the first of which is implied by the other two) are usually abbreviated in the expression "σ is a free face of Σ". Some complexes (like any 2-sphere, or the Dunce Hat, cf. Figure 3.1) have no free face.

If $C' := C - \Sigma - \sigma$, we say that the complex C *collapses onto* the complex C'. We also say that the complex C *collapses onto* the complex D, and write $C \searrow D$, if C can be reduced to D by a finite nonempty sequence of elementary collapses. (Thus a *collapse* refers to a sequence of elementary collapses, cf. Figure 3.2.) A *collapsible* complex is a complex that can be collapsed onto a single vertex. Since $C' := C - \Sigma - \sigma$ is a deformation retract of C, each collapse preserves the homotopy type. In particular, all collapsible complexes are contractible. The converse holds for 1-complexes: "contractible 1-complex" is a synonime of "tree", and trees

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can be collapsed by removing one leaf at the time. However, the Dunce Hat (Fig. 3.1) is a contractible 2-complex without free edges, and thus with no elementary collapse to start with. If we restrict the topology to balls, it is
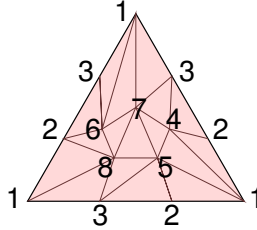
Figure 3.1: The Dunce Hat.

easy to see that every contractible 2-dimensional polytopal complex that is homeomorphic to a 2-ball is collapsible. But even this statement does not generalize to $d \geq 3$, as shown in 1964 by Bing [21], who gave an example of non-collapsible 3-ball.

Also, a connected 2-dimensional complex is collapsible if and only if it does *not* contain a 2-dimensional complex without a free edge. In particular, for 2-dimensional complexes, if $C \searrow D$ and D is not collapsible, then C is also not collapsible [76]. This holds no more for complexes C of dimension larger than two: for example, there are collapsible simplicial 3-balls that can be collapsed onto the Dunce Hat as well (see Section 5.4.) Due to the work of Cohen [41], a complex C is contractible if and only if some collapsible complex D collapses also onto C.

By a result of Dong [48, Lemma 17, p. 1116], for every integer $d > 0$, if a contractible d-complex C is shellable then C is also collapsible. This implication is also known to be strict: as early as 1958 Mary E. Rudin [126] obtained a non-shellable 3-ball by subdividing a tetrahedron (see Example 1.3.4), and nine years later Chillingworth [40] showed that every subdivided tetrahedron is collapsible. Thus Rudin's 3-ball is collapsible, contractible, constructible, but not shellable.

Note that a shellable 3-ball B has some boundary facet σ such that the removal of σ and of the unique tetrahedron Σ containing it deforms the 3-ball B into another (shellable) 3-ball B'.[1] Constructible or collapsible balls might not satisfy this property: for example, any elementary collapse deforms Rudin's ball into a contractible complex that is not homeomorphic to a 3-ball any more.

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[1]In [154] the tetrahedron Σ with this property is called "free facet", and a 3-ball without free facets is called "strongly non-shellable". We will refrain from this notation to avoid any possible confusion with the notion of "free face".

3.1   Collapsing a manifold minus a facet

Definition 3.1.1. A *natural labeling* of a rooted tree T (on n vertices) is a bijection

$$b : V(T) \longrightarrow \{1, \ldots, n\}$$

such that the root is mapped to 1, and if v is not the root, then there exists a vertex w adjacent to v such that $b(w) < b(v)$.

Definition 3.1.2. By a *facet-killing sequence* for a d-dimensional complex C we mean a sequence $C_0, C_1, \ldots, C_{t-1}, C_t$ of complexes such that $t = f_d(C)$, $C_0 = C$ and C_{i+1} is obtained by an elementary collapse that removes a free $(d-1)$-face σ of C_i, together with the unique d-face Σ containing σ.

If C is a d-complex and D is a lower-dimensional complex such that $C \searrow D$, then there exists a facet-killing sequence $C_0, \ldots, C_t$ for C such that $C_t \searrow D$. In other words, the collapse of C onto D can always be rearranged so that the pairs $((d-1)$- face, d- face) are removed first. In particular, for any d-complex C, the following are equivalent:

1. there exists a facet-killing sequence for C;
2. there exists a k-complex D with $k \le d - 1$ such that $C \searrow D$.

Let M be a manifold; fix a facet Δ of M and a spanning tree T of the dual graph of M. Viewing T as rooted tree with root Δ, we may collapse the d-complex $C - \Delta$ along T, obtaining a lower dimensional complex K^T. More precisely, any natural labeling of the rooted tree T naturally induces a facet-killing collapse of $M - \Delta$: The i-th facet to be collapsed is the node of T labelled by $i + 1$.

A crucial remark is that the $(d-1)$-complex obtained collapsing along T does not depend on the natural labeling, nor on the Δ chosen: it is simply the pure $(d-1)$-dimensional subcomplex of M formed by all the $(d-1)$-faces of M that are not intersected by T.

Definition 3.1.3 ($\mathbf{K^T}$). Let M be a manifold and T a spanning tree of its dual graph. We call K^T the pure $(d-1)$-dimensional subcomplex of M formed by all the $(d-1)$-faces of M that are not intersected by T.

In case M is simplicial, it is easy to compute the number of facets of K^T: Every facet of M contains $d + 1$ ridges and every ridge lies in exactly two facets, so M has $\frac{d+1}{2} \cdot N$ ridges, $N - 1$ of which are hit by T. Therefore K^T has exactly $\frac{d+1}{2}N - (N - 1)$ facets, all of them of dimension $d - 1$. Analogously, in case M is cubical each of the N facets of M contains $2d$

ridges, so we get that K^T has exactly $d \cdot N - (N-1)$ facets (all of them of dimension $d-1$).

We have seen that for any facet Δ of M, $\quad M - \Delta$ collapses "along T" onto K^T. On the other hand, any collapse of a d-manifold M minus a facet Δ onto a complex of dimension at most $d-1$ proceeds along a dual spanning tree T. To see this, fix a collapsing sequence; we may assume that the collapse of $M - \Delta$ is ordered so that the pairs $(\,(d-1)$-face, d-face$\,)$ are removed first. Whenever

1. σ is the intersection of the d-faces Σ and Σ' of M, and
2. the pair (σ, Σ) is one of the elementary collapses in the collapsing sequence of $M - \Delta$,

draw an oriented arrow from the barycenter of Σ to the barycenter of Σ'. This yields a directed spanning tree T of the dual graph of M, where Δ is the root. Indeed, T is *spanning* because all d-simplices of $M - \Delta$ are removed in the collapse; it is *acyclic*, because the barycenter of each d-simplex of $M - \Delta$ is reached by exactly one arrow; it is *connected*, because the only free $(d-1)$-faces of $M - \Delta$, where the collapse can start at, are the proper $(d-1)$-faces of the "missing simplex" Δ. We will say that the collapsing sequence *acts along the tree T* (in its top-dimensional part). Thus the complex K^T appears as intermediate step of the collapse: It is the complex obtained after the $(N-1)$-st pair of faces has been removed from $M - \Delta$.
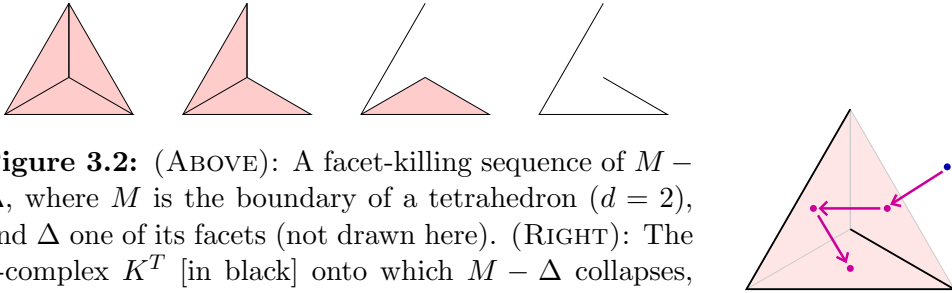
Figure 3.2: (ABOVE): A facet-killing sequence of $M - \Delta$, where M is the boundary of a tetrahedron ($d = 2$), and Δ one of its facets (not drawn here). (RIGHT): The 1-complex K^T [in black] onto which $M - \Delta$ collapses, and the directed spanning tree T [in purple] along which the collapse above acts.

What we argued so far can be rephrased as follows:

Proposition 3.1.4. *Let M be a d-manifold, Δ a d-face of M. Let C be any k-dimensional subcomplex of M, with $k \le d - 2$. Then,*

$$M - \Delta \searrow C \Longleftrightarrow \exists\, T \text{ s.t. } K^T \searrow C.$$

The right-hand side in the equivalence of Proposition 3.1.4 does not depend on the Δ chosen. So, for any d-polytope Δ, either $M - \Delta$ is collapsible for every Δ, or $M - \Delta$ is not collapsible for any Δ.

Take a d-manifold M, a facet Δ of M, and a rooted spanning tree T of the dual graph of M, with root Δ. By Proposition 3.1.4, natural labelings of T are in bijection with collapses $M - \Delta \searrow K^T$ (the i-th facet to be collapsed is the node of T labelled $i + 1$).

If we fix M and Δ, but not T, the previous reasoning yields a bijection among the set of all facet-killing sequences of $M - \Delta$ and the set of natural labelings of spanning trees of M, rooted at Δ.

Remark 3.1.5. For the results in this section it is not necessary for M to be a manifold: It suffices that M is a strongly-connected polytopal complex such that every ridge of M lies in two facets of M. Also, suppose a polytopal complex C is connected but not strongly-connected, and let k be the number of "strongly-connected components" of C (which is the same as the number of connected components of the dual graph of C). If every ridge of C lies in two facets, then C collapses onto a $(d-1)$-complex after the removal of exactly k faces. (These k faces may be chosen arbitrarily, provided we pick one per each strongly-connected component.)

3.2 Collapse depth

Definition 3.2.1 (Collapse depth)**.** The *collapse depth* of a d-complex C is defined as

$$\operatorname{cdepth}(C) := d - \min\{\dim D \; : \; (C - \Delta) \searrow D, \text{ for some } d\text{-face } \Delta \text{ of } C\}.$$

In other words, $\operatorname{cdepth}(C) \geq c$ if and only if C minus a facet collapses onto a complex of dimension $\dim C - c$. We showed in the previous section that every manifold has collapse depth greater or equal than one. Recall that when C is a manifold, the complex of minimal dimension onto which $C - \Delta$ collapses does not depend on the Δ chosen.

Definition 3.2.2. Let K be a d-manifold, A an r-face in K, and $\hat{A}$ the barycenter of A. Consider the barycentric subdivision $sd(K)$ of K. The *dual* A^* of A is the subcomplex of $sd(K)$ given by all flags

$$A = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_r$$

where $r = \dim A$, and $\dim A_{i+1} = \dim A_i + 1$ for each i.

A^* is a cone with apex $\hat{A}$ [110, pp. 377-380], and thus collapsible by Proposition 3.4.1.

Lemma 3.2.3 ("Newman's theorem"[77, pp. 29-30][110, pp. 377-380]).
Let K be a PL d-manifold (without boundary) and let A be a polytope in K of dimension r. Then

- A^* *is a* $(d-r)$-*ball, and*
- *if A is a face of an* $(r+1)$-*polytope B, then B^* is a* $(d-r-1)$-*sub-complex of* ∂A^*.

Theorem 3.2.4. *Every PL manifold M is* $(\operatorname{cdepth}(M)-1)$-*connected.*

Proof. Let M be a PL d-manifold; suppose that $M - \Delta$ collapses onto some t-complex. We can assume that the collapse of $M - \Delta$ is ordered so that:

– first all pairs (d-face, $(d-1)$-face) are collapsed;
– then all pairs (($d-1$)-face , $(d-2)$-face) are collapsed;
– $\vdots$
– finally, all pairs (($t+1$)-face, t-face) are collapsed.

Let us put together all the faces that appear above, maintaining their order, to form a unique list of faces

$$A_1, A_2, \ldots, A_{2E-1}, A_{2E}.$$

In this list A_1 is a free face of A_2; A_3 is a free face of A_4 with respect to the complex $M - A_1 - A_2$; and so on. In general, A_{2i-1} is a face of A_{2i} for each i, and in addition, if $j > 2i$, A_{2i-1} is not a face of A_j.

We set $X_0 = A_0 := \hat{\Delta}$ and define a finite sequence $X_1, \ldots, X_E$ of CW complexes as follows:

$$X_j := \bigcup \left\{ A_i^* \text{ s.t. } i \in \{0, \ldots, 2j\} \right\}, \qquad \text{for } j \in \{1, \ldots, E\}.$$

In other words, $X_j = X_{j-1} \cup A_{2j-1}^* \cup A_{2j}^*$. By Lemma 3.2.3, A_{2j-1}^* is a $(d-r)$-ball that contains in its boundary the $(d-r-1)$-ball A_{2j}^*, where $r = \dim A_{2j-1}$. Thus $|X_j|$ is just $|X_{j-1}|$ with a $(d-r)$-cell attached via a cell in its boundary, and such an attachment does not change the homotopy type. Since X_0 is a point, it follows that X_E is contractible.

Now, let us list by (weakly) decreasing dimension all the faces of M that did not previously appear in the list $A_1, A_2, \ldots, A_{2E-1}, A_{2E}$. We name the elements of such new list

$$A_{2E+1}, A_{2E+2}, \ldots, A_F,$$

where $\sum_{i=1}^{d} f_i(M) = F + 1$ because all faces appear in $A_0, \ldots, A_F$.

Correspondingly, we define a new sequence of CW complexes setting $Y_0 := X_E$ and $Y_h := Y_{h-1} \cup A_{2E+h}^*$.

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Since $\dim A_{2E+h} \leq \dim A_{2E+1} = t$, we have that $|Y_h|$ is just $|Y_{h-1}|$ with a cell of dimension at least $d - t$ attached via its whole boundary. Let us consider the homotopy groups of the Y_h 's: Recall that Y_0 was homotopy equivalent to a point and that by construction Y_{2E-F} coincides with $sd(M)$. Therefore $\pi_j(Y_h) = 0$, for all h and for each j in $\{1, \ldots, d - t - 1\}$.

The conclusion follows taking t maximal with respect to the property that $M - \Delta$ collapses onto a t-complex: in this case $t = d - \text{cdepth}(M)$, whence $\pi_j(M) = 0$ for each $j \in \{1, \ldots, \text{cdepth}(M) - 1\}$. $\qquad\square$

Corollary 3.2.5. *Let $d \geq 3$. Let M be a PL d-manifold. If $\text{cdepth}(M) \geq d - 1$ then M is a d-sphere and $\text{cdepth}(M) = d$.*

Proof. By Theorem 3.2.4, $\pi_i(M) = 0$ for all $i \leq d - 2$. It is an outstanding result of algebraic topology that this implies M is a d-sphere (provided $d \geq 3$). In fact, by Hurewicz' theorem, by the Universal Coefficient Theorem for cohomology and by Poincaré duality, $\pi_i(M) = H_i(M) = 0$ for all i in $\{1, \ldots, d - 1\}$. Every simply connected manifolds is orientable: Thus by Hurewicz' theorem $\pi_d(M) = H_d(M) = \mathbb{Z}$. So M is a homotopy d-sphere. This implies that M is a d-sphere by the generalized Poincaré conjecture, which was proven (in inverse chronological order) for $d = 3$ by Perelman [117] [118] [88] [36] [107] [108], for $d = 4$ by Freedman [56] and for $d \geq 5$ by Smale [130] on the beaches of Rio de Janeiro [131] [155].

A d-sphere minus a facet yields a contractible complex and contractibility is preserved throughout a collapse. Since all contractible 1-complexes are collapsible, a d-sphere has collapse depth $\geq d - 1$ if and only if it has collapse depth d. $\qquad\square$

Remark 3.2.6. The validity of the proof above does not rely on Perelman's proof of the Poincaré conjecture. When $d = 3$, in fact, Corollary 3.2.5 boils down (via Theorem 5.2.6) to Corollary 1.6.7, which was proven by Durhuus and Jonsson with elementary methods.

When $d = 3$, also Theorem 3.2.4 boils down via Theorem 5.2.6 to a result of Durhuus and Jonsson, namely, "all locally constructible 3-manifolds are simply connected" (cf. Lemma 1.6.3).

Remark 3.2.7. While every manifold with collapse depth greater than k is k-connected, not every k-connected manifold has collapse depth greater than k: We will show in Corollary 4.3.9 that for each $d \geq 3$, some d-spheres have cdepth $= 1$. (All d-spheres are $(d-1)$-connected.) In Chapter 5 we will characterize the manifolds with cdepth ≥ 2 in terms of local constructibility.

Remark 3.2.8. Corollary 3.2.5 strengthens a result of Whitehead, "every collapsible d-manifold is a d-ball" [144, Thm. 23, Cor. 1]. The assumption $d \geq 3$ is crucial: Any triangulated torus has collapse depth one.

3.3   Collapses and products

In this section we investigate the relation between collapses and products. We have seen in Corollary 3.2.5 that all 3-manifolds with cdepth ≥ 2 are spheres: Here we show, for each $d \geq 4$, that d-manifolds with cdepth ≥ 2 might as well be *products* of spheres.

Recall that the product of two polytopal complexes $X \times Y$ is a polytopal complex whose nonempty cells are the products $P_\alpha^i \times P_\beta^j$, where P_α^i ranges over the nonempty polytopes of X and P_β^j ranges over the nonempty polytopes of Y (cf. Ziegler [153, p.10]). In general, the product of two simplicial complexes is not a simplicial complex; however, the product of two cubical complexes is a cubical complex.

Proposition 3.3.1 (Cohen [41, p. 254]). *Let A and B be two polytopal complexes. If A collapses onto a complex C_A then $A \times B$ collapses onto $C_A \times B$.*

Proof. Let $B_1, \ldots, B_M$ be an ordered list of all the faces of B, ordered by weakly decreasing dimension. Let (σ_1^A, Σ_1^A) be the first pair of faces appearing in the collapse of A onto C_A. We perform the M collapses $(\sigma_1^A \times B_1, \Sigma_1^A \times B_1),\ \ldots, (\sigma_1^A \times B_M, \Sigma_1^A \times B_M)$, in this order. It is easy to check that each of the steps above is a legitimate collapse: When we remove $\sigma_1^A \times B_i$ all the faces of the type $\sigma_1^A \times \beta$ containing $\sigma_1^A \times B_i$ have already been removed, because in the list $B_1, \ldots, B_M$ the face β appears before B_i. On the other hand, σ_1^A is a *free* face of Σ_1^A, thus no face of the type $\alpha \times B_i$ may contain $\sigma_1^A \times B_i$ other than $\Sigma_1^A \times B_i$.

Next, we consider the *second* pair of faces (σ_2^A, Σ_2^A) that appears in the collapse of A onto C_A and we repeat the procedure above, and so on: In the end, the only faces left are those of $C_A \times B$. $\square$

Corollary 3.3.2 (cf. Welker [143, Theorem 2.6]). *If A is collapsible, then $A \times B$ collapses onto a copy of B. In particular, $A \times B$ is collapsible if both A and B are collapsible.*

The converse of the latter implication is false: the product of an interval with the Dunce Hat is collapsible [150].

Now, consider a 1-sphere S consisting of four edges. The 2-complex $S \times S$ is a cubical torus; after the removal of a facet, it collapses onto the union of a meridian and a parallel. (In fact, a punctured torus topologically retracts to a bouquet of two circles.) In the next result, we give an explanation to this fact in the generality of polytopal complexes (even if the same proof work also for regular CW complexes).

Proposition 3.3.3. *Let A and B be two polytopal complexes. Let Δ_A (resp. Δ_B) be a facet of A (resp. B). If $A - \Delta_A$ collapses onto some complex C_A and if $B - \Delta_B$ collapses onto some complex C_B then $(A \times B) - (\Delta_A \times \Delta_B)$ collapses onto $(A \times C_B) \cup (C_A \times B)$.*

Proof. We start by forming three ordered lists of pairs of faces.

Let (σ_1, Σ_1), ..., (σ_U, Σ_U) be the list of the removed pairs of faces in the collapse of A minus Δ_A onto C_A. (We assume that higher dimensional faces are collapsed first.) Analogously, let (γ_1, Γ_1), ..., (γ_V, Γ_V) be the list of all the removed pairs in the collapse of B minus Δ_B onto C_B. Let then $B_1, \ldots, B_W$ be the list of all the faces of B that are not in C_B, ordered by weakly decreasing dimension. The desired collapsing sequence for $(A \times B) - (\Delta_A \times \Delta_B)$ consists of $U + 1$ distinct phases:

PHASE 0: We remove from $(A \times B) - (\Delta_A \times \Delta_B)$ the V pairs of faces $(\Delta_A \times \gamma_1, \Delta_A \times \Gamma_1), (\Delta_A \times \gamma_2, \Delta_A \times \Gamma_2), \ldots, (\Delta_A \times \gamma_V, \Delta_A \times \Gamma_V)$, in this order. Analogously to the proof of Proposition 3.3.1, one sees that all these removals are elementary collapses. They wipe away the "Δ_A-layer" of $A \times B$, but not entirely: The faces $\alpha \times \beta$ with β in C_B are still present. What we have written is in fact a collapse of $(A \times B) - (\Delta_A \times \Delta_B)$ onto the complex $((A - \Delta_A) \times B) \cup (\Delta_A \times C_B)$.

PHASE 1: We take the first pair (σ_1, Σ_1) in the first list and we perform the W elementary collapses $(\sigma_1 \times B_1, \Sigma_1 \times B_1)$, ..., $(\sigma_1 \times B_W, \Sigma_1 \times B_W)$. This way we remove (with the exception of $\Sigma_1 \times C_B$) the Σ_1-layer of $A \times B$, where Σ_1 is the first facet of A to be collapsed away in $A - \Delta_A \searrow C_A$.

$\vdots$

PHASE j: We consider (σ_j, Σ_j) and proceed as in Phase 1, performing W collapses to remove the Σ_j-layer of $A \times B$ (except $\Sigma_j \times C_B$).

$\vdots$

PHASE U: We consider (σ_U, Σ_U) and proceed as in Phase 1, performing W collapses to remove the Σ_U-layer of $A \times B$ (except $\Sigma_U \times C_B$).

Eventually, the only faces of $A \times B$ left are those of $A \times C_B \cup C_A \times B$. $\square$

Corollary 3.3.4. *Given s polytopal complexes $A_1, \ldots, A_s$, suppose that each A_i after the removal of a facet collapses onto some lower-dimensional complex C_i. Then the complex $A_1 \times \ldots \times A_s$ after the removal of a facet collapses onto*

$$(C_1 \times A_2 \times \ldots \times A_s) \cup (A_1 \times C_2 \times A_3 \times \ldots \times A_s) \cup \ldots \cup (A_1 \times \ldots \times A_{s-1} \times C_s).$$

Proof. It follows directly from Proposition 3.3.3, by induction on s. $\square$

Corollary 3.3.5. *Given s manifolds $M_1, \ldots, M_s$, one has that*

$$\mathrm{cdepth}(M_1 \times M_2 \times \ldots \times M_s) \ \geq \ \min\{\mathrm{cdepth}\, M_1, \mathrm{cdepth}\, M_2, \ldots, \mathrm{cdepth}\, M_s\}.$$

Proof. By definition, each M_i minus a facet collapses onto a subcomplex C_i of dimension $\dim M_i - \mathrm{cdepth}\, C_i$ $(i = 1, 2)$. By Corollary 3.3.4, the union of complexes onto which the product of the M_i minus a facet collapses has dimension $\sum_{i=1}^{s} \dim M_i - \min\{\mathrm{cdepth}\, C_1, \ldots, \mathrm{cdepth}\, C_n\}$. $\qquad\square$

Corollary 3.3.6. *Let C_d denote the boundary complex of the $(d+1)$-cube $(d \geq 2)$. Then:*
  – *the cubical 4-manifold $C_2 \times C_2$ has collapse depth equal to two;*
  – *the cubical $(d+2)$-manifold $C_2 \times C_d$ has collapse depth not smaller than two;*
  – *the cubical 2d-manifold $C_d \times C_d$ has collapse depth not smaller than d.*

Proof. The collapse depth of $C_2 \times C_2$ cannot be larger than two, because $\pi_2(C_2 \times C_2)$ is non-trivial (compare Theorem 3.2.4). Since the collapse depth of C_d is d, the collapse depth of $C_d \times C_k$ is at least $\min(d, k)$, by Corollary 3.3.5. $\qquad\square$

3.4 Collapses and cones

It is well known that taking cones induces collapsibility:

Proposition 3.4.1 (Welker [143, Prop. 2.4]). *The join of two complexes C_1 and C_2 is collapsible if at least one of the C_i is collapsible. In particular, given a vertex v and an arbitrary polytopal complex C, the cone $v * C$ is collapsible.*

What about coning off the boundary of a 3-ball? When B is a *collapsible* 3-ball, coning off the boundary of B yields a sphere that collapses after the removal of a facet:

Proposition 3.4.2. *Let B be a 3-ball and let $S_B = B \cup v * \partial B$. Let C be a complex of dimension ≤ 2 such that $B \searrow C$. For every facet Δ of S_B, one has $S_B - \Delta \searrow C$.*

Proof. By Proposition 3.1.4, it suffices to prove that $S_B - \Delta \searrow C$ for *some* facet Δ. Choose a $(d-1)$-face σ in the boundary ∂B and consider $\Delta = v * \sigma$. As B collapses onto C, if we show that $(B \cup v * \partial B) - v * \sigma$ collapses onto B we are done.

Since all 2-balls are collapsible, there is some vertex P in ∂B such that $\partial B - \sigma \searrow P$. This induces a collapse of $v * \partial B - v * \sigma$ onto $\partial B \cup v * P$, according to the correspondence

$$\sigma \text{ is a free face of } \Sigma \iff v * \sigma \text{ is a free face of } v * \Sigma.$$

Furthermore, in such collapse the removed pairs of faces are all of the form $(v * \sigma, v * \Sigma)$; thus, the facets of ∂B are removed together with subfaces and not with superfaces. This means that the freeness of the faces in ∂B is not needed; so when we glue back B, the collapse

$$v * \partial B - v * \sigma \searrow \partial B \cup v * P$$

can be read off as

$$B \cup v * \partial B - v * \sigma \searrow B \cup v * P.$$

Collapsing the edge $v * P$ down to P, we conclude. $\square$

Remark 3.4.3. The converse of Proposition 3.4.2 does not hold: S_B minus a facet might be collapsible even if B is not collapsible (cf. Remark 5.3.5). See also Proposition 5.3.10 for a tricky variant of Proposition 3.4.2.

Proposition 3.4.2 can be extended to d-manifolds with (strongly connected) boundary:

Proposition 3.4.4. *Let $d > t \geq 2$ be two integers. Let B be a d-manifold with non-empty boundary and let $M_B = B \cup v * \partial B$. Suppose B collapses onto some $(d-t)$-complex. If $\mathrm{cdepth}(\partial B) \geq t$, then $\mathrm{cdepth}(M_B) \geq t$.*

Proof. For each facet σ of ∂B, the $(d-1)$-complex $\partial B - \sigma$ collapses onto some $(d-1-t)$-complex P, which does not depend on σ. Reasoning as in the proof of Proposition 3.4.2, one obtains a collapse

$$B \cup v * \partial B - v * \sigma \searrow B \cup v * P.$$

By assumption, B collapses onto some $(d-t)$-complex. Gluing the $(d-t)$-complex $v * P$ onto B at P does not interfere with such collapse. (Some k-faces of B might not be free any more, but their dimension k is too small to matter.) Therefore, $B \cup v * P$ still collapses onto a $(d-t)$-complex. $\square$

Remark 3.4.5. If a ball B collapses onto $\partial B - \sigma$ and $\partial B - \sigma$ is collapsible, then B is collapsible, too. Apart from this implication, these three properties seem to be independent:

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▷ since all 2-balls are collapsible, for each non-collapsible 3-ball B one
   has that $\partial B - \sigma$ is collapsible and B does not collapse onto it;

▷ for each 3-sphere S such that S minus a facet is not collapsible (see
   Example 5.3.4), given a new vertex v one has that $B := v * S$ is a
   collapsible 4-ball such that ∂B minus a facet is not collapsible; anyway,
   B does *not* collapse onto $\partial B - \sigma$ (cf. Prop. 1.7.2);

▷ in Theorem 6.3.6 we will give an example of a ball B such that B is
   collapsible, $\partial B - \sigma$ is collapsible, but B does not collapse onto $\partial B - \sigma$.

We do not know any non-collapsible 4-ball B that collapses onto $\partial B - \sigma$.

3.5 Collapses and subcomplexes

The manifolds that have maximal collapse depth are spheres, by Corollary
3.2.5. Lickorish [93] proved that if $S - \Delta$ is collapsible and $\mathfrak{L}$ is any $(d-2)$-
dimensional subcomplex of a d-sphere S, then the fundamental group of
$|S| - |\mathfrak{L}|$ has a presentation with (at most) $f_d(\mathfrak{L})$ generators.

   We present here a strengthened version of Lickorish's result. We show
that if M is a d-manifold and $cd(M) \geq k$, the $(k-1)$-th homotopy groups of
the complements of any $(d-k)$-subcomplex of M cannot be too complicated
to present:

Theorem 3.5.1. *Let t, d be integers with $0 \leq t \leq d-2$, and let M be a PL
d-manifold. Suppose that, for some facet Δ of M, $M - \Delta$ collapses onto
a t-dimensional complex. For each t-dimensional subcomplex $\mathfrak{L}$ of M, the
homotopy group*

$$\pi_{d-t-1}\left(|M| - |\mathfrak{L}|\right)$$

has a presentation with exactly $f_t(\mathfrak{L})$ generators, while the homotopy groups

$$\pi_i\left(|M| - |\mathfrak{L}|\right), \quad 1 \leq i < d-t-1,$$

are all trivial.

Proof. As in the proof of Theorem 3.2.4, let us form the list of faces

$$A_1, A_2, \ldots, A_{2E-1}, A_{2E}.$$

In such a list A_{2i-1} is a face of A_{2i} for each i; also, if $j > 2i$ then A_{2i-1} is
not a face of A_j. Set $X_0 = A_0 := \hat{\Delta}$ and define a finite sequence $X_1, \ldots, X_E$
of (not necessarily regular) CW complexes as follows:

$$X_j := \bigcup \left\{A_i^* \text{ s.t. } i \in \{0, \ldots, 2j\} \text{ and } A_i \notin \mathfrak{L}\right\}, \quad \text{for } j \in \{1, \ldots, E\}.$$

None of the A_{2i}'s can be in $\mathfrak{L}$, because $\mathfrak{L}$ is t-dimensional and $\dim A_{2i} \geq \dim A_{2E} = t + 1$ for each i. However, exactly $f_t(\mathfrak{L})$ of the A_{2i-1}'s are in $\mathfrak{L}$. Consider how X_j differs from X_{j-1}. There are two cases:

(1) If A_{2j-1} is not in $\mathfrak{L}$,

$$X_j = X_{j-1} \ \cup \ A_{2j-1}^* \ \cup \ A_{2j}^*.$$

By Lemma 3.2.3, setting $r = \dim A_{2j-1}$, A_{2j-1}^* is a $(d-r)$-ball that contains in its boundary the $(d-r-1)$-ball A_{2j}^*. Thus $|X_j|$ is just $|X_{j-1}|$ with a $(d-r)$-cell attached via a cell in its boundary, and such an attachment does not change the homotopy type.

(2) If A_{2j-1} is in $\mathfrak{L}$, then
$$X_j = X_{j-1} \ \cup \ A_{2j}^*.$$

As this occurs only when $\dim A_{2j-1} = t$, we have that $\dim A_{2j} = t + 1$ and $\dim A_{2j} = d - t - 1$; hence $|X_j|$ is just $|X_{j-1}|$ with a $(d-t-1)$-cell attached via its whole boundary.

Only in case (2) the homotopy type of $|X_j|$ changes at all, and this case (2) occurs exactly $f_t(\mathfrak{L})$ times. Since X_0 is one point, it follows that X_E is homotopy equivalent to a bouquet of $f_t(\mathfrak{L})$ many $(d-t-1)$-spheres.

Now, as in the proof of Theorem 3.2.4, let us list by (weakly) decreasing dimension all the faces of M that did *not* appear in the list $A_1, A_2, \ldots, A_{2E-1}, A_{2E}$. We name the elements of the new list

$$A_{2E+1}, A_{2E+2}, \ldots, A_F.$$

Correspondingly, we define a new sequence of subcomplexes of $sd(M)$ setting $Y_0 := X_E$ and

$$Y_h := \begin{cases} Y_{h-1} & \text{if } A_{2E+h} \in \mathfrak{L}, \\ Y_{h-1} \ \cup \ A_{2E+h}^* & \text{otherwise.} \end{cases}$$

Since $\dim A_{2E+h} \leq \dim A_{2E+1} = t$, we have that $|Y_h|$ is just $|Y_{h-1}|$ with possibly a cell of dimension at least $d-t$ attached via its whole boundary. Y_0 is homotopy equivalent to a bouquet of $f_t(\mathfrak{L})$ $(d-t-1)$-spheres; therefore, for each h and for each j in $\{1, \ldots, d-t-1\}$, one has that $\pi_j(Y_h) = 0$.

Moreover, the higher-dimensional cell attached to $|Y_{h-1}|$ to get $|Y_h|$ corresponds to the addition of relators to a presentation of $\pi_{d-t-1}(Y_{h-1})$ to get a presentation of $\pi_{d-t-1}(Y_h)$. This means that for all h the group $\pi_{d-t-1}(Y_h)$ is generated by (at most) $f_t(\mathfrak{L})$ elements.

The conclusion follows from the fact that Y_{2E-F} is the subcomplex of $sd(M)$ consisting of all simplices of $sd(M)$ that have no face in $\mathfrak{L}$; and it is known [110, Lemma 70.1][93, Lemma 1] that such a complex is a deformation retract of $|M| - |\mathfrak{L}|$. $\qquad\square$

In particular, if $M - \Delta$ collapses onto a $(d-2)$-dimensional complex, the fundamental group $\pi_1\left(|M| - |\mathfrak{L}|\right)$ has a presentation with $f_t(\mathfrak{L})$ generators, for each $(d-2)$-dimensional subcomplex $\mathfrak{L}$ of M.

Chapter 4

Knots

A (non-trivial) *knot* in a 3-sphere is a closed curve that neither intersects itself nor bounds a disc. The *trefoil* knot (also known among sailors as "overhand knot" or "thumb knot", see Figure 4.2) yields a classical example; another one is the *connected sum* of two trefoils, which is obtained[1] by cutting out a tiny arc from each and them sewing the resulting curves together along the boundary of the cutouts. (See Figure 4.3.)

All the knots we consider are *tame*, that is, realizable as 1-dimensional subcomplexes of some triangulated 3-sphere. A *regular projection* for a tame knot $\mathfrak{L}$ is an orthogonal projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ such that:

(1) the preimage of any point of $p(\mathfrak{L})$ contains at most two points of $\mathfrak{L}$;
(2) there are only finitely many points of $p(\mathfrak{L})$ (called *crossings*) whose preimage contains two points of $\mathfrak{L}$;
(3) the preimage of a crossing contains no vertex of $\mathfrak{L}$.

Regular projections always exist [83, pp. 7–8]: if we assume that the projection determined by the z-axis is regular, we can distinguish the two points in the preimage of each crossing as the *overcrossing* and the *under-*

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[1] The reader should be warned that two distinct knots — the so-called "granny knot" and the "square knot" — may be formed by summing two trefoils [125, pp. 40–41]. The reason is that two trefoils can be merged with consistent or with opposite orientations. However, the granny knot and the square knot have the same group. Since our excursion in knot theory will be exclusively focused on knot groups, we will talk about "connected sums of knots" without specifying the orientation of the summands — in fact, as far as the group is concerned, it does not matter.

crossing, according to the obvious convention that the z-coordinate of the undercrossing should be smaller than the z-coordinate of the overcrossing.

A *diagram* of a knot is its image under a regular projection. Given two diagrams of the same knot, it is always possible to pass from one diagram to the other via a finite sequence of *Reidemeister moves*, described in Figure 4.1. In general, two knots are *of the same type* if a diagram of the first one is related to a diagram of the second one by a finite sequence of Reidemeister moves [83]. In other words, Reidemeister moves are local and ergodic — like the Pachner moves we encountered in Section 1.2.
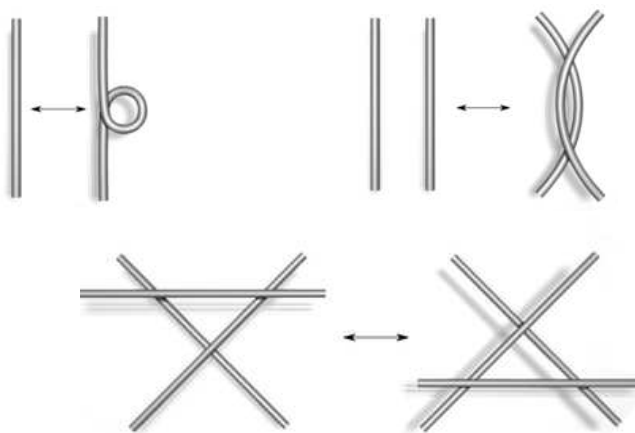
Figure 4.1: The three Reidemeister moves (from Wikipedia).

4.1 Knot groups

A knot is *m-complicated* if the fundamental group of the complement of the knot in the 3-sphere has a presentation with $m + 1$ generators, but no presentation with m generators. By "at least m-complicated" we mean "k-complicated for some $k \geq m$". There exist arbitrarily complicated knots: Goodrick [60] showed that the connected sum of m trefoil knots is at least m-complicated.

A more common measure of how tangled a knot can be is the bridge index, that can be defined as follows [83, p. 18]. Let $D_{\mathfrak{L}}$ be a diagram of a knot $\mathfrak{L}$. A *bridge* is a subarc of $D_{\mathfrak{L}}$ which contains only overcrossings (cf. Figure 4.2). The *bridge number* of a diagram $D_{\mathfrak{L}}$ is the minimum number of disjoint bridges which together include all overcrossings. The bridge number depends on the diagram chosen. For example, the standard diagram of the trefoil knot has three bridges, but there is also a diagram

Figure 4.2: Two diagrams of the trefoil knot with bridge number three resp. two. Distinct bridges are colored differently.

with bridge number two: See Figure 4.2 above. The *bridge index* of a knot $\mathfrak{L}$ is the minimum of the bridge numbers of all diagrams of $\mathfrak{L}$.

If a knot has bridge index b, the fundamental group of the knot complement admits a presentation with b generators and $b-1$ relations [83, p. 82]. In other words, the bridge index of a t-complicated knot is at least $t + 1$. As a matter of fact, the connected sum of t trefoil knots is t-complicated, and its bridge index is exactly $t + 1$ [52]. More generally, the bridge index of the connected sum of two knots equals the sum of their bridge indices, minus one: See Figure 4.3 below.
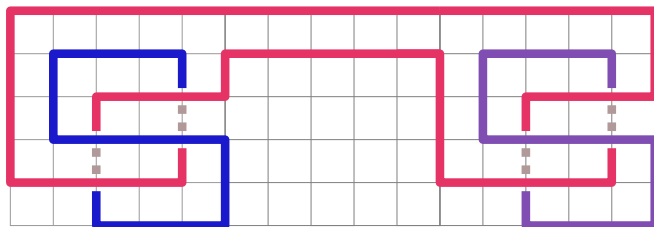
Figure 4.3: The bridge index of the connected union of two trefoils is three.

4.1.1 Spinning

A $(d-2)$-knot ($d \geq 3$) is a subcomplex $\mathfrak{L}$ homeomorphic to a $(d-2)$-sphere of a triangulated d-sphere S. The *group* of $\mathfrak{L}$ is the fundamental group of the complement of $\mathfrak{L}$ inside S. The "spinning" process, introduced by Artin [10] in 1925, lifts a $(d-2)$-knot to a $(d-1)$-knot preserving the group: we quote the passage in which Zeeman [151, p. 478] explained how to do it. (Here $d = 3$ and $\mathfrak{L}$ is the trefoil knot.)

> The formula $\mathrm{Spin}(D^1) = S^2$ means map the arc D^1 onto a meridian of S^2 and, keeping ∂D^1 fixed at the poles, multiply the in-

terior of D^1 by S^1, or in other words spin the meridian about the poles to form S^2.

Similarly $\mathrm{Spin}(D^n) = S^{n+1}$ means keep ∂D^n fixed and multiply the interior of D^n by S^1. In particular, $\mathrm{Spin}(D^3) = S^4$.

Now in D^3 draw an arc D^1 running from the North pole to the South pole via a trefoil knot. The spinning process induces $S^2 = \mathrm{Spin}(D^1) \subset \mathrm{Spin}(D^3) = S^4$, which is the Artin 2-knot.

The Artin 2-knot can be triangulated with exactly five vertices and six triangles. In fact, let us subdivide the arc D^1 by inserting a new vertex x_1 in the middle of it, so that x_1 is not on the North-South axis. Let x_2 resp. x_3 be the points in $\mathbb{R}^3$ obtained rotating x_1 of $\frac{\pi}{3}$ resp. $\frac{2\pi}{3}$ around the North-South axis. The triangulation we seek has

$$\{P_N, x_1, x_2\},\ \{P_N, x_1, x_3\},\ \{P_N, x_2, x_3\},$$
$$\{P_S, x_1, x_2\}, \{P_S, x_1, x_3\}\ \text{and}\ \{P_S, x_2, x_3\}$$

as facets, where P_N is the North pole and P_S is the South pole. By a result of Bing [20], this triangulation of S^2 can be completed to a triangulation of the whole S^4. The knot groups of the Artin 2-knot and the trefoil knot are the same; in general, the spinning process "lifts" knot groups one dimension up.

Note that the way we constructed the knotted surface above did not depend on the chosen knot. This justifies the following new result:

Proposition 4.1.1. *Every 2-knot obtained from spinning a 1-knot can be realized with only five vertices and six triangles in some simplicial 4-sphere.*

However, there are 2-knots whose group is not the fundamental group of the complement of any knot in a 3-sphere. (See Kawauchi [83, p. 190].) Hence, there are more 2-knot groups than knot groups: Therefore, not all 2-knots arise from spinning.

4.2 Putting knots inside 3-spheres or 3-balls

Given an arbitrary knot $\mathfrak{L}$, in this section we show how to build:

- a 3-ball where the knot is realized with a single interior edge plus a boundary path;
- a 3-sphere where the knot $\mathfrak{L}$ is realized with three edges;
- a 3-ball where the knot $\mathfrak{L}$ is realized with three edges.

4.2.1   Knotted spanning edges in 3-balls

A *spanning edge* of a 3-ball B is an interior edge that has both endpoints on the boundary ∂B. An $\mathfrak{L}$-*knotted spanning edge* of a 3-ball B is a spanning edge $[x, y]$ such that some simple path on ∂B between x and y completes the edge to a (non-trivial) knot $\mathfrak{L}$. From the simply-connectedness of 2-spheres it follows that the knot type does not depend on the boundary path chosen; in other words, the knot is determined by the edge.

More generally, a *spanning arc* is a path of interior edges in a 3-ball B, such that both extremes of the path lie on the boundary ∂B. If every path on ∂B between the two endpoints of a spanning arc completes the latter to a knot $\mathfrak{L}$, the arc is called $\mathfrak{L}$-*knotted*. Note that the relative interior of the arc is allowed to intersect the boundary of the 3-ball; compare Ehrenborg–Hachimori [52].

For any positive integer l, there exists a 3-ball with a knotted spanning arc of exactly l edges. To see this, take a sufficently large pile of cubes, and drill a tubular hole from the top to the bottom along a trefoil knot. Stop l steps before "perforating" the 3-ball completely (that is, stop when you are at a distance of l edges from the bottom). The result is a 3-ball with a knotted spanning arc of l edges; any boundary path completes such arc to a trefoil knot.

The previous model can be easily triangulated yielding an example of a *simplicial* ball with a knotted spanning edge. Also, the trefoil knot might be replaced with any knot $\mathfrak{L}$ you fancy, yielding a 3-ball with an $\mathfrak{L}$-knotted spanning arc. For example, $\mathfrak{L}$ might be a connected sum of m trefoil knots:

> *For any positive integers l and m, there is a cubical/simplicial 3-ball with a knotted spanning arc so that* (1) *the arc consists of l consecutive edges and* (2) *the knot is m-complicated.*

It is also known that knotted spanning edges can be 'summed':

Lemma 4.2.1. *Let $\mathfrak{L}_1, \ldots, \mathfrak{L}_h$ be knots, and let $B_1, \ldots, B_h$ be 3-balls. Suppose that each B_i contains an $\mathfrak{L}_i$-knotted spanning arc of l_i edges. Then there exists a simplicial 3-ball B containing a $(\sum_{i=1}^{h} \mathfrak{L}_i)$-knotted spanning arc of $\sum_{i=1}^{h} l_i$ edges.*

Proof. We prove the statement only in the simplicial case, the general case being analogous. For $i = 1, \ldots h$, let x_i and y_i denote the endpoints of the spanning knotted arc inside B_i. For each i, choose triangles $\{v_i, w_i, x_i\}$ and $\{y_i, z_i, t_i\}$ in the boundary of ∂B_i. Now, merge all the B_i's by identifying x_{i+1} and y_i, for each i. The resulting 3-complex C is a cactus of 3-balls.

To obtain the requested 3-ball, attach onto C square pyramids with apex $x_{i+1} \equiv y_i$ and basis $\{t_i, z_i, v_{i+1}, w_{i+1}\}$, for each i. To obtain a *simplicial* 3-ball, it suffices to subdivide each pyramid into two simplices. □

4.2.2 Knotted 3-balls and 3-spheres

From a knotted spanning arc inside a 3-ball, one can easily produce a knotted sphere: We just have to "cone off the boundary". In fact, given a 3-ball B and a new point v, consider $S_B = B \cup (v * \partial B)$: If B is simplicial, so is S_B. (If B is cubical, S_B is not cubical, because all the facets of the form $v * \sigma$, with $\sigma \in \partial B$, are square pyramids; however, it can be made cubical by dicing [105] [12] [128, p. 37]. Note that dicing increases the number of edges in the knot.)

By definition, the sphere S_B contains the three edges $[v, x], [x, y], [v, y]$, but it does not contain the face $\{v, x, y\}$. The spanning edge $[x, y]$ closes up to a triangular 1-complex, which is *knotted*: In fact, the knot is the same in S_B as in B, because there is an obvious homotopy between the two-edges-path $[x, v], [v, y]$ on S_B and the path on the boundary of B that closed up $[x, y]$ to a knot. Similarly, any $\mathfrak{L}$-knotted spanning arc of l edges closes up to a knotted $(l + 2)$-gon with the same knot type.

Taking out a facet from a knotted 3-sphere, one produces a 3-ball that contains a complete knot (and not just a knotted spanning edge!) in its 1-skeleton. Such ball cannot be rectilinearly embedded in $\mathbb{R}^3$; see Lutz [101] for a small example (a trefoil-knotted 3-ball with 12 vertices and 37 facets).

> *For any positive integers l and m, there exist a simplicial 3-sphere and a simplicial 3-ball with a knotted spanning arc so that (1) the knot consists of $l + 2$ consecutive edges and (2) the knot is m-complicated.*

Recall that when B is a shellable (resp. constructible) 3-ball, S_B is a shellable (resp. constructible) 3-sphere, in view of Lemma 1.7.4; when B is collapsible, then S_B minus a facet is collapsible, by Proposition 3.4.2. In Section 4.3 we will see that shellability and constructibility are incompatible with the presence of a knot, while knotted 3-balls might be collapsible.

A simplicial 3-sphere can have an arbitrarily complicated knot in its 1-skeleton; however, the connected union of many trefoils can only be realized in a 3-sphere with many facets. Recently King [85] [86] demonstrated that any knot in a 3-sphere with N facets must have bridge index smaller than $2^{196 N^2}$. In particular, all knots in a sphere with N facets are at most 2^{196N^2}-complicated.

King's result extends to knotted balls or balls with knotted spanning edges, in the following way. Among all the simplicial 3-balls with N facets, the trees of tetrahedra are the ones with the biggest number of boundary triangles, namely $2N+2$. (See Lemma 2.1.1.) Thus if B is a simplicial 3-ball with N tetrahedra, S_B has at most $3N+2$ tetrahedra. So, all knots and all knotted spanning edges in a 3-ball with N facets are at most $2^{9 \cdot 196 N^2}$-complicated.

4.3   Knots versus collapsibility

Knot theory and the theory of elementary collapses are closely related. In the Sixties, the work by Bing, Goodrick, Lickorish and Martin has shown that out of the 3-balls with a knotted spanning edge some are collapsible and some are not, depending on how complicated the knot is.

The same holds for knotted spheres: after the removal of a facet, some collapse and some do not, depending on the intricacy of the knot.

4.3.1   Knots that are simple enough

Let us start by guaranteeing[2] that balls with knotted spanning edges can be collapsible.

Theorem 4.3.1 (Lickorish–Martin [96], Hamstrom–Jerrard [71]). *Let $\mathfrak{L}$ be any 2-bridge knot (for example, the trefoil knot). There exists a collapsible triangulated 3-ball B with an $\mathfrak{L}$-knotted spanning edge.*

Theorem 4.3.1 extends to knotted spanning arcs by concatenating knotted spanning edges as in Lemma 4.2.1:

Theorem 4.3.2. *Let $B_1, \ldots, B_h$ be collapsible triangulated 3-balls. Suppose that each B_i contains in its 1-skeleton an $\mathfrak{L}_i$-knotted spanning arc of l_i edges. Then there exists a collapsible triangulated 3-ball B with a $\left(\sum_{i=1}^h \mathfrak{L}_i\right)$-knotted spanning arc of $\sum_{i=1}^h l_i$ edges in its 1-skeleton.*

Proof. Every collapsible ball collapses onto a tree, and every tree can be collapsed onto any of its vertices. By induction on h this implies that any cactus of collapsible 3-balls is collapsible. Given a cactus of collapsible balls, if we thicken their junctions a bit to obtain a ball — for example by attaching pyramids, as in the proof of Lemma 4.2.1 — we still maintain the collapsibility property: In fact, the added pyramids can be collapsed away.

---

[2]For the proof of the next statement, see the proof of Theorem 6.3.6.

Thus the sum of the B_i described in Lemma 4.2.1 is collapsible, provided each B_i is collapsible. □

Corollary 4.3.3. *For every positive integer m, there exist:*

(1) *a collapsible simplicial 3-ball B_m with an $\mathfrak{L}$-knotted spanning arc consisting of m edges, where $\mathfrak{L}$ is the connected sum of m trefoils;*

(2) *a simplicial 3-sphere S_m with an m-complicated $(m+2)$-gonal knot in its 1-skeleton, such that $S_m - \Delta$ is collapsible for every facet Δ of S_m.*

Proof. B_m is obtained via Theorem 4.3.2, summing m copies of the collapsible 3-ball B with a knotted spanning edge given by Theorem 4.3.1. The 3-sphere S_m is then obtained coning off the boundary of B_m: by Theorem 3.4.2, since B_m is collapsible, $S_m - \Delta$ is collapsible for every Δ. □

Remark 4.3.4. With the notation of Cor. 4.3.3, if B has i interior vertices then B_m has exactly mi interior vertices. Note also that the knotted spanning arc produced in Theorem 4.3.2 intersects in its relative interior the boundary of the ball B_m exactly $(m-1)$ times.

4.3.2 Knots that are complicated enough

In the Sixties, Bing [21] showed that a knotted ball cannot be collapsible, if the knot is sufficiently complicated: A sum of two trefoils is enough. (See also Goodrick [60] and Kearton–Lickorish [84].)

Theorem 4.3.5 (Bing–Goodrick). *A simplicial 3-ball with an $\mathfrak{L}$-knotted spanning edge cannot be collapsible, if the knot $\mathfrak{L}$ has bridge index larger than two.*

Later Lickorish [93] showed that if the knot $\mathfrak{L}$ is even more complicated (a sum of three trefoils would do), the sphere obtained coning off the boundary of the ball does not become collapsible after the removal of any facet. This is a stronger result than claiming that the knotted ball cannot be collapsible, in view of Proposition 3.4.2.

Theorem 4.3.6 (Lickorish [93]). *Let B be a 3-ball with an $\mathfrak{L}$-knotted spanning edge, and $S_B := B \cup v * \partial B$. If the knot $\mathfrak{L}$ is at least 3-complicated, neither B nor $S_B - \Delta$ are collapsible, for any facet Δ of S.*

(In fact, there is a 3-ball B with a 2-complicated knot such that B is collapsible, but the sphere S_B does *not* become collapsible after the removal of any facet. See Proposition 5.3.10.)

The previous result was achieved through a careful understanding of what goes on during a collapse of $S - \Delta$: In particular, Lickorish was able to read off from the collapsing sequence crucial information on the knot group.

Theorem 4.3.7 (Lickorish [93]). *Let $\mathfrak{L}$ be an m-edges knot in a simplicial 3-sphere S. Suppose that $S - \Delta$ is collapsible, for some facet Δ of S. Then $|S| - |\mathfrak{L}|$ is homotopy equivalent to a connected CW complex with one 0-cell and at most m 1-cells. In particular, the fundamental group of $|S| - |\mathfrak{L}|$ admits a presentation with m generators.*

Recall that in Section 3.2 we proved a stronger statement:

Theorem 3.5.1. *Let t, d with $0 \le t \le d - 2$, and let S be a PL d-sphere. Suppose that $S - \Delta$ collapses onto a t-complex, for some facet Δ of S. Then, for each t-dimensional subcomplex $\mathfrak{L}$ of S, the homotopy group*

$$\pi_{d-t-1}\left(|S| - |\mathfrak{L}|\right)$$

has a presentation with exactly $f_t(\mathfrak{L})$ generators.

Now we are repaid of our efforts in replacing Theorem 4.3.7 by Theorem 3.5.1: Via the latter, we can reach the following results.

Corollary 4.3.8. *Let B be a 3-ball with an $\mathfrak{L}$-knotted spanning edge, the knot $\mathfrak{L}$ being the sum of six (or more) trefoil knots. Then spinning B one gets a 4-sphere S with an embedded knotted surface such that $S - \Delta$ is not collapsible onto any 2-complex, for any facet Δ of S.*

Proof. Juxtapose Theorem 3.5.1 and Proposition 4.1.1. □

Corollary 4.3.9. *Let k be a non-negative integer. Let S be a 3-sphere with an $m \cdot 2^k$-complicated m-edges knot. The k-th suspension of S is a PL $(k+3)$-sphere such that:*

1. *$S - \Delta$ is not collapsible;*
2. *$S - \Delta$ does not collapse onto any $(d-2)$-complex (in other words, $\mathrm{cdepth}(S) = 1$).*

Proof. Let S' be the k-th suspension of S and let $\mathfrak{L}'$ be the subcomplex of S' obtained taking the k-th suspension of the m-gonal knot $\mathfrak{L}$. Since $|S| - |\mathfrak{L}|$ is a deformation retract of $|S'| - |\mathfrak{L}'|$, they have the same homotopy groups. In particular, the fundamental group of $|S'| - |\mathfrak{L}'|$ has no presentation with $m \cdot 2^{d-3}$ generators.

Since $f_{d-2}(\mathfrak{L}') = 2^{d-3} \cdot f_1(\mathfrak{L}) = m \cdot 2^{d-3}$, via Theorem 3.5.1 we conclude. □

4.4 Knots versus shellability

Since for contractible complexes "shellable" implies "collapsible", knots in spheres and balls are obstructions to shellability as well. For example: Bing's ball contains a double-trefoil knotted spanning edge, so it cannot be collapsible by Theorem 4.3.5, so it cannot be shellable. To prevent shellability, however, a single trefoil would be enough: the first example ever of non-shellable ball, due to Furch [57], had a knotted spanning edge with the same knot type of the trefoil. A 3-ball with a trefoil-knotted spanning edge might be collapsible (see Section 4.3.1). However, Furch's ball cannot be constructible, in view of the following result.

Theorem 4.4.1 (Hachimori–Ziegler [69, Ex. 1&3, Lemmas 1&4]).
Let B be a 3-ball with a knotted spanning arc consisting of m edges.

- *(i) if $m = 1$ or $m = 2$, B cannot be shellable nor constructible, but it might be collapsible;*
- *(ii) if $m = 3$, B might be shellable and constructible (for example if the knot is the trefoil, cf. [146]), but it cannot be vertex decomposable;*
- *(iii) if $m \geq 4$ and B is simplicial, B might be vertex decomposable (for example if the knot is the trefoil, cf. [145]).*

The 3-ball B we obtained in Theorem 4.3.2 is constructible if and only if each B_i is constructible [65, Lemma 1]. Thus via Theorem 4.3.2 and Theorem 4.4.1 we may produce examples of:

- *constructible* 3-balls with a double-trefoil-knotted spanning arc of six edges (for example, the sum of two constructible balls with three-edges trefoil-knotted spanning arcs (cf. [69, Example 1]), and
- *non-constructible* 3-balls with a double-trefoil-knotted spanning arc of six edges (for example, the sum of a ball with a five-edges trefoil-knotted spanning arc, which might be constructible, with a ball with a knotted spanning edge, which cannot be constructible by [69, Lemma 1]).

Therefore, constructibility can be *excluded*, but not *decided*, by looking at the tightness of a knot. As a matter of fact, there are non-constructible balls that have no knot at all (e.g. Bing's house with two rooms, cf. Example 6.3.3).

Some simplicial 3-balls contain an entire triangular knot $\mathfrak{L}$ in their 1-skeleton. In order to obtain such a "knotted" ball, start with any 3-ball B with an $\mathfrak{L}$-knotted spanning edge, cone off the boundary of B obtaining a knotted sphere S_B, and finally subtract a facet from S_B. We have seen in Section 4.3.1 that some knotted balls are collapsible; furthermore, the

collapsibility of B (strictly) implies the collapsibility of $S_B - \Delta$, by Proposition 3.4.2 (resp. Remark 3.4.3). However, balls with a knotted triangle cannot be constructible, no matter how complicated the knot is:

Theorem 4.4.2 (Hachimori–Ziegler [69, Ex. 2&4, Thms. 3&5]).
Let B be a 3-ball with a knotted m-gon in its 1-skeleton.

  (i) If $m = 3$, B cannot be shellable nor constructible.
 (ii) If $m = 4$ or $m = 5$, B might be shellable and constructible (for example if the knot is a trefoil, cf. [148]), but (in the simplicial case) B cannot be vertex decomposable.
(iii) If $m \geq 6$ and B is simplicial, B might be vertex decomposable (for example if the knot is a trefoil, cf. [147]).

Note that there is a discrepancy between this hierarchy and the hierarchy of Theorem 4.4.1: if B has a 2-edges knotted spanning arc, B cannot be shellable, but S_B minus a facet might be shellable, because the arc is closed up to a quadrilateral knot. (Compare Remark 1.7.5.) The discrepancy will be partially explained by Proposition 5.3.10.

A 3-sphere S is shellable if and only if $S - \Delta$ is shellable for some facet Δ. A 3-sphere S is constructible if and only if $S - \Delta$ is constructible for some (or equivalently, for each) facet Δ [69, Theorem 4]. A simplicial 3-sphere is vertex decomposable if and only if the deletion of some vertex yields a vertex decomposable 3-ball. Using these facts, one can easily lift Theorem 4.4.2 to spheres:

Theorem 4.4.3 (Hachimori–Ziegler [69, Ex. 2&4, Thms. 3&5]).
Let S be a 3-sphere with a knotted m-gon.

  (i) if $m = 3$, S cannot be shellable nor constructible;
 (ii) if $m = 4$ or $m = 5$, S might be shellable and constructible (for example if the knot is a trefoil), but (in the simplicial case) S cannot be vertex decomposable;
(iii) if $m \geq 6$ and S is simplicial, S might be vertex decomposable.

These results partially extend to 3-spheres with knots of prescribed intricacy. Ehrenborg and Hachimori [52] showed in 2001 that a 3-sphere (or a 3-ball) containing a knot of m edges

– (in the simplicial case) cannot be vertex decomposable, if the bridge index of the knot exceeds $\frac{m}{3}$, and
– cannot be shellable, if the bridge index of the knot exceeds $\frac{m}{2}$.

The latter claim was later strengthened by Hachimori and Shimokawa [68], who proved that a regular CW complex homeomorphic to a 3-sphere or a

3-ball cannot be *constructible* if it contains a knot on m edges whose bridge index exceeds $\frac{m}{2}$. These bounds are sharp by Theorems 4.4.2 and 4.4.3, since the trefoil knot has bridge index two.

Also, the bridge index of a t-complicated knot is at least $t+1$, so if a knot is at least $\lfloor \frac{m}{3} \rfloor$-complicated then its bridge index automatically exceeds $\frac{m}{3}$. Thus the results above imply the following statement, which will be used to establish our Theorem 5.3.12.

Proposition 4.4.4 (Ehrenborg–Hachimori–Shimokawa). *A 3-sphere or 3-ball containing a knot of m edges*

– *cannot be constructible, if the knot is at least $\lfloor \frac{m}{2} \rfloor$-complicated, and*
– *(in the simplicial case) cannot be vertex decomposable, if the knot is at least $\lfloor \frac{m}{3} \rfloor$-complicated.*

Furthermore, both these lower bounds are sharp.

Remark 4.4.5. An analogous bound holds for 3-balls with an $\mathfrak{L}$-knotted spanning arc of $m - 2$ edges: If the knot $\mathfrak{L}$ is at least $\lfloor \frac{m}{2} \rfloor$-complicated, then B cannot be constructible. (Otherwise S_B would be constructible, a contradiction with Proposition 4.4.4.) This derived bound is not sharp, though: When $m = 2$, it claims that a knotted spanning arc of two edges obstructs constructibility provided the knot is at least 2-complicated, while we know from Theorem 4.4.1 that already a 1-complicated knot is enough.

4.5   Barycentric subdivisions versus knots

Performing a barycentric subdivision on a knotted 3-sphere doubles the number of edges in the knot, simply because every edge of the sphere is subdivided into two subedges. The knot type remains unchanged: Subdividing a 3-ball with an $\mathfrak{L}$-knotted spanning arc of m edges one gets a 3-ball with an $\mathfrak{L}$-knotted spanning arc of $2m$ edges. Therefore, the obstructions to collapsibility, shellability and vertex decomposability discussed in the previous sections will sooner or later vanish, if we perform sufficiently many barycentric subdivisions.

Indeed, it was shown by Zeeman [152, Chapters I and III] in the Sixties that every ball becomes collapsible after a sufficient number of barycentric subdivision. Later Bruggesser and Mani showed that after sufficiently many subdivisions, any ball gets shellable (and performing one more subdivision it gets even vertex decomposable).

Is there an integer r such that the r-th barycentric subdivision of every simplicial 3-ball is collapsible?

The answer is negative, as established by Goodrick [60]: If B is a simplicial 3-ball with a knotted spanning edge and the bridge index of the knot is bigger than $2^r + 1$, then the r-th subdivision of B is not collapsible. Lickorish and Martin [96] showed that Goodrick's result is best possible: for any knot $\mathfrak{L}$ of bridge index $\leq 2^r + 1$, they constructed a 3-ball with collapsible r-th barycentric subdivision and with an $\mathfrak{L}$-knotted spanning edge. Also, Goodrick's result extends to higher dimensions: Kearton and Lickorish [84, Theorem 2], bridging a gap in Goodrick's original proof [60], showed that for all $r \in \mathbb{N}$, for all $d \geq 3$, there exists a d-ball whose r-th barycentric subdivision is not collapsible.

A similar results holds for spheres: for any integer r, there exists a knot $\mathfrak{L}_r$ complicated enough (cf. Theorem 4.3.7) such that any 3-sphere S_r that contains an $\mathfrak{L}_r$-knotted triangle must satisfy the following condition: The removal of any facet from the r-th barycentric subdivision of S_r is not collapsible. In particular, the r-th barycentric subdivision of S_r cannot be shellable, either.

Chapter 5

Locally constructible manifolds

In this chapter, we characterize local constructibility in terms of the collapse depth: The LC manifolds are precisely the manifolds with cdepth ≥ 2. Thus while all LC 3-manifolds are spheres (Corollary 1.6.7), an LC d-manifold ($d \geq 4$) might be a product of spheres (cf. Example 5.2.8), or perhaps something more complicated[1]. However, by Lemma 1.6.3 all LC d-pseudomanifolds are simply connected.

Our characterization enables us to prove the following hierarchy for d-spheres announced in the introduction:

Theorem 5.0.1. *For all $d \geq 3$, we have the following inclusion relations between families of simplicial d-spheres:*

$$\{\text{vertex dec.}\} \subsetneq \{\text{shellable}\} \subseteq \{\text{constructible}\} \subsetneq \{\text{LC}\} \subsetneq \{\text{all } d\text{-spheres}\}.$$

Proof. The first two inclusions are known (see Chapter 1). The third inclusion follows from Lemma 5.1.1; its strictness is shown via Theorem 5.3.7. Finally, Theorem 5.3.2 establishes the strictness of the fourth inclusion for all $d \geq 3$. $\square$

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[1] Using Frank Lutz's greedy collapsing algorithm, after the submission of the present thesis we found out that Kühnel's vertex-minimal triangulation of the complex projective plane, with 8 vertices and 36 facets, has collapse depth two. Therefore, LC 4-manifolds are not just spheres or product of spheres. The details will be found elsewhere.

5.1 Constructible complexes are LC

Here we show that all constructible pseudomanifolds are LC.

Lemma 5.1.1. *Let C be a d-pseudomanifold. If C can be split in the form $C = C_1 \cup C_2$, where C_1 and C_2 are LC d-pseudomanifolds and $C_1 \cap C_2$ is a strongly connected $(d-1)$-pseudomanifold, then C is LC.*

Proof. Notice first that $C_1 \cap C_2 = \partial C_1 \cap \partial C_2$. In fact, every ridge of C belongs to at most two facets of C, so that every $(d-1)$-face σ of $C_1 \cap C_2$ is contained in exactly one d-face of C_1 and in exactly one d-face of C_2.

Each C_i is LC; let us fix a local construction for each of them, and call T_i the tree along which C_i is locally constructed. Choose some $(d-1)$-face σ in $C_1 \cap C_2$, which thus specifies a $(d-1)$-face in the boundary of C_1 and of C_2. Let C' be the manifold with boundary obtained attaching C_1 to C_2 along the two copies of σ. C' can be locally constructed along the tree obtained by joining T_1 and T_2 by an edge across σ: Just redo the same moves of the local constructions of the C_i's. So C' is LC.

If $C_1 \cap C_2$ consists of one polytope only, then $C' \equiv C$ and we are already done. Otherwise, by the strongly connectedness assumption, the facets of $C_1 \cap C_2$ can be labeled $0, 1, \ldots, s$ so that:

- the facet labeled by 0 is σ;
- each facet labeled by $k \geq 1$ is adjacent to some facet labeled j with $j < k$.

Now for each $i \geq 1$, glue together the two copies of the facet i inside C'. All these gluings are *local* because of the labeling chosen, and we eventually obtain C. Thus, C is LC. $\square$

Constructible complexes are strongly connected, simply connected and even $(d-1)$-*connected* (see Björner [24, pp. 1846–1848, 1854]). As a matter of fact, any constructible d-pseudomanifold must be either a d-ball or a d-sphere [68, Proposition 1.4]. Thus from Lemma 5.1.1 we obtain for d-complexes that

$$\{\text{constructible}\} \subseteq \{\text{LC}\}.$$

Since all shellable complexes are constructible, using Theorems 2.5.1 and 2.6.3 we arrive to the following conclusions:

Corollary 5.1.2. *For fixed $d \geq 2$, there are exponential upper and lower bounds for the number of constructible d-spheres and d-balls with N facets, with bounded facet complexity.*

Corollary 5.1.3. *For all $d \geq 4$, shellable simplicial d-spheres are exponentially many when counted with respect to the number N of facets, but more than exponentially many when counted with respect to the number n of vertices.*

Proof. Kalai [81] and Lee [90] showed that for $d \geq 4$, there are at least

$$2^{\Omega\left(n^{\left\lfloor \frac{d}{2} \right\rfloor}\right)}$$

simplicial shellable d-spheres with n vertices; however, it follows from Theorem 2.5.1 that there are at most $2^{d^2 \cdot N}$ simplicial shellable d-spheres with N facets. $\qquad\square$

The containment {constructible} $\subseteq$ {LC} is strict: Let C_1 be a cubical 3-ball with 17 facets obtained from a $2 \times 3 \times 3$ pile of cubes by performing an elementary collapse (see Figure 5.1 below). Let C_2 be a $1 \times 3 \times 3$ pile of cubes. Glue C_1 and C_2 together along the 2-dimensional annulus consisting of the external squares of a 3×3 face of C_2.

Figure 5.1: Gluing the two 3-balls above along the green 2-dimensional region yields an LC, non-constructible 3-manifold with boundary, called *fake cube*.

C_1 and C_2 are both shellable. Thus $C_1 \cup C_2$ is a cubical 3-manifold with boundary that is LC (by Lemma 5.1.1) but not 2-connected (because it retracts to a 2-sphere). Therefore, it cannot be constructible.

This argument can be generalized to produce many examples of complexes that are LC but not $(d-1)$-connected, and hence not constructible. However, none of these examples will be a sphere (or a ball). We will obtain examples of nonconstructible LC spheres (resp. balls) from Theorem 5.3.7 (resp. from Theorem 6.2.4).

From an algebraic point of view, the fake cube of Figure 5.1 is Buchsbaum, but not Cohen–Macaulay (see e.g. [25, p. 194] [24, p. 1855] for the definitions). By the work of Hibi and Björner [25, Lemma 1, p. 194] [74,

p. 98], the union of two Buchsbaum (resp. Cohen-Macaulay) d-complexes whose intersection is a Buchsbaum (resp. Cohen-Macaulay) $(d-1)$-complex yields a Buchsbaum (resp. Cohen-Macaulay) d-complex. As a matter of fact, the green annulus in Figure 5.1 is strongly connected and Buchsbaum, but not Cohen–Macaulay.

On the contrary, a *pinched* annulus is neither Buchsbaum nor Cohen–Macaulay. Based on this, we can produce examples of LC complexes that are neither Buchsbaum nor Cohen–Macaulay. Let us start with two simplicial (shellable) 3-balls on 7 vertices consisting of 7 tetrahedra, as indicated in Figure 5.2. Let us glue them together in the strongly connected green subcomplex in their boundary (which uses 5 vertices and 4 triangles, and is homeomorphic to a pinched annulus).

Figure 5.2: Gluing the mirroring balls along the green 2-dimensional region yields an LC 3-complex, in which the link at the top vertex is neither LC nor simply connected.

The resulting simplicial complex C, on 9 vertices and 14 tetrahedra, is LC by Lemma 5.1.1, but the link of the top vertex is a 2-complex that retracts to a 1-sphere. In particular, the link of the top vertex is strongly connected (compare Proposition 1.7.3) but not LC. Therefore, the LC class is not closed under taking links.

Here is an example of pseudomanifold that is *not* LC. Let C_1 and C_2 be the two simplicial 3-balls on 7 vertices consisting of 7 tetrahedra, as indicated in Figure 5.3.

Figure 5.3: Gluing the mirroring balls along the green subcomplex yields a 3-pseudomanifold that *cannot* be LC, for topological reasons. (It is "weakly LC" in the sense of Mogami [106].)

Again, let us glue them together in the green subcomplex of their boundary (which uses 5 vertices and 4 triangles) to obtain a simplicial complex C on 9 vertices and 14 tetrahedra.

The main difference with Figure 5.2 is that this time the green subcomplex is pure and connected, but not strongly connected. A priori this is not enough to conclude that C is not LC. However, the boundary of C consists of two 2-spheres that intersect in two distinct points. Via Theorem 1.6.6, this implies that C cannot be LC.

Nonetheless, C is "weakly LC" in the sense of Mogami (see Section 2.6.4), in virtue of the following analogue of Lemma 5.1.1:

Lemma 5.1.4. *Let C be a d-pseudomanifold. If C can be split in the form $C = C_1 \cup C_2$, where C_1 and C_2 are weakly-LC d-pseudomanifolds and $C_1 \cap C_2$ is a connected $(d-1)$-pseudomanifold, then C is weakly-LC.*

Proof. Recall that weakly-LC CW complexes are obtained from a tree of d-polytopes by local gluings and/or by Mogami moves (cf. Section 2.6.4). Pick a $(d-1)$-dimensional face σ in $C_1 \cap C_2$. Let T be the tree of simplices obtained merging T_1 and T_2 across σ. Since $C_1 \cap C_2$ is connected, its facets (possibly of smaller dimension than $d-1$) can be labeled $0, \ldots, s$ so that the facet labeled by 0 is σ and each facet labeled by $k \geq 1$ has a vertex in common with some facet labeled j, with $j < k$. One then concludes as in Lemma 5.1.1. $\square$

In 1970, Lickorish [92] proved that some (simplicial) 3-balls do not contain any 2-disc properly embedded as subcomplex. (Clearly such 3-balls are not constructible.) His argument proceeded as follows: given a 3-ball K with an embedded disk that divides K into two 3-balls K_1 and K_2, if each K_i collapses to $\partial K_i - \sigma$ for each facet σ of ∂K_i, then also K collapses to $\partial K - \sigma$ for each facet σ of ∂K. Hence, if K is the "smallest" example of a (simplicial) 3-ball that does not collapse onto its boundary minus a facet, then K cannot have any disc embedded as subcomplex. ("Smallest" means here "with the smallest number of facets".)

Via Corollary 6.2.6, we will characterize LC 3-balls as the balls B that collapse onto $\partial B - \sigma$, for some boundary facet σ. Thus, a posteriori, Lickorish's argument can be read as follows:

> *by gluing together two LC 3-balls K_1 and K_2 alongside a 2-ball in their boundary, one obtains a 3-ball that is still LC.*

This is a special case of our Lemma 5.1.1. (Actually, gluing together an LC 3-ball and a *collapsible* 3-ball alongside a 2-ball one also gets an LC 3-ball.) In particular, a smallest example C of a *non-LC* 3-ball cannot contain a 2-disc properly embedded as subcomplex.

Analogously, a smallest example B of a *non-constructible* 3-ball cannot contain a 2-disc properly embedded as subcomplex. After Lemma 5.1.1, one can extend Lickorish's argument as follows: Let us call *pseudomanifold with LC topology* a pseudomanifold whose underlying space is a 3-sphere with a finite number of cacti of 3-balls removed. Let P be the smallest example of pseudomanifold with LC topology that is *not* LC. This P cannot contain a 2-disc properly embedded a subcomplex; moreover, for any 2-sphere S properly embedded as subcomplex, the intersection $S \cap \partial P$ must contain at least two points. (In fact, suppose $S \cap \partial P$ is empty or consisting of one vertex; then S divides P into one 3-ball and one 3-pseudomanifold with LC topology, and both have to be LC by the minimality of P. By Lemma 5.1.1, this would imply that P is LC too, a contradiction.)

5.2   Characterization of LC manifolds

Take a d-manifold M, a facet Δ of M, and a rooted spanning tree T of the dual graph of M, with root Δ. Recall that K^T is the subcomplex of the faces of M that are not intersected by T. Cutting M open along K^T we obtain a tree of facets T_N, whose dual graph is precisely T.

Now, T_N can be assembled one polytope at the time, according to a natural labeling of T; we have seen in Section 3.1 that natural labelings of T correspond to collapses $M - \Delta \searrow K^T$ (the i-th facet to be collapsed is the node of T labelled $i + 1$; see Proposition 3.1.4).

Since any possible local construction for M needs to start at some tree of polytopes (or equivalently at some spanning tree T), we have a bijection among the following sets:

1. the set of all facet-killing sequences of $M - \Delta$;
2. the set of all natural labelings of spanning trees of M, rooted at Δ;
3. the set of the first parts $(T_1, \ldots, T_N)$ of local constructions for M, with $T_1 = \Delta$.

To understand also the second part of a local construction combinatorially, we introduce a variant of the "facet-killing sequence" notion.

Definition 5.2.1. A *pure facet-massacre* of a pure d-dimensional complex P is a sequence $P_0, P_1, \ldots, P_{t-1}, P_t$ of (pure) complexes such that $t = f_d(P)$, $P_0 = P$, and P_{i+1} is obtained by P_i removing:

(a) a free $(d-1)$-face σ of P_i, together with the unique facet Σ containing σ, and

(b) all inclusion-maximal faces of dimension smaller than d that are left after the removal of type (a) or, recursively, after removals of type (b).
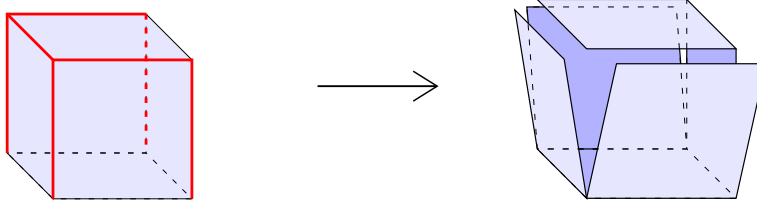
Figure 5.4: Cutting a dice open along the 1-complex K^T (in red) yields the tree of squares T. The reverse procedure – folding up the dice by local gluings, starting from the tree of squares T – can be viewed as a facet-massacre of K^T.

In other words, the (b) steps remove lower-dimensional facets until we get a pure complex. Since $t = f_d(P)$, P_t has no facets of dimension d left, nor inclusion-maximal faces of smaller dimension; hence P_t is empty. The other P_i's are pure complexes of dimension d. Notice that the step $P_i \longrightarrow P_{i+1}$ is not a collapse, and does not preserve the homotopy type in general. Of course $P_i \longrightarrow P_{i+1}$ can be "factorized" in an elementary collapse followed by a removal of a finite number of k-faces, with $k < d$. However, this factorization is not unique, as the next example shows.

Example 5.2.2. Let P be the full square $\{1, 2, 3, 4\}$. P admits four different facet-killing collapses (each edge can be chosen as free face), but it admits only one pure facet-massacre, namely $P_0 = P$, $P_1 = \emptyset$.

Lemma 5.2.3. *Let P be a pure d-dimensional polytopal complex. Every facet-killing sequence of P naturally induces a unique pure facet-massacre of P. All pure facet-massacres of P are induced by some (possibly more than one) facet-killing sequence.*

Proof. The map consists in taking a facet-killing sequence C_0, ..., C_t, and then in "cleaning up" the C_i by recursively killing the inclusion-maximal faces of dimension smaller than d. As the previous example shows, this map is not injective. It is surjective essentially because the removed lower-dimensional faces are of dimension "too small to be relevant". In fact, their dimension is at most $d - 1$, hence their presence can interfere only with the freeness of faces of dimension at most $d - 2$; so the list of all removals of the form $((d-1)$-face, d-face$)$ in a facet-massacre yields a facet-killing sequence. $\square$

Theorem 5.2.4. *Let M be a d-manifold; fix a spanning tree T of the dual graph of M. The second part of a local construction for M along T corresponds bijectively to a facet-massacre of K^T.*

Proof. Fix M and T; T_N and K^T are determined by this. Let us start with a local construction $(T_1, \ldots, T_{N-1},)\, T_N, \ldots, T_k$ for M along T. Topologically, $M = T_N/\sim$, where $\sim$ is the equivalence relation determined by the gluing (two distinct points of T_N are equivalent if and only if they will be identified in the gluing). Moreover, $K^T = \partial T_N/\sim$, by the definition of K_T.

Define $P_0 := K_T = \partial T_N/\sim$, and $P_j := \partial T_{N+j}/\sim$. We leave it to the reader to verify that $k - N$ and $f_d(K^T)$ are the same integer; in particular $P_D = \partial T_k/\sim = \partial M/\sim = \emptyset$.

In the first LC step, $T_N \to T_{N+1}$, we remove from the boundary a free ridge r, together with the unique pair σ', σ'' of facets of ∂T_N sharing r. At the same time, r and the newly formed face σ are sunk into the interior. This step $\partial T_N \longrightarrow \partial T_{N+1}$ naturally induces an analogous step $\partial T_{N+j}/\sim \longrightarrow \partial T_{N+j+1}/\sim$, namely, the removal of r and of the (unique!) $(d-1)$-face σ containing it.

In the j-th LC step, $\partial T_{N+j} \longrightarrow \partial T_{N+j+1}$, we remove from the boundary a ridge r together with a pair σ', σ'' of facets sharing r; moreover, we sink into the interior a lower-dimensional face F if and only if we have just sunk into the interior all faces containing F. The induced step from $\partial T_{N+j}/\sim$ to $\partial T_{N+j+1}/\sim$ is precisely a "facet-massacre" step.

For the converse, we start with a "facet-massacre" $P_0, \ldots, P_D$ of K^T, and we have $P_0 = K_T = \partial T_N/\sim$. The unique $(d-1)$-face σ_j killed in passing from P_j to P_{j+1} corresponds to a unique pair of (adjacent!) $(d-1)$-faces σ_j', σ_j'' in ∂T_{N+j}. Gluing them together is the LC move that transforms T_{N+j} into T_{N+j+1}. $\qquad\square$

Remark 5.2.5. We recall that:

- the first part of a local construction of a manifold M along a tree T corresponds to a facet-killing collapse of $M - \Delta$ (that ends up in K^T);
- the second part of a local construction along a tree T corresponds to a pure facet-massacre of K^T;
- a single facet-massacre of K^T corresponds to (possibly) many facet-killing sequences of K^T;
- by Proposition 3.1.4, there exists a facet-killing sequence of K^T if and only if K^T collapses onto some $(d-2)$-dimensional complex C.

Summing up, the following are equivalent:

1. M is locally constructible along T;
2. K^T collapses onto some $(d-2)$-dimensional complex C;
3. K^T has a facet-killing sequence.

What if we do not fix the tree T?

Theorem 5.2.6. *Let M be a d-manifold ($d \geq 3$). The following are equivalent:*

1. *M is LC;*
2. *for some spanning tree T of the dual graph of M, K^T is collapsible onto some $(d-2)$-dimensional complex C;*
3. *there exists a $(d-2)$-dimensional complex C such that for every facet Δ of M, $M - \Delta \searrow C$;*
4. *for some facet Δ of M, $M - \Delta$ is collapsible onto a $(d-2)$-dimensional complex C;*
5. *cdepth$(M) \geq 2$.*

Proof. M is LC if and only if M can be locally constructed along some tree T: So the equivalence of (1) and (2) is a straightforward consequence of Remark 5.2.5. On the other hand, the equivalence of (2) and (3) follows from the fact that $M - \Delta \searrow K^T$, where K^T is independent of the choice of Δ. Obviously (3) implies (4), and (5) is equivalent to (4) by definition.

To show that (4) implies (2), we take a collapse of $M - \Delta$ onto some $(d-2)$-complex C. By Proposition 3.1.4, if T is the tree along which the collapse acts, one has that $M - \Delta$ collapses onto K^T and K^T collapses onto C. $\square$

Corollary 5.2.7. *The product of LC manifolds (of dimension at least two) is an LC manifold.*

Proof. By Corollary 3.3.5, if cdepth$(M_1) \geq 2$ and cdepth$(M_2) \geq 2$ then cdepth$(M_1 \times M_2) \geq 2$. The conclusion follows by Theorem 5.2.6. $\square$

By Lemma 1.6.3, every LC (pseudo)manifold is simply connected. However, the next example shows that the higher homotopy groups of an LC manifold might be non-trivial.

Example 5.2.8. Let C be the boundary of the 3-cube; the product $C \times C$ is a cubical 4-manifold homeomorphic to $S^2 \times S^2$. Now $\pi_2(S^2 \times S^2) \neq 0$, so $C \times C$ is not homeomorphic to a 4-sphere. However, $\pi_1(S^2 \times S^2) = 0$ and by Corollary 3.3.4 $C \times C$ minus a facet collapses onto a 2-complex: Therefore, $C \times C$ is an LC (simply connected) 4-manifold.

Example 5.2.9. Kühnel's 8-vertex triangulation of the complex projective plane is LC; see the footnote on page 89.

5.3 Applications to d-spheres

Apparently in passing from manifolds to spheres Theorem 5.2.6 tells us nothing new: The LC spheres are the spheres that after the removal of a facet collapse onto some $(d-2)$-dimensional subcomplex. Indeed there is a hidden bonus information: This subcomplex is certainly contractible, because a sphere minus a facet is contractible and the homotopy type is preserved under collapses. This information becomes relevant in case the subcomplex has dimension one: in fact, all contractible 1-complexes are collapsible.

Corollary 5.3.1. *Let S be a 3-sphere. Then the following are equivalent:*

1. S is LC;
2. K^T is collapsible, for some spanning tree T of the dual graph of S;
3. $S - \Delta$ is collapsible for every facet Δ of S;
4. $S - \Delta$ is collapsible for some facet Δ of S.

Proof. This follows from Theorem 5.2.6, together with the fact that the complex onto which $S - \Delta$ collapses is contractible and 1-dimensional: Therefore, it is collapsible. □

Finally, the efforts we made in Section 3.2 and the characterization above enable us to answer Durhuus–Jonsson's conjecture for all dimensions.

Theorem 5.3.2. *For every $d \geq 3$, not all d-spheres are LC.*

Proof. It follows from Corollary 4.3.9, together with Theorem 5.2.6. □

We even have a constructive explanation:

Theorem 5.3.3. *The k-th suspension of a 3-sphere with an m-gonal knot cannot be LC, if the knot is at least $m \cdot 2^k$-complicated.*

Example 5.3.4. Analogously to the "Furch–Bing ball", we drill a hole into a finely triangulated 3-ball along a triple pike dive of three consecutive trefoils; we stop drilling one step before destroying the property of having a ball. (See Figure 5.5). We then add a cone over the boundary. The resulting sphere has a triangular knot which is a connected sum of three trefoil knots. By Goodrick [60], the triple trefoil is 3-complicated, so S cannot be LC.

Remark 5.3.5. The idea for the knotted 3-sphere above goes back to Lickorish's 1991 paper [93, p. 530]. In the same paper he announced (without proof given) that "with a little ingenuity" one could get a sphere S with
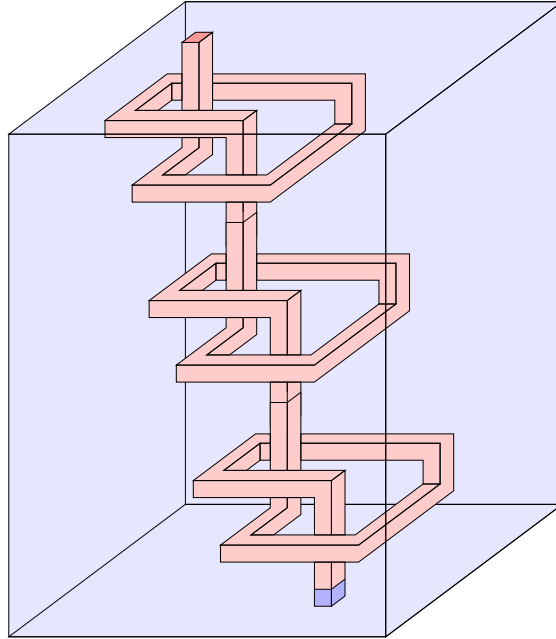
Figure 5.5: A 3-ball B with a knotted spanning edge where the knot is the triple trefoil. (It is produced out of a nicely triangulated 3-ball by drilling a tubular hole downwards, along a triple-trefoil knot, yet stopping one edge before hitting the bottom.) By coning off the boundary of B, we obtain a 3-sphere that contains a knot and is *not* LC.

a 2-complicated triangular knot (the double trefoil), such that $S - \Delta$ is collapsible. (We will prove Lickorish's claim in Proposition 5.3.10.) Such a 3-sphere is LC by Corollary 5.3.1; on the other hand, if v is any vertex of the triangular knot, S can be viewed as the cone (with apex v) over the 3-ball $A := \mathrm{del}_S v$, which contains a double-trefoil-knotted spanning edge. This yields an example of a 3-ball A which is not collapsible (by Theorem 4.3.5), such that S_A minus a facet is collapsible. (Compare with Proposition 3.4.2.)

Example 5.3.6. The 4-sphere with a knotted surface described in Corollary 4.3.8, which is obtained spinning a 6-ple trefoil knot (see Proposition 4.1.1), cannot be LC.

The next result shows that being constructible is strictly stronger than being locally constructible. In fact, locally constructible spheres may contain knots in their 1-skeleton.

Theorem 5.3.7. *For every $d \geq 3$, not all LC d-spheres are constructible.*

Proof. First we show that it suffices to prove the claim for $d = 3$. In fact, if S is an LC d-sphere, $v * S$ is an LC $(d + 1)$-ball by Proposition 1.7.2. The suspension $(v * S) \cup (w * S)$ is then LC by Lemma 5.1.1, since the intersection $(v * S) \cap (w * S)$ coincides with S, which is strongly connected. On the other hand, the suspension of a non-constructible sphere is a non-constructible sphere [69, Corollary 2]. (This is an easy consequence of the fact that in a constructible complex, all vertex links are constructible.) Thus, the d-th suspension of a non-constructible LC 3-sphere is a non-constructible LC $(d + 3)$-sphere.

Let us now show that non-constructible LC 3-spheres exist. Hachimori [69] [66, p. 54] showed that a knotted 3-sphere cannot be constructible. However, by Proposition 3.4.2 and Corollary 5.3.1, coning off the boundary of a collapsible 3-ball yields an LC 3-sphere. Thus coning off the boundary of a collapsible 3-ball with a knotted spanning edge (see Theorem 4.3.1) yields a knotted 3-sphere that is LC. $\square$

Lemma 5.3.8. *For every positive integer m, there exists a simplicial LC 3-sphere with an m-complicated $(m + 2)$-gonal knot.*

Proof. Straightforward from Corollary 4.3.3, part (2), and from Corollary 5.3.1. $\square$

Based on Lickorish's claim (see Remark 5.3.5), which we prove in Proposition 5.3.10, we show now that the bound of Lemma 5.3.8 can be beaten: A 3-sphere with an $(m+1)$-complicated $(m+2)$-gonal knot can still be LC.

Lemma 5.3.9. *Let Q be a simplicial 3-polytope, let A be a 1-sphere disjoint from Q and let v, x be two new vertices. The pyramid $P = |v * x * A|$ admits a triangulation that*

(1) *contains a copy Q' of Q embedded as subcomplex, so that the intersection of Q' with ∂P consists of the two vertices v and x;*
(2) *collapses onto $Q' \cup (x * A)$;*
(3) *coincides with the triangulation $v * x * A$ on the boundary of P.*

Proof. Inscribe the polytope $Q \subset \mathbb{R}^3$ in some pyramid P, so that Q touches P at the apex v and at the midpoint x of the basis of P. We can assume that x coincides with the origin, that the polygonal basis $|x * A|$ of P lies on the plane $z = 0$, that such basis contains two "antipodal" vertices p_1 and p_2 (see Figure 5.6) and that p_1 is not in the affine hull of any facet of P (cf. [153, p. 240]).
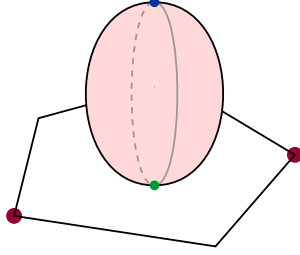
Figure 5.6: Given a polytope Q (pink) in $\mathbb{R}^3$, call v its top vertex (blue) and x its bottom vertex (green). Take a 1-sphere $A \subset \mathbb{R}^3$ so that x is coplanar with A and two vertices p_1 and p_2 (red) of A are "antipodal" with respect to x. If A is chosen 'large enough"' then the pyramid $|v * x * A|$ (which is the convex hull of $v \cup A$) contains Q and admits a triangulation that collapses onto $Q \cup (x * A)$.

Let Q_1 (resp. Q_2) be the subcomplex of ∂Q given by the facets that are visible (resp. not visible) from p_1. Clearly Q_1 and Q_2 are 2-balls that intersect in a 1-sphere. The triangulation of P we seek is obtained by completing the triangulation

$$(p_1 * Q_1) \ \cup \ (p_1 * Q_2)$$

via further cones, using the vertices of A as apices. $\qquad\square$

Proposition 5.3.10 (announced by Lickorish [93, p. 530]). *Let B be a simplicial 3-ball with an $\mathfrak{L}$-knotted spanning edge. There exists a simplicial 3-sphere $S_{B,B}$ that*

(1) *contains the 3-edge knot $2\mathfrak{L}$ in its 1-skeleton, where $2\mathfrak{L}$ denotes the connected sum of $\mathfrak{L}$ with itself;*
(2) *after the removal of a facet collapses onto two copies of B, glued together in a single vertex of ∂B.*

In particular, if B is collapsible, then $S_{B,B}$ is LC.

Proof. Choose a vertex x in the boundary of B. Let $A := \operatorname{link}_{\partial B} x$ and let P be the pyramid $|v * x * A|$. We claim that there is a new triangulation $\tilde{P}$ of P such that

(1) $\tilde{P}$ contains a copy B' of B, so that the two copies of x are identified and the $\mathfrak{L}$-knotted spanning edge of B' goes from x to v;
(2) $\tilde{P}$ collapses onto $B' \cup (x * A)$;
(3) $\tilde{P}$ coincides with $v * x * A$ on ∂P.

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In fact, even if the copy B' of our knotted ball B is *not* a convex polytope, its boundary $\partial B'$ is combinatorially equivalent to the boundary of some simplicial polytope Q. Note that Q contains the endpoints x', y' of the spanning edge of B', but it need not contain an edge that connects x' to y'. Let us apply Lemma 5.3.9, making sure that the points x' resp. y' of Q coincide with the vertices x resp. v of P. We then re-triangulate the interior of Q, replacing its triangulation with the original triangulation of B'.

The claim is thus proven. Since the 3-sphere S_B can be decomposed as

$$S_B \;=\; v * x * A \;\cup\; v * \mathrm{del}_{\partial B} x \;\cup\; B,$$

the union

$$\tilde{P} \;\cup\; v * \mathrm{del}_{\partial B} x \;\cup\; B$$

yields another triangulated 3-sphere, which we will call $S_{B,B}$. Inside $S_{B,B}$ the knotted spanning edges of B and B' are concatenated, so that the knot $\mathfrak{L}$ is "doubled".

Now, if τ is a triangle of ∂B, it is easy to see that the 2-ball $\partial B - \tau$ collapses onto the closed star of x in ∂B. This means that $v * \partial B - v * \tau$ collapses onto $v * x * A$. By attaching B and observing that this does not interfere with the collapse, we obtain

$$S_B - v * \tau \;\searrow\; v * x * A \;\cup\; B.$$

The collapse above does not depend on how the interior of $v * x * A$ is triangulated. Therefore, by replacing $v * x * A$ with $\tilde{P}$, we obtain

$$S_{B,B} - v * \tau \;\searrow\; \tilde{P} \cup B.$$

But by construction $\tilde{P} \searrow B' \cup_x (x * A)$ (where "$\cup_x$" means "union along x") and $x * A \subset B$. Therefore, $\tilde{P} \cup B \searrow B' \cup_x B$, which implies

$$S_{B,B} - v * \tau \;\searrow\; B' \cup_x B.$$

When B is collapsible, $B' \cup_x B$ is also collapsible; $S_{B,B}$ is then LC by Corollary 5.3.1. □

Proposition 5.3.11. *For every positive integer m, there exists a simplicial LC 3-sphere $S_{B_m,B}$ with an $(m+1)$-complicated $(m+2)$-gonal knot in its 1-skeleton.*

Proof. Let B_m be the collapsible 3-ball with an $\mathfrak{L}$-knotted spanning arc of m edges (where $\mathfrak{L}$ stands for the m-ple trefoil) that we constructed in Corollary 4.3.3, part (1). Let x be one of the two endpoints of such knotted spanning arc.

Analogously to the proof of 5.3.10,

– we cone off the boundary of B_m,
– we re-triangulate the cone over the star of x by inserting a copy B' of a collapsible 3-ball with a trefoil-knotted spanning edge, and
– we call $S_{B_m,B}$ the obtained simplicial 3-sphere.

This $S_{B_m,B}$ contains an $(m+1)$-ple trefoil knot realized with $m+2$ edges. Since

$$S_{B_m,B} - v * \tau \ \searrow \ B' \cup_x B_m$$

for some boundary triangle τ (not containing x) of B_m, we conclude by Corollary 5.3.1. □

Therefore,

– Theorem 4.4.3 by Hachimori–Ziegler,
– Proposition 4.4.4 by Ehrenborg–Hachimori–Shimokawa,
– Remark 5.3.5/Prop.- 5.3.10 by Lickorish,
– our Lemma 5.3.8, Proposition 5.3.11 and Theorem 5.3.3,

all blend into the following new hierarchy:

Theorem 5.3.12. *A simplicial 3-sphere that contains a non-trivial knot consisting of*

3 *edges, 1-complicated*	*is not constructible, but can be LC.*
3 *edges, 2-complicated*	*is not constructible, but can be LC.*
3 *edges, 3-complicated or more*	*is not LC.*
4 *edges, 1-complicated*	*is not vertex dec., but can be shellable.*
4 *edges, 2- or 3-complicated*	*is not constructible, but can be LC.*
4 *edges, 4-complicated or more*	*is not LC.*
5 *edges, 1-complicated*	*is not vertex dec., but can be shellable.*
5 *edges, 2-, 3- or 4-complicated*	*is not constructible, but can be LC.*
5 *edges, 5-complicated or more*	*is not LC.*
6 *edges, 1-complicated*	*can be vertex decomposable.*
6 *edges, 2-complicated*	*is not vertex dec., but can be LC.*
6 *edges, 3-, 4- or 5-complicated*	*is not constructible, but can be LC.*
6 *edges, 6-complicated or more*	*is not LC.*
⋮	⋮
m *edges, k-complicated, $k \geq \lfloor \frac{m}{3} \rfloor$*	*is not vertex decomposable.*
m *edges, k-complicated, $k \geq \lfloor \frac{m}{2} \rfloor$*	*is not constructible.*
m *edges, k-complicated, $k \leq m - 1$*	*can be LC.*
m *edges, k-complicated, $k \geq m$*	*is not LC.*

Furthermore, every 3-sphere becomes LC after becoming sufficiently many barycentric subdivisions; however, there is no fixed number r of subdivisions that is sufficient to make *all* 3-spheres LC. (Compare Section 4.5.)

For the sake of completeness, we also give a "bridge index" version of Theorem 5.3.12. Recall that any knot group with bridge index b has a presentation with $b-1$ generators, but it is an open question whether such a presentation is minimal with respect to the number of generators.

Theorem 5.3.13. *A simplicial 3-sphere with a non-trivial knot $\mathfrak{L}$ with*

$\quad$ *3 edges, $b(\mathfrak{L}) = 2$ $\quad$ is not constructible, but can be LC.*

$\quad$ *3 edges, $b(\mathfrak{L}) = 3$ $\quad$ is not constructible, but can be LC.*

$\quad$ *3 edges, $b(\mathfrak{L}) \geq 4$ $\quad$ can be non-LC.*

$$\vdots \qquad\qquad \vdots$$

$\quad$ *m edges, $b(\mathfrak{L}) > \frac{m}{3}$ $\quad$ is not vertex decomposable.*

$\quad$ *m edges, $b(\mathfrak{L}) > \frac{m}{2}$ $\quad$ is not constructible.*

$\quad$ *m edges, $b(\mathfrak{L}) \leq m$ $\quad$ can be LC.*

$\quad$ *m edges, $b(\mathfrak{L}) > m$ $\quad$ can be non-LC.*

5.4 Computer-generated examples

We have seen in Section 1.2 that while 3-spheres are algorithmically recognizable, PL d-spheres are not recognizable for any $d \geq 5$. Nevertheless, most of the classes of manifolds we have encountered so far are algorithmically recognizable. For example, if we want to decide whether a complex has collapse depth c or not, we can try all possible sequences of elementary collapses and record the dimension of the complexes in which we get stuck.

When it comes to find *efficient* algorithms, on the other hand, the situation is not so rosy: There is currently (see e.g. [109]) no efficient algorithm to decide whether a given simplicial d-complex is shellable (resp. collapsible) or not, unless $d \leq 2$ [46]. For each d, both problems are in **NP**, because if somebody gave us a shelling (resp. a collapsing sequence) we would be able to double-check its correctness in polynomial time.

So, deciding local constructibility of a d-manifold is difficult for $d > 2$. (When $d = 2$ the problem boils down to a computation of the Euler characteristic, since all LC 2-manifolds are 2-spheres and the other way around.) The best method seems to use our Theorem 5.2.6: choose a spanning tree T (easy), compute K^T (easy) and then check the collapsibility of K^T (which is $(d-1)$-dimensional). However, while a positive answer leads to the conclusion, a negative answer requires "backtracking", i.e. another try with a different spanning tree.

An algorithm to decide local constructibility of 3-balls without interior vertices will be described in Remark 6.3.5: It is essentially due to Hachimori, even if the original (incorrect) version of the algorithm was designed to test constructibility of 3-balls without interior vertices.

In 1993, Eğecioğlu and Gonzalez [54] studied the following related problem:

Definition 5.4.1 (Erasure)**.** For a polytopal d-complex C, the *erasure* $er(C)$ is the minimal number of d-faces whose removal makes C collapsible onto a lower dimensional complex.

Collapsibility problem

   INSTANCE: A pair (C, k) where C is a simplicial 2-complex,
   and k is a nonnegative integer.
   QUESTION: Is $er(C) \leq k$?

By reducing it to the vertex cover problem, Eğecioğlu and Gonzalez [54] proved that the collapsibility problem is NP-complete. Furthermore, they showed that the collapsibility problem is MAX-SNP-hard, i.e. "a NP-hard problem for which any polynomial approximation algorithm can lead to a result arbitrarily far from the optimum" [91, p. 226]. This is not a contradiction with what we said above ("shellability and collapsibility of 2-complexes are not difficult to decide"), because we were reasoning with respect to the number N of facets, and not with respect to this integer k.

Following Eğecioğlu and Gonzalez, some approximation results have been obtained by Lewiner–Lopes–Tavares [91] and by Joswig–Pfetsch [79] in the extended framework of discrete Morse theory. Recently Engström [53] obtained interesting results using Fourier transforms. From his data [53, Table 4, p. 51] one can derive for example that the erasure of the Dunce Hat is one.

In general, if φ is a discrete Morse function on a d-complex C and $c_k(\varphi)$ is the number of k-dimensional critical cells of C, then the erasure of C is bounded above by $c_d(\varphi)$.

In Subsections 5.4.1 and 5.4.2, we will show how greedy-collapsing algorithms may sometimes succeed in determining the local constructibility of balls or spheres. Both subsections 5.4.1 and 5.4.2 are joint work with Frank H. Lutz.
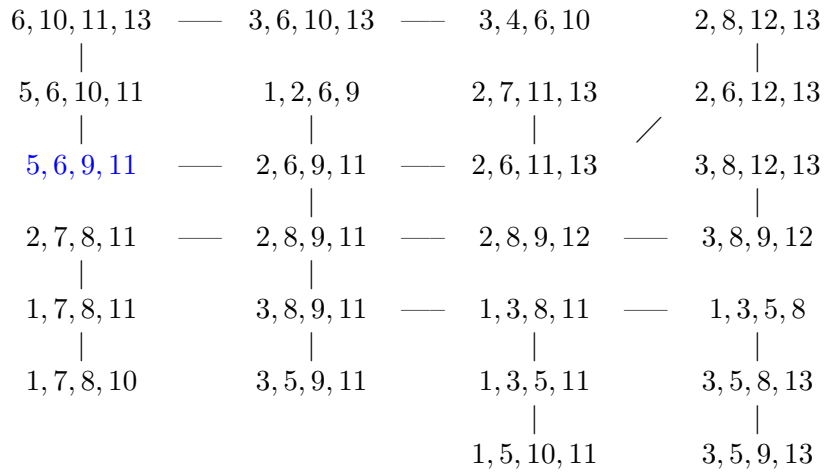
5.4.1 LC knotted spheres and balls

The PL simplicial knotted 3-sphere $S^3_{13,56}$ described by Lutz [101] has only 13 vertices and 56 tetrahedra. Using a greedy-collapsing algorithm, we show here that the removal of a facet makes $S^3_{13,56}$ collapsible. For the sake of brevity, we will only write a facet-killing sequence of $S^3_{13,56}$. The collapsibility of the resulting complex is then easy to verify, because such complex is 2-dimensional.

Consider the following connected subgraph T_1 of the dual graph of $S^3_{13,56}$:

$$
\begin{array}{cccccccccc}
 & & & & 3,6,11,13 & & 1,7,11,13 & & 2,5,7,13 & \\
 & & & & | & & | & & | & \\
 & & & & 3,6,7,12 & & 1,7,9,13 & & 5,7,9,13 & \\
 & & & & | & & | & & | & \\
1,2,9,12 & & 1,2,6,12 & - & 1,6,7,12 & - & 1,6,7,9 & - & 5,6,7,9 & \\
 & \diagdown & & & | & & & & | & \\
1,9,10,13 & - & 1,9,10,12 & - & 1,7,10,12 & & 2,4,5,7 & - & 4,5,6,7 & \\
| & & & & | & & & \diagup & | & \\
1,10,11,13 & & 2,3,7,10 & - & 3,7,10,12 & & 3,4,6,7 & & 4,5,6,10 & \\
 & \diagup & | & & | & & & & | & \\
2,7,8,10 & & 2,3,4,7 & & 3,9,10,12 & & 1,5,8,10 & & 2,4,5,10 & \\
 & & | & & | & & & \diagdown & | & \\
 & & 2,3,4,10 & & 3,9,10,13 & & 2,5,8,13 & - & 2,5,8,10 &
\end{array}
$$
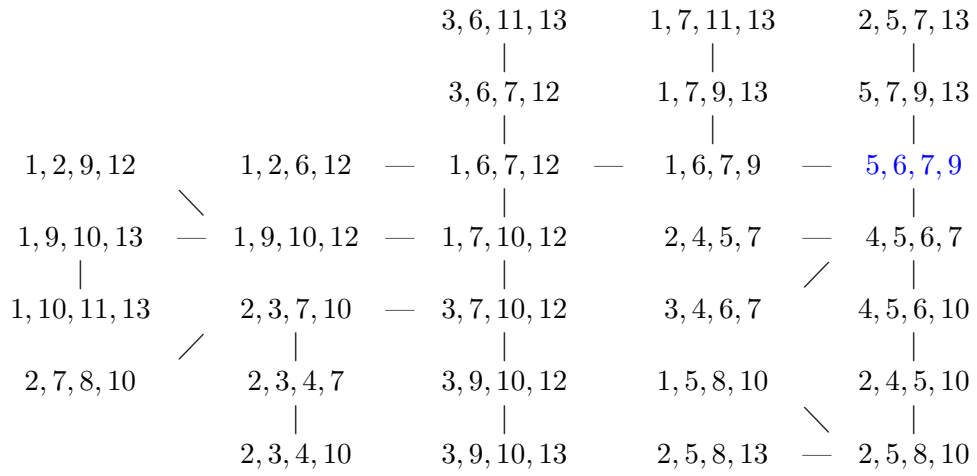
T_1 consists of 30 nodes and 29 edges. Consider now the connected subgraph T_2 of the dual graph of $S^3_{13,56}$ formed by the following 26 nodes and 25 edges:

$$
\begin{array}{cccccccc}
6,10,11,13 & -\!\!- & 3,6,10,13 & -\!\!- & 3,4,6,10 & & 2,8,12,13 \\
| & & & & & & | \\
5,6,10,11 & & 1,2,6,9 & & 2,7,11,13 & & 2,6,12,13 \\
| & & | & & | & \diagup & \\
5,6,9,11 & -\!\!- & 2,6,9,11 & -\!\!- & 2,6,11,13 & & 3,8,12,13 \\
| & & | & & & & | \\
2,7,8,11 & -\!\!- & 2,8,9,11 & -\!\!- & 2,8,9,12 & -\!\!- & 3,8,9,12 \\
| & & | & & & & \\
1,7,8,11 & & 3,8,9,11 & -\!\!- & 1,3,8,11 & -\!\!- & 1,3,5,8 \\
| & & | & & | & & | \\
1,7,8,10 & & 3,5,9,11 & & 1,3,5,11 & & 3,5,8,13 \\
 & & & & | & & | \\
 & & & & 1,5,10,11 & & 3,5,9,13
\end{array}
$$

We can merge T_1 and T_2 together by adding an edge between the blue-coloured nodes, i.e. between $\{5, 6, 7, 9\}$ and $\{5, 6, 9, 11\}$. The result is a spanning tree T of the dual graph of $S^3_{13,56}$. The corresponding 2-complex K^T is given by the following 54 triangles:

$\{1, 2, 12\}$	$\{1, 3, 5\}$	$\{1, 5, 8\}$	$\{1, 5, 10\}$	$\{1, 7, 10\}$	$\{1, 7, 11\}$
$\{1, 8, 10\}$	$\{1, 8, 11\}$	$\{1, 9, 13\}$	$\{1, 10, 11\}$	$\{1, 11, 13\}$	$\{2, 3, 10\}$
$\{2, 4, 5\}$	$\{2, 4, 7\}$	$\{2, 4, 10\}$	$\{2, 5, 13\}$	$\{2, 5, 7\}$	$\{2, 6, 12\}$
$\{2, 7, 8\}$	$\{2, 7, 11\}$	$\{2, 7, 13\}$	$\{2, 8, 10\}$	$\{2, 8, 12\}$	$\{2, 8, 13\}$
$\{2, 9, 12\}$	$\{3, 4, 6\}$	$\{3, 4, 7\}$	$\{3, 4, 10\}$	$\{3, 5, 9\}$	$\{3, 5, 11\}$
$\{3, 6, 7\}$	$\{3, 6, 13\}$	$\{3, 7, 12\}$	$\{3, 8, 9\}$	$\{3, 8, 13\}$	$\{3, 9, 12\}$
$\{3, 9, 13\}$	$\{3, 10, 13\}$	$\{3, 12, 13\}$	$\{4, 6, 10\}$	$\{5, 6, 10\}$	$\{5, 8, 13\}$
$\{5, 9, 11\}$	$\{5, 9, 13\}$	$\{5, 10, 11\}$	$\{6, 11, 13\}$	$\{6, 12, 13\}$	$\{7, 8, 10\}$
$\{7, 9, 13\}$	$\{7, 11, 13\}$	$\{8, 12, 13\}$	$\{9, 10, 12\}$	$\{9, 10, 13\}$	$\{10, 11, 13\}$.

With the help of the computer, we checked that K^T collapses onto the 2-ball given by the triangles $\{1, 2, 6\}$, $\{1, 2, 9\}$ and $\{1, 6, 9\}$. In particular, K^T is collapsible. Therefore, $S^3_{13,56}$ can be locally constructed along T.

Being knotted, $S^3_{13,56}$ cannot be constructible (cf. Theorem 4.4.3); moreover, the removal of a single facet from $S^3_{13,56}$ yields a non-constructible 3-ball. Is there a facet Δ such that $S^3_{13,56} - \Delta$ is LC? We will answer the question positively in Theorem 6.2.4.

5.4.2 Some spheres are LC, but not extensively LC

If we want to locally construct a 3-sphere, the choice of the spanning tree of the dual graph does matter. In fact, it may occur that a 3-sphere is locally constructible along *some* dual spanning tree, but not along *every* dual spanning tree.

Recall that the Dunce Hat is the complex D on the vertices $1, \ldots, 8$ given by the following facets:

$\{1, 2, 4\}$	$\{1, 2, 5\}$	$\{1, 2, 8\}$	$\{1, 3, 6\}$	$\{1, 3, 7\}$	$\{1, 3, 8\}$	$\{1, 4, 5\}$
$\{1, 6, 7\}$	$\{2, 3, 5\}$	$\{2, 3, 6\}$	$\{2, 3, 7\}$	$\{2, 4, 6\}$	$\{2, 7, 8\}$	$\{3, 5, 8\}$
$\{4, 5, 7\}$	$\{4, 6, 7\}$	$\{5, 7, 8\}$.

The Dunce Hat can be realized as subcomplex of a 3-sphere. In fact, let S_D

be the simplicial 3-sphere on 12 vertices given by the following 40 facets:

$$\{0,1,2,3\ \} \quad \{0,1,2,5\ \} \quad \{0,1,3,8\ \} \quad \{0,1,8,11\} \quad \{0,2,3,5\ \}$$
$$\{0,3,5,8\ \} \quad \{0,5,8,11\} \quad \{0,1,5,11\} \quad \{1,2,4,5\ \} \quad \{1,2,4,11\}$$
$$\{1,2,8,10\} \quad \{1,2,8,11\} \quad \{1,3,6,7\ \} \quad \{1,3,6,9\ \} \quad \{1,3,7,10\}$$
$$\{1,3,8,9\ \} \quad \{1,4,5,11\} \quad \{1,6,7,9\ \} \quad \{1,7,9,10\} \quad \{1,8,9,10\}$$
$$\{2,3,5,9\ \} \quad \{2,3,6,9\ \} \quad \{2,3,6,11\} \quad \{2,3,7,10\} \quad \{2,3,7,11\}$$
$$\{1,2,3,10\} \quad \{2,4,5,9\ \} \quad \{2,4,6,9\ \} \quad \{2,4,6,11\} \quad \{2,7,8,10\}$$
$$\{2,7,8,11\} \quad \{3,5,8,9\ \} \quad \{3,6,7,11\} \quad \{4,5,7,9\ \} \quad \{4,5,7,11\}$$
$$\{4,6,7,9\ \} \quad \{4,6,7,11\} \quad \{5,7,8,9\ \} \quad \{5,7,8,11\} \quad \{7,8,9,10\}.$$

It is then easy to check that $D \hookrightarrow S_D$.

Proposition 5.4.2. *There exists a spanning tree T of the dual graph of S_D, such that K^T collapses onto the Dunce Hat.*

Proof. Let T be the tree that goes across the following 39 triangles of S_D:

$$\{2,4,5\} \quad \{2,5,9\} \quad \{2,3,9\} \quad \{2,6,9\} \quad \{4,6,9\} \quad \{4,7,9\}$$
$$\{5,7,9\} \quad \{5,8,9\} \quad \{3,8,9\} \quad \{1,3,9\} \quad \{1,6,9\} \quad \{1,8,9\}$$
$$\{8,9,10\} \quad \{7,8,10\} \quad \{2,7,10\} \quad \{3,7,10\} \quad \{1,7,10\} \quad \{2,8,10\}$$
$$\{1,2,10\} \quad \{1,2,3\} \quad \{3,6,7\} \quad \{3,7,11\} \quad \{2,3,11\} \quad \{2,6,11\}$$
$$\{4,6,11\} \quad \{4,7,11\} \quad \{5,7,11\} \quad \{7,8,11\} \quad \{2,8,11\} \quad \{1,2,11\}$$
$$\{1,4,11\} \quad \{1,5,11\} \quad \{0,1,5\} \quad \{0,2,5\} \quad \{0,3,5\} \quad \{0,5,8\}$$
$$\{0,8,11\} \quad \{0,3,8\} \quad \{0,1,3\}.$$

None of the 17 faces of the Dunce Hat appears in such list. Since the facets of K^T are the 41 triangles of S_D that are *not* hit by T, the Dunce Hat is a full-dimensional subcomplex of K^T. Being K^T 2-dimensional, this suffices to conclude that K^T is not collapsible. (In fact, K^T collapses onto D). $\square$

Proposition 5.4.3. *There exists a spanning tree T' of the dual graph of S_D such that $K^{T'}$ is collapsible.*

Proof. Let T' be the tree that goes across the following 39 triangles of S_D:

$$\{6,7,9\} \quad \{4,6,9\} \quad \{2,4,9\} \quad \{2,4,5\} \quad \{1,2,4\} \quad \{1,6,7\}$$
$$\{4,6,7\} \quad \{4,7,11\} \quad \{3,6,7\} \quad \{1,3,6\} \quad \{2,5,9\} \quad \{1,3,7\}$$
$$\{2,3,9\} \quad \{2,3,6\} \quad \{4,5,11\} \quad \{4,7,9\} \quad \{3,7,10\} \quad \{1,7,9\}$$
$$\{3,7,11\} \quad \{7,9,10\} \quad \{2,3,10\} \quad \{2,7,10\} \quad \{1,9,10\} \quad \{3,5,9\}$$
$$\{2,4,6\} \quad \{2,8,10\} \quad \{5,8,9\} \quad \{1,5,11\} \quad \{1,2,5\} \quad \{2,3,5\}$$
$$\{3,5,8\} \quad \{1,3,9\} \quad \{0,3,8\} \quad \{0,1,2\} \quad \{2,7,11\} \quad \{7,8,11\}$$
$$\{5,8,11\} \quad \{0,1,8\} \quad \{1,8,11\}.$$

The reader can check that $K^{T'}$ is collapsible. $\square$

By Corollary 5.3.1, S_D is locally constructible along T', but it is not locally constructible along T.

Corollary 5.4.4. *Given an LC sphere S, not every sequence of local gluings may be completed to a local construction of S. Therefore, while locally constructing an LC sphere, we may get stuck.*

5.5 Extension to k-LC manifolds

Recall that a k-LC d-manifolds is obtained from a tree of d-polytopes by gluing together $2k$ boundary facets pairwise, and then by matching the remaining boundary facets according to local gluings. Some of the results we saw in Section 5.2 generalize to the class of k-LC d-manifolds. This yields useful applications, like an exponential upper bound for simplicial 2-manifolds with bounded genus.

Let us start with a straightforward extension of Theorem 5.2.6, which relates the k-LC notion with the "erasure" notion by Eğecioğlu–Gonzalez (cf. Def. 5.4.1):

Theorem 5.5.1. *Let $k, d \in \mathbb{N}$, with $d \geq 2$, and let M be a d-manifold. The following are equivalent:*

(1) *M is k-LC;*
(2) *$er(K^T) \leq k$, for some spanning tree T of the dual graph of M;*
(3) *there is a spanning tree T of the dual graph of M and there are k $(d-1)$-dimensional faces $\sigma_1, \ldots, \sigma_k$ such that $K^T - \sigma_1 - \sigma_2 - \ldots - \sigma_k$ collapses onto a $(d-2)$-complex.*

After Theorem 5.5.1, the k-LC notion can be explained by means of a game. Suppose we are given a d-manifold M with one d-cell removed, while our goal is to remove all the $(d-1)$-cells from it. The rules of the game are the following:

– removing a $(d-1)$-cell via an elementary collapse costs us nothing;
– otherwise, removing a $(d-1)$-cell costs one dollar.

Different strategies may result in different costs. LC d-manifolds are characterized by the fact that we can win the game for free by playing optimally. A d-manifold is k-LC if and only if we can win the game with a budget of k dollars.

Theorem 5.5.2. *An orientable surface M is $(2g)$-LC if and only if its genus is at most g. In particular, the smallest integer k for which an orientable surface is k-LC is always even.*

Proof. Fix an orientable surface M_g of genus $g > 0$ and a spanning tree T of its dual graph. Choose a facet σ of M_g: The 2-manifold with boundary $M_g - \sigma$ collapses along T onto the 1-complex K^T, which in turn collapses onto some leafless connected graph G^T. (Note that G^T depends on the spanning tree chosen, but not on the facet σ chosen.) Since collapses preserve the homotopy type, G^T is homotopy equivalent to $M_g - \sigma$, which retracts to a wedge of $2g$ 1-spheres. In particular:

- the removal of $2g$ edges makes G^T contractible; therefore, K^T minus the same $2g$ edges is collapsible. By Theorem 5.5.1, M_g is $(2g)$-LC;
- the removal of fewer edges does not make G^T contractible, thus M_g is not $(2g - 1)$-LC.

$\square$

Corollary 5.5.3. *Simplicial orientable 2-manifolds with bounded genus are exponentially many, both with respect to the number of facets N and to the number of vertices n.*

Proof. By Theorem 5.5.2, all orientable 2-manifolds with genus bounded by $\frac{k}{2}$ are k-LC. By Theorem 2.6.5, simplicial 2-manifolds with bounded genus are exponentially many with respect to N. To prove an exponential bound with respect to n, it suffices to focus on manifolds of *fixed* genus, since a finite sum of exponential bounds yields an exponential bound. If M has genus g, using the Euler equation we may write $n = \frac{N}{2} - 2g + 2$, which depends on N linearly: Therefore, what is exponential in N is also exponential in n. $\square$

The next result generalizes Theorem 3.5.1:

Theorem 5.5.4. *Let $k, d \in \mathbb{N}$, with $d \geq 2$. Let K be a simplicial k-LC d-manifold. For each $(d-2)$-dimensional subcomplex $\mathfrak{L}$ of K, the fundamental group of $|K| - |\mathfrak{L}|$ has a presentation with exactly $f_{d-2}(\mathfrak{L}) + k$ generators.*

Proof. Analogous to the proof of Theorem 3.5.1: We repeat it for the sake of completeness. Choose a tetrahedron of K and call it A_0. By Theorem 5.5.1, there is a dual spanning tree T of K such that the $(d-1)$-complex K^T, after the removal of k faces, collapses onto a $(d-2)$-complex. Thus we can write down:

- a list of $N-1$ pairs $((d-1)$-face, d-face$)$ that form the collapse of $K - A_0$ onto K^T;
- a list of k faces $\sigma_1, \ldots, \sigma_k$, of dimension $d-1$;

– a list of P pairs[2] of the type $((d-2)$-face, $(d-1)$-face), which form the collapse of K^T minus k faces onto a $(d-2)$-complex;
– a list of all the remaining faces, ordered by decreasing dimension.

Let us put together all the faces that appear above, maintaining their order, to form a single list of simplices

$$A_1, A_2, \ldots, A_{2N-1}, A_{2N-2},$$
$$A_{2N-1}, \ldots, A_{2N+k-2},$$
$$A_{2N+k-1}, A_{2N+k}, \ldots, A_{2M+k-1}, A_{2M+k},$$
$$A_{2M+k+1}, \ldots, A_{F-1},$$

where $F = \sum_{j=0}^{d} f_j(K)$ counts the number of nonempty simplices of K.

In such a list A_1 is a free face of A_2; A_3 is a free face of A_4 with respect to the complex $K - A_1 - A_2$; and so on. In general, for each i in $\{1, \ldots, 2M+k\}$, A_i is not a face of A_j for any $j > i + 1$. Note that A_i may or may not be a face of A_{i+1}, depending on the value of i: for example A_1 is a face of A_2, but A_2 is not a face of A_3.

Whenever A_i is a face of A_{i+1}, we will consider the two faces as an "indivisible pair". For example, (A_1, A_2) form an indivisible pair and so do (A_{2N-3}, A_{2N-2}); on the contrary, A_{2N-1} is "all by himself". (The indivisible pairs are exactly the pairs of faces that were collapsed together.)

The idea is now to consider the subcomplex of $sd(K)$ consisting of all simplices of $sd(K)$ that have no face in $\mathfrak{L}$; such a complex is a deformation retract of $|K| - |\mathfrak{L}|$. We can build this complex step by step as follows:

– we start with the point $\hat{A}_0$;
– we attach one at a time (or two at a time, in case they form an indivisible pair) the ordered dual cells $\hat{A}_i$, provided A_i is not in $\mathfrak{L}$.

How does each attachment affect the homotopy type? There are five cases to consider:

(I) A_i is a $(d-1)$-cell and forms an indivisible pair with A_{i+1}. This means that A_{i+1} is a d-cell; thus neither A_i, nor A_{i+1} may belong to $\mathfrak{L}$, which is $(d-2)$-dimensional. By Newman's theorem (Lemma 3.2.3), A_i^* is a 1-cell that contains in its boundary the 0-cell A_{i+1}^*. Thus our attachment consists in attaching an edge along one of its vertices; this does not change the homotopy type of the complex.

---

[2]This integer P can be computed explicitly. The total number of $(d-1)$-faces of $K - A_0$ is $\frac{(d+1)N}{2}$. Out of these faces, $N-1$ resp. k have been removed in the first resp. second phase. The remaining $\frac{dN-N+2-2k}{2}$ faces of dimension $d-1$ are removed in the third phase: Hence $P = \frac{dN-N+2-2k}{2}$.

(II) A_i is a $(d-2)$-cell not in $\mathfrak{L}$ and forms an indivisible pair with A_{i+1}. This means that A_{i+1} is a $(d-1)$-cell, thus it cannot belong to $\mathfrak{L}$. This time we are attaching a 2-cell A_i^* together with a 1-cell A_{i+1}^* in its boundary. Such an attachment does not change the homotopy type.

(III) A_i is a $(d-2)$-cell that does belong to $\mathfrak{L}$ and forms an indivisible pair with A_{i+1}. The situation is similar to the previous one, but this time we are attaching only the 1-cell A_{i+1}^*. Such an attachment does change the homotopy type: it creates a loop.

(IV) A_i is a $(d-1)$-cell and does not form an indivisible pair with A_{i+1}, nor with A_{i-1}. Since A_i is all by himself, we are attaching a single 1-cell A_i^* to the complex, creating a loop.

(V) A_i is a k-cell, with $k \leq d-2$, and does not form an indivisible pair with A_{i+1}. Thus we are attaching to the CW complex a single cell A_i^*, of dimension at least two.

Only in the last three cases the homotopy type changes at all; yet since we are interested in the number of *generators* in a presentation for the first homotopy group of the complex (and not in the number of relators), we may focus only on the number of *loops* in the model, so that case (V) may be neglected. Now, case (III) occurs exactly $f_{d-2}(\mathfrak{L})$ times, while case (IV) occurs exactly k times. Therefore, the fundamental group of the subcomplex of all simplices of $sd(K)$ that have no face in $\mathfrak{L}$ has a presentation with $k + f_{d-2}(\mathfrak{L})$ generators; but such a subcomplex is a deformation retract of $|K| - |\mathfrak{L}|$, so we are done. □

Corollary 5.5.5. *Fix an integer $d \geq 3$. Let S be a 3-sphere with an m-gonal knot in its 1-skeleton, so that the knot is at least $(m \cdot 2^{d-3} + k)$-complicated. Then the $(d-3)$-rd suspension of S is a PL d-sphere that is not k-LC.*

Corollary 5.5.6. *A 3-sphere with a $(k+m)$-complicated m-gonal knot cannot be k-LC.*

Chapter 6

Locally constructible manifolds with boundary

In this Chapter, in order to reach the hierarchy for d-balls collected in Theorem 6.0.1, we give a combinatorial characterization of LC d-manifolds with boundary (Theorem 6.1.9); it is a bit more complicated, but otherwise analogous to the characterization given in Theorem 5.0.1.

Theorem 6.0.1. *For simplicial d-balls, we have the following hierarchy:*

$$\{\text{vertex dec.}\} \subsetneq \{\text{shellable}\} \subsetneq \{\text{constructible}\} \subsetneq \{\text{LC}\} \subsetneq$$
$$\subsetneq \{\text{collapsible onto a } (d-2)\text{-complex}\} \subsetneq \{\text{all } d\text{-balls}\}.$$

Proof. The first two inclusions are known. We have already seen that all constructible complexes are LC (Lemma 5.1.1). Every LC d-ball is collapsible onto a $(d-2)$-complex by Corollary 6.2.1.

Let us see next that all inclusions are strict for $d = 3$: For the first inclusion this follows from Lockeberg's example of a 4-polytope whose boundary is not vertex decomposable. For the second inclusion, take Ziegler's non-shellable ball from [154], which is constructible by construction. A non-constructible 3-ball that is LC will be provided by Theorem 6.2.4. A collapsible 3-ball that is not LC will be given in Theorem 6.3.6. Finally, Bing [21] and Goodrick [60] showed that some 3-balls are not collapsible.

To show that the inclusions are strict for all $d \geq 3$, we argue as follows. For the first four inclusions we get this from the case $d = 3$, since

– cones are always collapsible (cf. Proposition 3.4.1);

- the cone $v * B$ is vertex decomposable resp. shellable resp. constructible if and only if B is (cf. Lemma 1.7.1);
- the cone $v * B$ is LC if and only if B is (cf. Proposition 1.7.2).

For the last inclusion and $d \geq 3$, we look at the d-balls obtained by removing a facet from a non-LC d-sphere. These exist by Corollary 4.3.9; they do not collapse onto a $(d-2)$-complex by Theorem 5.3.2. $\qquad\square$

When $d = 3$, "collapsible onto a $(d-2)$-complex" is the same as "collapsible": in fact, if a 3-ball collapses onto a 1-complex C, this C has to be contractible, yet all contractible 1-complexes are collapsible. Thus for $d = 3$ the hierarchy above yields the following result (valid also in the non-simplicial case):

Corollary 6.0.2. *For* 3-*balls,*

$$\{\text{shellable}\} \subsetneq \{\text{constructible}\} \subsetneq \{\text{LC}\} \subsetneq \{\text{collapsible}\} \subsetneq \{\text{all 3-balls}\}.$$

6.1   Characterization of local constructibility

The arguments of Section 3.1 can be extended to manifolds with boundary; the idea is to consider collapses that preserve the boundary faces. We start with a relative version of facet-killing sequences and facet-massacres.

Definition 6.1.1. Let P a pure d-complex. Let Q be a proper subcomplex of P, either pure d-dimensional or empty. A *facet-killing sequence of* (P, Q) is a sequence $P_0, P_1, \ldots, P_{t-1}, P_t$ of simplicial complexes such that $t = f_d(P) - f_d(Q)$, $P_0 = P$, and P_{i+1} is obtained from P_i by removing a pair (σ, Σ) such that σ is a free $(d-1)$-face of Σ that does not lie in Q.

It is easy to see that P_t has the same $(d-1)$-faces as Q. The version of facet killing sequence given in Definition 3.1.2 is a special case of this one, namely the case when Q is empty.

Definition 6.1.2. Let P a pure d-dimensional complex. Let Q be either the empty complex, or a pure d-dimensional proper subcomplex of P. A *pure facet-massacre of* (P, Q) is a sequence $P_0, P_1, \ldots, P_{t-1}, P_t$ of (pure) complexes such that $t = f_d(P) - f_d(Q)$, $P_0 = P$, and P_{i+1} is obtained from P_i by removing:

(a) a pair (σ, Σ) such that σ is a free $(d-1)$-face of Σ, with σ not in Q, and

(b) all inclusion-maximal faces of dimension smaller than d that are left after the removal of type (a) or, recursively, after removals of type (b).

Necessarily $P_t = Q$ (and when $Q = \emptyset$ we recover the notion of facet-massacre of P that we introduced in Definition 5.2.1). It is easy to see that a step $P_i \longrightarrow P_{i+1}$ can be factorized (not in a unique way) into an elementary collapse followed by the removal of some k-faces ($k < d$) which makes P_{i+1} a pure complex. Thus, a single pure facet-massacre of (P, Q) corresponds to many facet-killing sequences of (P, Q).

We will study the pair $(P, Q) = (K^T, \partial M)$, where M is a d-manifold with boundary, T is a spanning tree of its dual graph, and K^T is defined as follows:

Definition 6.1.3. Given a d-manifold M and a spanning tree T of its dual graph, we denote by K^T the pure $(d-1)$-dimensional subcomplex of M formed by all $(d-1)$-faces of M that are not intersected by T.

Note that K^T contains ∂M as subcomplex, because any spanning tree of the dual graph of a manifold with boundary does not intersect the boundary faces. Let b be the number of $(d-1)$-faces in the boundary ∂M, and let N be as usual the number of facets of M.

Lemma 6.1.4. *Under the previous notations, $M - \Delta \searrow K^T$ for any facet Δ of M. Moreover:*

- *if M is simplicial, K^T has $D + \frac{b}{2}$ facets, where $D := \frac{d-1}{2}N + 1$;*
- *if M is cubical, K^T has $E + \frac{b}{2}$ facets, where $E := (d-1)N + 1$.*

Proof. $M - \Delta$ collapses onto K^T along the tree T. Since T hits exactly $N-1$ interior $(d-1)$-faces of M, the number of facets of K^T is by definition

$$f_{d-1}(K^T) = f_{d-1}(M) - (N-1),$$

with the convention that $f_d(K)$ counts the number of d-faces of a complex K. The integer $f_{d-1}(M)$ can be determined double-counting the facet/ridge adjacencies of M: every interior ridge of M lies in exactly two facets, while each of the b boundary ridges lies in exactly one facet. Since all the N facets of M contain the same number A of ridges ($A = d+1$ in the simplicial case, $A = 2d$ in the cubical case), it follows that

$$AN = 2\left(f_{d-1}(M) - b\right) + 1 \cdot b,$$

whence we get $f_{d-1}(M) = \frac{AN+b}{2}$ and finally

$$f_{d-1}(K^T) = \frac{AN + b}{2} - (N - 1) = \frac{(A-2)N}{2} + 1 + \frac{b}{2}.$$

$\square$

We introduce another convenient piece of terminology.

Definition 6.1.5 (seepage)**.** Let M be a d-manifold. A *seepage* is a $(d-1)$-dimensional subcomplex C of M whose $(d-1)$-faces are exactly given by the list of the boundary facets of M.

A seepage is not necessarily pure; actually there is only one pure seepage, namely ∂M itself. Since K^T contains ∂M, a collapse of K^T onto a seepage must remove all the $(d-1)$-faces of K^T that are not in ∂M: This is what we called a facet-killing sequence of $(K^T, \partial M)$.

Proposition 6.1.6. *Let M be a d-manifold with boundary, and Δ a d-simplex of M. Let C be a seepage of ∂M. Then,*

$$M - \Delta \searrow C \iff \exists\, T \text{ s.t. } K^T \searrow C.$$

Proof. Analogous to the proof of Proposition 3.1.4. The crucial assumption is that no face of ∂M is removed in the collapse (since all of the boundary faces are still present in the final complex C). $\qquad\square$

If we fix a spanning tree T of the dual graph of M, we have then a 1-1 correspondence between the following sets:

1. the set of collapses $M - \Delta \searrow K^T$;
2. the set of natural labelings of T, where Δ is labelled by 1;
3. the set of the first parts $(T_1, \ldots, T_N)$ of local constructions for M, with $T_1 = \Delta$.

Theorem 6.1.7. *Let M be a d-manifold with boundary; fix a facet Δ and a spanning tree T of the dual graph of M, rooted at Δ. The second part of a local construction for M along T corresponds bijectively to a facet-massacre of $(K^T, \partial M)$.*

Proof. Let us start with a local construction $[T_1, \ldots, T_{N-1},]\, T_N, \ldots, T_k$ for M along T. Topologically, $M = T_N/\!\sim$, where $\sim$ is the equivalence relation determined by the gluing, and $K^T = \partial T_N/\!\sim$.

The complex K^T is pure $(d-1)$-dimensional, and contains the boundary ∂M. All the $(d-1)$-faces in $K^T - \partial M$ represent gluings. The local construction $T_1, \ldots, T_{N-1}, T_N, \ldots, T_k$ produces M from T_N in $k - N$ steps, each removing a pair of facets from the boundary. It is easy to see that $k - N$ equals the number of facets of K^T.

Define $P_0 := K_T = \partial T_N/\!\sim$, and $P_j := \partial T_{N+j}/\!\sim$ $(j = 1, \ldots, k - N)$. In the first LC step, $T_N \to T_{N+1}$, we remove from the boundary a free ridge r, together with the unique pair σ', σ'' of facets of ∂T_N sharing r. At the same

time, r and the newly formed face σ are sunk into the interior; so obviously neither σ nor r will appear in ∂M. This step $\partial T_N \longrightarrow \partial T_{N+1}$ naturally induces an analogous step $\partial T_{N+j}/\sim \longrightarrow \partial T_{N+j+1}/\sim$, namely, the removal of r and of the unique $(d-1)-$face σ containing it, with r not in ∂M.

Thus from each local construction we obtain a pure facet-massacre of $(K^T, \partial M)$. Conversely, let us start with a "facet-massacre" $P_0, \ldots, P_{k-N}$ of K^T, which is $(d-1)$-dimensional; again $P_0 = K_T = \partial T_N/\sim$; the unique $(d-1)$-face σ_j killed in passing from P_j to P_{j+1} corresponds to a unique pair of $(d-1)$ (adjacent!) faces σ'_j, σ''_j in ∂T_{N+j}; gluing them together is the LC move that transforms T_{N+j} into T_{N+j+1}. $\qquad\square$

Summing up, for fixed T and Δ:

- the first part of a local construction along T corresponds to a collapse of $M - \Delta$ onto K^T;
- the second part of a local construction along T can be viewed as a pure facet massacre of $(K^T, \partial M)$;
- a single facet massacre of $(K^T, \partial M)$ corresponds to many facet-killing sequences of $(K^T, \partial M)$;
- a facet-killing sequence of $(K^T, \partial M)$ is a collapse of K^T onto some seepage.

Thus, M can be locally constructed along a tree T if and only if K^T collapses onto some seepage. What if we do not fix the tree T or the facet Δ?

Lemma 6.1.8. *Let M be a d-manifold with non-empty boundary; let σ be a $(d-1)$-face in the boundary ∂M, and let Σ be the unique facet of M containing σ. Let C be a subcomplex of M. If C contains ∂M, the following are equivalent:*

(1) $M - \Sigma \searrow C$;
(2) $M - \Sigma - \sigma \searrow C - \sigma$;
(3) $M \searrow C - \sigma$;

Proof. (1) and (2) are clearly equivalent. In the collapse $M \searrow C - \sigma$, the boundary face σ must have been removed together with Σ; we can assume that this elementary collapse was the first to be performed. Thus (3) implies (2). The implication (2) $\Rightarrow$ (3) is obvious. $\qquad\square$

Theorem 6.1.9. *Let M be a d-manifold with boundary. The following are equivalent:*

1. *M is LC;*
2. *K^T collapses onto some seepage C, for some spanning tree T of the dual graph of M;*

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3. *there exists a seepage C such that for every facet Δ of M one has $M - \Delta \searrow C$;*
4. *$M - \Delta \searrow C$, for some facet Δ of M, and for some seepage C;*
5. *there exists a seepage C such that for every facet σ of ∂M one has $M \searrow C - \sigma$;*
6. *$M \searrow C - \sigma$, for some facet σ of ∂M, and for some seepage C.*

Proof. The equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ are established analogously to the proof of Theorem 5.2.6. Lemma 6.1.8 implies that $3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 4$. $\qquad\square$

Remark 6.1.10. In order to extend the previous results to the case where M is a pseudomanifold (or a "strongly connected simplicial complex in which every ridge lies in two facets"), one needs to take care of the following example: Let M be a pinched annulus obtained by identifying two "distant" vertices in a tree of triangles T_N. The 2-complex M is not simply connected and thus not LC; however, M minus a facet collapses onto the boundary ∂M (which coincides with K^T, where T is the dual graph of T_N).

In general, let M' be a d-pseudomanifold obtained from M by identifying two k-dimensional boundary faces, $k < d$. If M minus a facet collapses onto the boundary ∂M, then M' minus a facet also collapses onto the boundary $\partial M'$: The collapsing sequence is the same.

6.2   Application to d-balls

Theorem 6.1.9 has many interesting consequences when applied to d-balls. The first advantage in considering balls instead of generic manifolds with boundary is that the boundary of a d-ball is a $(d-1)$-manifold, while the boundary of an arbitrary d-manifold might be disconnected. In particular, the boundary of a d-ball has collapse depth greater or equal than one.

Corollary 6.2.1. *Every LC d-ball collapses onto a $(d-2)$-complex.*

Proof. By Theorem 6.1.9, the ball B collapses onto the union of the boundary of B minus a facet with some $(d-2)$-complex. The boundary of B is a $(d-1)$-sphere; thus the boundary of B minus a facet is a $(d-1)$-ball that can be collapsed down to dimension $d-2$, and the additional $(d-2)$-complex will not interfere. $\qquad\square$

Remark 6.2.2. By Proposition 3.4.4 and Theorem 5.2.6, if a d-manifold with boundary B collapses onto some $(d-2)$-complex and in addition ∂B is an LC $(d-1)$-manifold, then $B \cup v * \partial B$ is an LC d-manifold.

Removing any facet Δ from a 3-sphere S we obtain a 3-ball $S - \Delta$. The combinatorial topology of d-balls and of d-spheres are intimately related:

– A d-sphere S is shellable if and only if $S - \Delta$ is shellable for some facet Δ of S. (To see this, take as Δ the last facet in the shelling order.) It is an open question whether this is equivalent to "$S - \Delta$ is shellable for *all* facets Δ" [69, p. 166].

– Hachimori and Ziegler [69, Theorem 4] showed that a 3-sphere S is constructible if and only if $S - \Delta$ is constructible for some facet Δ of S, if and only if $S - \Delta$ is constructible for *all* facets Δ of S. Their argument is specific for dimension three: For d-spheres we only know that the constructibility of $S - \Delta$ for some Δ implies S is constructible.

– We have shown in Corollary 5.3.1 that a d-sphere S is LC if and only if $S - \Delta$ is collapsible onto a $(d-2)$-complex for some facet Δ of S, if and only if $S - \Delta$ is collapsible onto a $(d-2)$-complex for *all* facets Δ of S, if and only if there is a dual spanning tree T of S such that K^T collapse onto a $(d-2)$-complex.

– A d-sphere S is LC if $S - \Delta$ is LC for some Δ, by Corollary 6.2.1 and Theorem 5.2.6 (or by Lemma 5.1.1.)

The next result yields a partial converse of the last fact above.

Lemma 6.2.3. *Let Δ be a facet of a d-sphere S; let δ be a facet of $\partial\Delta$. Then $S - \Delta$ is an LC d-ball if and only if there is a dual spanning tree T of S such that K^T collapses onto the union of $(\partial\Delta - \delta)$ with some $(d-2)$-complex.*

Proof. Straightforward from Theorem 6.1.9. $\qquad\square$

When any of the two equivalent conditions of Lemma 6.2.3 is met, Δ is a *leaf* of the tree T, because all the boundary facets of Δ except δ belong to K^T. For $d = 3$, Lemma 6.2.3 boils down to "$S - \Delta$ is LC if and only K^T collapses onto the 2-ball $(\partial\Delta - \delta)$, for some T".

Theorem 6.2.4. *For every $d \geq 3$, not all (simplicial) constructible d-balls are LC.*

Proof. If B is a non-constructible LC d-ball, $v * B$ is a non-constructible LC $(d+1)$-ball; thus, it suffices to prove the claim for $d = 3$.

Let $S^3_{13,56}$ be Lutz's simplicial 3-sphere [101] described in Section 5.4.1. Since it contains a 3-edge knot, $S^3_{13,56}$ cannot be constructible. However, there exists a spanning tree T of $S^3_{13,56}$ such that K^T is collapsible; moreover, the facet $\Delta := \{1, 2, 6, 9\}$ is a leaf of such tree and K^T collapses onto $\Delta - \{2, 6, 9\}$ (cf. Section 5.4.1). Thus by Lemma 6.2.3 the 3-ball $B_{13,55} := S^3_{13,56} - \Delta$ is LC. Being knotted, $B_{13,55}$ cannot be constructible. $\qquad\square$

Remark 6.2.5. The LC non-constructible 3-ball $B_{13,55}$ contains plenty of interior points: Compare Proposition 6.3.2.

Applying Theorem 6.1.9 to 3-balls, we are able to answer the question of Hachimori (see e.g. [66, pp. 54, 66]) of whether all constructible 3-balls are collapsible.

Corollary 6.2.6. *Let B be a 3-ball. The following are equivalent:*

1.   *B is LC;*
2.   *$K^T \searrow \partial B$, for some spanning tree T of the dual graph of B;*
3.   *$B - \Delta \searrow \partial B$, for every facet Δ of B;*
4.   *$B - \Delta \searrow \partial B$, for some facet Δ of B;*
5.   *$B \searrow \partial B - \sigma$, for every facet σ of ∂B;*
6.   *$B \searrow \partial B - \sigma$, for some facet σ of ∂B.*

Proof. When B has dimension 3, any seepage C of ∂B is a 2-complex containing ∂B, plus some edges and vertices. Now, $B - \Delta$ is homotopy equivalent to a S^2, and collapses onto C; thus C is also homotopy equivalent to S^2. Therefore, C can only be ∂B with some trees attached (see Figure 6.1), whence we conclude that $C \searrow \partial B$. □
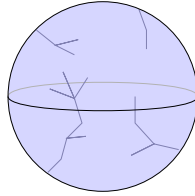

Figure 6.1: A seepage of a 3-ball.

Corollary 6.2.7. *All LC 3-balls are collapsible.*
    In particular, all constructible 3-balls are collapsible.

Proof. By Corollary 6.2.6, an LC 3-ball collapses to some 2-ball $\partial B - \sigma$; but all 2-balls are collapsible. □

For example, the four 3-balls described by Ziegler, Lutz, Grünbaum and Rudin (see Section 1.3) are all collapsible. When $d \geq 4$, we do not know whether constructible d-balls are all collapsible or not. All shellable d-balls are collapsible by [48, Lemma 17, p. 1116]. Also, all constructible d-balls are collapsible onto a $(d-2)$-complex by Corollary 6.2.1.

Note that the locally constructible 3-balls with N facets are precisely the 3-balls that admit a "special collapse", namely such that after the first elementary collapse, in the next $N-1$ collapses, no triangle of ∂B is collapsed

away. Such a collapse acts along a dual (directed) tree of the ball, whereas a generic collapse acts along an acyclic graph that might be disconnected.

One could argue that maybe "special collapses" are not that special: Perhaps every collapsible 3-ball has a collapse that removes only one boundary triangle in its top-dimensional phase? This is not so: We will produce a counterexample in the next section (Theorem 6.3.6).

6.3   3-Balls without interior vertices

Here we show that a 3-ball with all vertices on the boundary cannot contain any knotted spanning edge if it is LC, but might contain some if it is collapsible. We use this fact to establish our hierarchy for d-balls (Theorem 6.0.1).

Let us fix some notation first. Consider a local gluing $T_i \rightsquigarrow T_{i+1}$ in the local construction of some 3-complex P. Let σ be the interior k-gon of T_{i+1} (and also of P) obtained from the identification of adjacent k-gons σ', σ'' in the boundary of T_i. In other words, σ is the image of $\sigma' \cup \sigma''$ under the identification $T_i \twoheadrightarrow T_{i+1}$. Let K be the image of $\sigma' \cap \sigma''$ under this identification map. K is a 1-dimensional subcomplex of σ.

We claim that there are only three possible cases: either

(A) K consists of a single edge, or
(B) K contains (at least) two adjacent edges, or
(C) K is disconnected.

In fact, K must contain at least one edge e. If K coincides with e we are in case (A); if K contains e plus some isolated vertices, we are in case (C); if K contains several edges, and no two of these edges are adjacent, we are again in case (C); otherwise, we are in case (B). So the claim is proven. (Note that (A), (B) and (C) do not form a partition of all possible situations: Cases (B) and (C) may overlap.)

Correspondingly, the local gluing modifies the topology as follows:

– in case (A), T_{i+1} has the same topology of T_i and every vertex on the boundary of T_i lies also on the boundary of T_{i+1};
– in case (B), at least one boundary vertex of T_i (namely, the vertex of σ in between two adjacent edges of K) is sunk into the interior of T_{i+1};
– in case (C), T_{i+1} does not have the same topology of T_i, because the boundary ∂T_i is pinched in some point(s) and/or disconnected.

Now, let B be an LC 3-ball *without* interior vertices. Gluings of type (B) cannot occur in the local construction of B, because they would create interior vertices. Gluings of type (C) have to be followed by gluings of type

(B) to restore the ball topology; thus they cannot occur, either. This proves the following lemma:

Lemma 6.3.1. *Let B be an LC 3-pseudomanifold. The following are equivalent:*

(1) in some local construction for B all of the local gluings are of type (A)*;*
(2) in every local construction for B all of the local gluings are of type (A)*;*
(3) B is a 3-ball without interior vertices.

Note that there are LC 3-manifolds without interior vertices that are not 3-balls, like the *fake cube*, which is a $3 \times 3 \times 3$ pile of cubes with the central cube missing, cf. Figure 5.1. In the local construction of the fake cube most of the gluings are of type (A), but there is exactly one local gluing of type (C).

We will use Lemma 6.3.1 to obtain examples of non-LC 3-balls. We already know that non-collapsible balls are not LC, by Corollary 6.2.7: So a 3-ball with a knotted spanning edge cannot be LC if the knot is the sum of two or more trefoil knots. (See Section 4.3.2). What about balls with a spanning edge realizing a single trefoil knot?

Proposition 6.3.2. *An LC 3-ball without interior vertices does not contain any knotted spanning edge.*

Proof. By Lemma 6.3.1, an LC 3-ball B without interior vertices is obtained from a tree of polytopes via local gluings of type (A). A tree of polytopes has no interior edge. Each type (A) step preserves the existing spanning edges (because it does not sink any vertex into the interior) and creates one more spanning edge e, clearly unknotted (because the other $k - 1$ edges of the sunk k-gon form a boundary path that "closes up" the edge e onto a 1-sphere bounding a disc inside B). It is easy to verify that the subsequent local gluings of type (A) leave such edge e spanning and unknotted. $\square$

The presence of knots or knotted spanning edges is not the only obstruction to local constructibility:

Example 6.3.3. Bing's thickened house with two rooms [21, pp. 108-109] is a cubical 3-ball B with all vertices on the boundary, so that each cube "touches air in two components". In particular, every interior square of B has at most two edges on the boundary ∂B. Were B LC, every step in its local construction would be of type (A) (by Lemma 6.3.1); in particular, the last square to be sunk into the interior of B would have exactly three edges on the boundary of B, a contradiction. Thus Bing's thickened house with two rooms cannot be LC, even if it does not contain a knotted spanning

edge. The same results hold also for the triangulated version [21, p. 111] [64] of the same 3-ball. Bing's question [21, p. 111] whether such triangulation is collapsible or not remains open, even if, in virtue of Corollary 6.2.7, by showing that it cannot be LC we made a step in the direction of non-collapsibility.

Example 6.3.4. Furch's 3-ball [57, p. 73] [21, p. 110] can be realized without interior vertices (see e.g. Hachimori [64]). Since it contains a knotted spanning edge, by Proposition 6.3.2 Furch's ball is not LC. Bing's question [21, p. 111] whether Furch's ball is collapsible or not remains open.

Remark 6.3.5. In [65, Lemma 2], Hachimori claimed that any simplicial 3-ball C obtained from a simplicial constructible 3-ball C' via a type (A) step is constructible. This would imply that all LC simplicial 3-balls without interior vertices are constructible, which is stronger than Proposition 6.3.2 since constructible 3-balls do not contain any knotted spanning edge.

Unfortunately, Hachimori's proof [65, p. 227] is not satisfactory: If $C' = C'_1 \cup C'_2$ is a constructible decomposition of C', and C_i is the subcomplex of C with the same facets of C'_i, $C = C_1 \cup C_2$ need not be a constructible decomposition for C. (For example, if the two glued triangles both lie on $\partial C'_1$, and if the two vertices that the triangles do not have in common lie in $C'_1 \cap C'_2$, then $C_1 \cap C_2$ is not a 2-ball and one of the C_i's is not a 3-ball.)

At present we do not know whether Hachimori's claim is true or not: Does C' admit a different constructible decomposition that survives the local gluing of type (A)? On this depends the correctness of the algorithm [65, p. 227][66, p. 101] to test *constructibility* of 3-balls without interior vertices by cutting them open along interior k-gons with exactly $k - 1$ boundary edges.

However, we point out that Hachimori's algorithm can be validly used to decide the *local constructibility* of 3-balls without interior vertices: In fact, by Lemma 6.3.1, the algorithm proceeds by reversing the LC-construction of the ball.

We can now move on to complete the proof of our Theorem 6.0.1. Inspired by Proposition 6.3.2, we show that a *collapsible* 3-ball without interior vertices may contain a knotted spanning edge. Our construction is a tricky version of Lickorish–Martin's [96].

Theorem 6.3.6. *Not all collapsible* 3-*balls are LC.*

Proof. Start with a large $m \times m \times 1$ pile of cubes, triangulated in the standard way, and take away two distant cubes, leaving only their bottom squares X and Y. The 3-complex C obtained can be collapsed vertically

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onto its square basis; in particular, it is collapsible and it has no interior vertices.

Let C' be a 3-ball with two tubular holes drilled away, but where (1) each hole has been corked at a bottom with a 2-disk, and (2) the tubes are disjoint but intertwined, so that a closed path that passes through both holes and between these traverses the top resp. bottom face of C' yields a trefoil knot (see Figure 6.2).
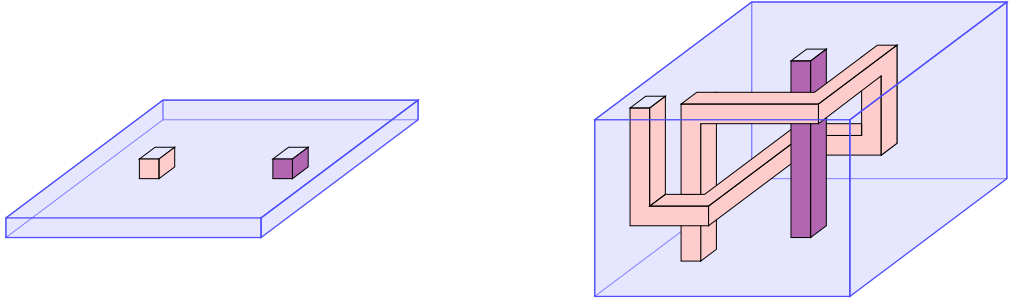

Figure 6.2: C and C' are obtained from a 3-ball drilling away two tubular holes, and then "corking" the holes on the bottom with 2-dimensional membranes.

C and C' are homeomorphic. Any homeomorphism induces on C' a collapsible triangulation with no interior vertices. X and Y correspond via the homeomorphism to the corking membranes of C', which we will call correspondingly X' and Y'. To get from C' to a ball with a knotted spanning edge we will carry out two more steps:

(i) create a single edge $[x', y']$ that goes from X' to Y';
(ii) thicken the "bottom" of C' a bit, so that C' becomes a 3-ball and $[x', y']$ becomes an interior edge (even if its extremes are still on the boundary).

We perform both steps by adding cones over 2-disks to the complex. Such steps preserve collapsibility, but in general they produce interior vertices; thus we choose "specific" disks with few interior vertices.

(i) Provided m is large enough, one finds a "nice" strip $F_1, F_2, \ldots, F_k$ of triangles on the bottom of C', such that $F_1 \cup F_2 \cup \cdots \cup F_k$ is a disk without interior vertices, F_1 has a single vertex x' in the boundary of X', while F_k has a single vertex y' in the boundary of Y', and the whole strip intersects $X' \cup Y'$ only in x' and y'. Then we add a cone to C', setting

$$C_1 := C' \cup (y' * (F_1 \cup F_2 \cup \cdots \cup F_{k-1})).$$

(An explicit construction of this type is carried out in Hachimori–Ziegler [69, pp. 164-165].) Thus one obtains a collapsible 3-complex C_1 with no interior vertex, and with a direct edge from X' to Y'.

(ii) Let R be a 2-ball inside the boundary of C_1 that contains in its interior the 2-complex $X' \cup Y' \cup [x', y']$, and such that every interior vertex of R lies either in X' or in Y'. Take a new point z' and define $C_2 := C_1 \cup (z' * R)$.

As $z' * R$ collapses onto R, C_2 is a collapsible 3-ball with a knotted spanning edge $[x', y']$. By Proposition 6.3.2, C_2 is not LC. $\qquad\square$

Corollary 6.3.7. *There exists a collapsible 3-ball B such that, for any boundary facet σ, the ball B does not collapse onto $\partial B - \sigma$.*

Corollary 6.3.8. *For each positive integer m, there exists a collapsible non-LC 3-ball B_m with an $\mathfrak{L}$-knotted spanning arc of m edges, the knot being the m-ple trefoil.*

Proof. It suffices to "sum" m copies of the 3-ball described in Theorem 6.3.6 according to Theorem 4.3.2: by Remark 4.3.4, the result is a collapsible 3-ball B_m without interior vertices. Since it contains a knot, B_m is not LC by Proposition 6.3.2. $\qquad\square$

Theorem 6.3.6 can be extended to higher dimensions by taking cones:

Corollary 6.3.9. *For every $d \geq 3$, not all collapsible d-balls are LC.*

Proof. All cones are collapsible by Proposition 3.4.1. If B is a non-LC d-ball, then $v * B$ is a non-LC $(d+1)$-ball by Proposition 1.7.2. $\qquad\square$

Furthermore, Chillingworth's theorem ("every geometric triangulation of a convex 3-dimensional polytope is collapsible") can be strengthened as follows.

Theorem 6.3.10 (Chillingworth [40]). *Every 3-ball embeddable as a convex subset of the Euclidean 3-space $\mathbb{R}^3$ is LC.*

Proof. The argument of Chillingworth for collapsibility runs showing that $B \searrow \partial B - \sigma$, where σ is any triangle in the boundary of B. A glance at Theorem 6.2.6 ends the proof. $\qquad\square$

Thus any subdivided 3-simplex is LC. If Hachimori's claim is true (see Remark 6.3.5), then any subdivided 3-simplex with all vertices on the boundary is also constructible. (So far we can only exclude the presence of knotted spanning edges in it: See Lemma 6.3.1.) However, a subdivided

3-simplex might be non-shellable even if it has all vertices on the boundary (Rudin's ball is an example).

Recently Crowley [44] has shown via discrete Morse theory that every non-positively curved complex is collapsible. Her results represent a "metric analogous" of Chillingworth's theorem; it would be interesting to know whether non-positively curved complexes need to be LC or not.

6.4 A hierarchy for knotted balls

We conclude the study of manifolds with boundary claiming an analogue for 3-balls of Theorem 5.3.12. We start off by saying that any 3-ball with a non-trivial knot cannot be rectilinearly embedded in $\mathbb{R}^3$.

Theorem 6.4.1. *A simplicial* 3-*ball with a non-trivial knot consisting of*

3 *edges,* 1-*complicated*	*is not constructible, but can be LC.*
3 *edges,* 2-*complicated*	*is not constructible, but can be collapsible.*
3 *edges,* 3-*complicated or more*	*is not LC and also not collapsible.*
4 *edges,* 1-*complicated*	*is not vertex dec., but can be shellable.*
4 *edges,* 2- *or* 3-*complicated*	*is not constructible, but can be collapsible.*
4 *edges,* 4-*complicated or more*	*is not LC and also not collapsible.*
5 *edges,* 1-*complicated*	*is not vertex dec., but can be shellable.*
5 *edges,* 2-, 3- *or* 4-*complicated*	*is not constructible, but can be collapsible.*
5 *edges,* 5-*complicated or more*	*is not LC and also not collapsible.*
6 *edges,* 1-*complicated*	*can be vertex dec.*
6 *edges,* 2-*complicated*	*is not vertex dec., but can be collapsible.*
6 *edges,* 3-, 4- *or* 5-*complicated*	*is not constructible, but can be collapsible.*
6 *edges,* 6-*complicated or more*	*is not LC and also not collapsible.*

$$\vdots \qquad \qquad \vdots$$

m *edges,* k-*complicated, with* $k \geq \lfloor \frac{m}{3} \rfloor$	*is not vertex decomposable.*
m *edges,* k-*complicated, with* $k \geq \lfloor \frac{m}{2} \rfloor$	*is not constructible.*
m *edges,* k-*complicated, with* $k \leq m - 1$	*can be collapsible.*
m *edges,* k-*complicated, with* $k \geq m$	*cannot be collapsible.*

Proof. Each line follows from the corresponding line of Theorem 5.3.12.

Since a 3-sphere S with a 1-complicated (or 2-complicated) knotted triangle can be LC by Theorem 5.3.12, it follows that for any facet Δ of S the 3-ball $S - \Delta$ is collapsible. (If Δ is chosen carefully, one might be able to conclude that $S - \Delta$ is LC: Compare Theorem 6.2.4.) Yet $S - \Delta$ contains the same knot of S. Furthermore, if B is a knotted collapsible 3-ball, then S_B is a knotted LC 3-sphere; yet a 3-sphere with an m-complicated knotted

m-gon cannot be LC; therefore, a 3-ball with an m-complicated knotted m-gon cannot be collapsible.

The remaining items are shown in analogous way, either applying Corollary 5.3.1 or using the fact that coning off the boundary of a ball preserves constructibility and vertex decomposability. $\square$

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