SYMMETRIC TENSORS AND THE GEOMETRY OF SUBVARIETIES OF $\mathbb{P}^N$.

Fedor Bogomolov* Bruno De Oliveira**

* Courant Institute for Mathematical Sciences **University of Miami

0. Introduction

In this article we prove vanishing and nonvanishing results about the space of twisted symmetric differentials of subvarieties $X \subset \mathbb{P}^N$, $H^0(X, S^m\Omega^1_X \otimes \mathcal{O}_X(k))$ with $k \leq m$ (via a geometric approach). Emphasis is given to the case of $k = m$ which is special and whose nonvanishing results on the dimensional range $\dim X > 2/3(N-1)$ are related to the space of quadrics containing X and lead to interesting geometrical objects associated to X, as for example the variety of all tangent trisecant lines of X. The same techniques give results on the symmetric differentials of subvarieties of abelian varieties. The paper ends with an application concerning the jumping of the twisted symmetric plurigenera, $Q_{\alpha,m}(X_t) = \dim H^0(X, S^m(\Omega^1_{X_t} \otimes \alpha K_{X_t}))$ along smooth families of projective varieties X_t. In particular, we show that even for α arbitrarily large the invariance of the twisted symmetric plurigenera, $Q_{\alpha,m}(X_t)$ does not hold.

P. Bruckman showed in [Br71] that there are no symmetric differentials on smooth hypersurfaces in $\mathbb{P}^N$, $N \geq 3$, via an explicit constructive approach. Later, F. Sakai with a cohomological approach using a vanishing theorem of Kobayashi and Ochai showed that a complete intersection $Y \subset \mathbb{P}^N$ with dimension $n > N/2$ has no symmetric differentials [Sa78]. In the early nineties M. Schneider [Sc92] using a similar approach, but with more general vanishing theorems of le Potier, showed that any submanifold $X \subset \mathbb{P}^N$ of dimension $n > N/2$ has no symmetric differentials of order m even if twisted by $\mathcal{O}_X(k)$, $H^0(X, S^m\Omega^1_X \otimes \mathcal{O}_X(k)) = 0$, where $k < m$.

This paper, using a predominantly geometric approach, goes further along in in the study of the vanishing and nonvanishing of the space of twisted symmetric differentials (some parts of this approach can be traced back to an announcement in the ICM of 1978 by the first author, [Bo78]). Our method involves the structure and properties of the tangent map for X. The tangent map is given by $f : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to \mathbb{P}^N$, where

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$f(\mathbb{P}(\widetilde{\Omega^1_X}(1))_x) = T_xX$ and T_xX is the embedded projective tangent space to X at x in $\mathbb{P}^N$. As the first application of our approach, we show that if the tangent map for X has a positive dimensional general fiber, then X has no symmetric differentials of order m even if twisted by $\mathcal{O}_X(m) \otimes L$, where L is any negative line bundle on X, i.e. $H^0(X, S^m[\Omega^1_X(1)] \otimes L) = 0$ (this includes Schneider's result).

The main application of our method is to describe the spaces of sections of the symmetric powers of the sheaf of twisted differentials $S^m[\Omega^1_X(1)]$, i.e. $H^0(X, S^m\Omega^1_X \otimes \mathcal{O}_X(m))$. In other words, we are tackling the problem described in the previous paragraph for $L = \mathcal{O}_X$ or equivalently the case $k = m$ not reached by Schneider's results and methods. The loss of the negativity of L makes the study of the existence of twisted symmetric differentials more delicate. In the case $k < m$, in "low" codimensions one has $H^0(X, S^m\Omega^1_X \otimes \mathcal{O}_X(k)) = 0$ which follows just from the positivity of the dimension of the general fiber of the tangent map associated to $X \subset \mathbb{P}^N$ (e.g $\dim X > N/2$). On the other hand the case $k = m$ one has nonvanishing $H^0(X, S^m\Omega^1_X \otimes \mathcal{O}_X(m))$ and our method needs that the tangent map to be surjective and connected (the connectivity of the fibers is the delicate point). A key result of this paper is theorem 1.3 showing that the connectedness of the fibers of the tangent map is guaranteed if $dimX > 2/3(N-1)$.

The space of twisted symmetric differentials, $H^0(X, S^m[\Omega^1_X(1)])$, is connected to classical algebraic geometric objects associated to X as the trisecant variety of X and the space of quadrics containing X. We show that the elements of $\mathbb{P}(H^0(X, S^m[\Omega^1_X(1)]))$ are in a 1-1 relation with the hypersurfaces $H \subset \mathbb{P}^N$ satisfying $X \subset H$ and $H \cap T_xX$ is a cone with vertex x. This in particular imply that the hypersurfaces must contain the subvariety C_XX of the trisecant variety of X consisting of the closure of the union of all tangent trisecant lines of X which intersect X in at least two distinct points. The variety C_XX plays a key role as can be seen below.

We show that if $X \subset \mathbb{P}^N$ has codimension 1 or 2 and dimension $n \geq 3$ then C_XX is the intersection of the quadrics containing X and coincides with $Trisec(X)$. The space of twisted symmetric differentials when $n > 2/3(N-1)$ satisfies $H^0(X, S^m[\Omega^1_X(1)]) \neq 0$ if and only if $Trisec(X) \neq \mathbb{P}^N$ (X is contained in a quadric). It follows from the proof that the space of twisted symmetric differentials on X are related to the space of quadrics containing X. As an application of the circle ideas behind the proof of $C_XX = Trisec(X)$ we give an alternative proof of the Zak's theorem stating that $Tan(X) = Sec(X)$ if $Sec(X)$ does not have the expected dimension.

Associated with X one also has higher level tangent trisecant varieties C^k_XX, see section 1.2 for details, which play an important role in higher codimensions. We show that if $X \subset \mathbb{P}^N$ has dimension $n > 2/3(N-1)$, then $H^0(X, S^m[\Omega^1_X(1)]) \neq 0$ if and only if all C^k_XX varieties associated X are not $\mathbb{P}^N$ (this holds in particular if X is contained in a quadric).

At the end of this article we discuss the invariance of the dimension of the space of twisted symmetric differentials in smooth families of projective varieties. In particular, we answer the following question of M. Paun: is $\dim H^0(X_t, S^m\Omega^1_{X_t} \otimes K_{X_t})$ locally invariant in smooth families? This invariance would be the natural extension of the result of Y-T. Siu on the invariance of plurigenera [Si98] to other tensors. We show that the invariance does not hold. The answer follows from the results on nonexistence twisted symmetric differentials on smooth hypersurfaces described above plus our previous result [BoDeO06] giving families of smooth hypersurfaces of degree $d \geq 6$ in $\mathbb{P}^3$ specializing to smooth surfaces with many symmetric differentials (these surfaces are resolution of nodal hypersurfaces surface with sufficiently many nodes). We also give a construction of a family of surfaces with K_{X_t} big such that for all α $\dim H^0(X_t, S^m(\Omega^1_{X_t} \otimes \alpha K_{X_t}))$ is not constant in the family if m sufficiently large. The best one can hope for is that the ampleness of K_{X_t} might be sufficient to guarantee the invariance of the twisted symmetric plurigenera, $Q_{\alpha,m}(X_t)$.

1. Symmetric differentials on subvarieties of $\mathbb{P}^N$ and of Abelian varieties

1.1 Preliminaries.

Let E be a vector bundle on X and $\mathbb{P}(E)$ be the projective bundle of hyperplanes of E. Recall the connection between $S^m E$ and $\mathcal{O}_{\mathbb{P}(E)}(m)$ which plays a fundamental role in the study of symmetric powers of a vector bundle. If $\pi : \mathbb{P}(E) \to X$ is usual projection map then the following holds: $\pi_* \mathcal{O}_{\mathbb{P}(E)}(m) \cong S^m E$ and

$$H^0(X, S^m E) \cong H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) \tag{1.1.1}$$

The following case persistently appears in our arguments. Let E be a vector bundle on X which is a quotient of $\bigoplus^{N+1} L$ where L is a line bundle on X:

$$q : \bigoplus^{N+1} L \to E \to 0 \tag{1.1.2}$$

Let $\mathbb{P}(E)$ and $\mathbb{P}(\bigoplus^{N+1} L)$ be the projective bundles of hyperplanes of E and $\bigoplus^{N+1} L$ respectively. The surjection q induces an inclusion and the isomorphism:

$$i_q : \mathbb{P}(E) \hookrightarrow \mathbb{P}(\bigoplus^{N+1} L)$$

$$i_q^* \mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} L)}(1) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$$

Recall that there is a natural isomorphism $\phi : \mathbb{P}(\bigoplus^{N+1} L) \to \mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X)$ for which $\phi^* \mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} \mathcal{O})}(1) \cong \mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} L)}(1) \otimes \pi^* L^{-1}$. The projective bundle $\mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X)$ is the product $X \times \mathbb{P}^N$, if p_2 denotes the projection onto the second factor, then $\mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} \mathcal{O})}(1) \cong p_2^* \mathcal{O}_{\mathbb{P}^N}(1)$. Concluding, the surjection q in (2) naturally induces a map $f_q = p_2 \circ \phi \circ i_q$ and the isomorphism:

$$f_q : \mathbb{P}(E) \to \mathbb{P}^N$$

$$f_q^* \mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* L^{-1} \tag{1.1.3}$$

Hence

$$H^0(X, S^m E) \cong H^0(\mathbb{P}(E), f_q^* \mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^* L^{\otimes m}) \tag{1.1.4}$$

It follows from (1.1.4) that the properties of the map $f_q : \mathbb{P}(E) \to \mathbb{P}^N$ have an impact on the existence of sections of the symmetric powers of E. The next result gives an example of this phenomenon and will play a role in our study of existence of symmetric differentials.

Lemma 1.1. *Let E be a vector bundle on a smooth projective variety X. If E is the quotient of a trivial vector bundle:*

$$q : \bigoplus^{N+1} \mathcal{O}_X \to E \to 0$$

and the induced map $f_q : \mathbb{P}(E) \to \mathbb{P}^N$ is surjective with connected fibers, then q induces the isomorphism:

$$H^0(X, S^m E) = H^0(X, S^m(\bigoplus^{N+1} \mathcal{O}_X))$$

($H^0(X, S^m E) = S^m[\mathbb{C}s_0 \oplus ... \oplus \mathbb{C}s_N]$ where $s_i = q(e_i)$, $\bigoplus^{N+1} \mathcal{O}_X = \bigoplus_{i=0}^N \mathcal{O}_X e_i$).

Proof. The isomorphism $f_q^* \mathcal{O}_{\mathbb{P}^N}(m) \cong \mathcal{O}_{\mathbb{P}(E)}(m)$, (1.1.3), and $H^0(X, S^m E) \cong H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m))$ give that:

$$H^0(X, S^m E) \cong H^0(\mathbb{P}(E), f_q^* \mathcal{O}_{\mathbb{P}^N}(m))$$

The next step is to relate $H^0(\mathbb{P}(E), f_q^* \mathcal{O}_{\mathbb{P}^N}(m))$ with $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$. If f_q is surjective then $f_q^* : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(\mathbb{P}(E), f_q^* \mathcal{O}_{\mathbb{P}^N}(m))$ is injective. If the map f_q also has connected fibers, then all sections in $H^0(\mathbb{P}(E), f_q^* \mathcal{O}_{\mathbb{P}^N}(m))$ descend to be sections in $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$, and the following holds:

$$H^0(\mathbb{P}(E), f_q^* \mathcal{O}_{\mathbb{P}^N}(m)) \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$$

The result then follows from the brake down of the map f_q, $f_q = p_2 \circ i_q$, plus $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \cong H^0(\mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X), p_2^* \mathcal{O}_{\mathbb{P}^N}(m)) \cong H^0(\mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X), \mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X)}(m))$ and $H^0(\mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X), \mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X)}(m)) \cong H^0(X, S^m(\bigoplus^{N+1} \mathcal{O}_X))$. □

1.2 Symmetric differentials on subvarieties of $\mathbb{P}^N$ and abelian varieties.

The following is short collection of facts about the sheaf of differentials that will help the reader understand our approach. The Euler sequence of $\mathbb{P}^N$ is:

$$0 \to \Omega^1_{\mathbb{P}^N} \to \bigoplus^{N+1} \mathcal{O}(-1) \to \mathcal{O} \to 0 \tag{1.2.1}$$

The Euler sequence expresses the relation, induced by the natural projection $p : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$, between the differentials of $\mathbb{C}^{N+1}$ and $\mathbb{P}^N$. A necessary condition for a differential ω of $\mathbb{C}^{N+1}$ to come from a differential of $\mathbb{P}^N$ is that the coefficients $h_0(z)$,...,$h_N(z)$ of $\omega = h_0(z)dz_0 + ... + h_N(z)dz_N$ must be homogeneous of degree -1. But the last condition is not sufficient, the differentials ω on $\mathbb{C}^{N+1}$ must be such that at any point $z \in \mathbb{C}^{N+1}$ their contraction with the vector $z_0\partial/\partial z_0 + ... + z_N\partial/\partial z_N$, i.e with the direction of the line from z to the origin, must be zero. To see this algebraically, the sheaf $\bigoplus^{N+1} \mathcal{O}(-1)$ in (1.2.1) is $\bigoplus^{N+1} \mathcal{O}(-1) = \mathcal{O}(-1)dz_0 + ... + \mathcal{O}(-1)dz_N$. The map $q : \bigoplus^{N+1} \mathcal{O}(-1) \to \mathcal{O} \to 0$ is defined sending dz_i to z_i. So locally, let us say on $U_i = \{z_i \neq 0\}$, $\Omega^1_{U_i}$ the kernel of the map q is spanned by the sections induced by the differentials $\frac{1}{z_i}dz_j - \frac{z_j}{z_i^2}dz_i$ on $p^{-1}(U_i)$.

The sheaf of differentials Ω^1_X is determined by (1.2.1) restricted to X:

$$0 \to \Omega^1_{\mathbb{P}^N}|_X \to \bigoplus^{N+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0 \tag{1.2.2}$$

and the conormal bundle exact sequence:

$$0 \to N^* \to \Omega^1_{\mathbb{P}^N}|_X \to \Omega^1_X \to 0 \tag{1.2.3}$$

The extension defined by (1.2.2) (which corresponds to a cocycle $\alpha \in H^1(X, \Omega^1_{\mathbb{P}^N}|_X)$) induces via the surjection in (1.2.3) the extension:

$$0 \to \Omega^1_X \to \widetilde{\Omega^1_X} \to \mathcal{O}_X \to 0 \tag{1.2.4}$$

The geometric description of the sheaf $\widetilde{\Omega^1_X}$ is that it is the sheaf on X associated to the sheaf of 1-forms on the affine cone $\hat{X} \subset \mathbb{C}^{N+1}$. The above exact sequences after twisted by $\mathcal{O}_X(1)$ fit in the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{\mathbb{P}^N}|_X(1) \arrow[r] \arrow[d] & \bigoplus^{N+1} \mathcal{O}_X \arrow[r] \arrow[d, "q"'] & \mathcal{O}_X(1) \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & \Omega^1_X(1) \arrow[r] \arrow[d] & \widetilde{\Omega^1_X}(1) \arrow[r] \arrow[d] & \mathcal{O}_X(1) \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd} \tag{1.2.5}$$

The middle vertical surjection of diagram (1.2.5) can be represented more explicitly by:

$$q : \bigoplus_{i=0}^{N} \mathcal{O}_X dz_i \to \widetilde{\Omega^1_X}(1) \tag{1.2.6}$$

The induced map $f : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to \mathbb{P}^N$ is such that for each $x \in X$:

$$f(\mathbb{P}(\widetilde{\Omega^1_X}(1))_x) = T_x X \tag{1.2.7}$$

where $T_x X$ is the embedded projective tangent space to X at x inside $\mathbb{P}^N$. For the obvious reasons f will be called the tangent map for X. The tangent map f induces a map from X to $G(n, N)$ which is exactly the Gauss map for X, $\gamma_X : X \to G(n, N)$.

Theorem A. *Let X be a smooth projective subvariety of $\mathbb{P}^N$. If the general fiber of the tangent map for X, $f : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to \mathbb{P}^N$, is positive dimensional, then $\forall m \geq 0$:*

$$H^0(X, S^m[\Omega^1_X(1)] \otimes L) = 0$$

if L is a negative line bundle on X.

Proof. It is sufficient to show that $H^0(X, S^m[\widetilde{\Omega^1_X}(1)] \otimes L) = 0$, since there is the inclusion $S^m[\Omega^1_X(1)] \hookrightarrow S^m[\widetilde{\Omega^1_X}(1)]$, induced from (1.2.5).

The projective bundle $\mathbb{P}(\widetilde{\Omega^1_X}(1))$ comes with two maps. The tangent map for X, $f : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to \mathbb{P}^N$, and the projection onto X, $\pi : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to X$. One also has the natural isomorphisms $\mathcal{O}_{\mathbb{P}(\widetilde{\Omega^1_X}(1))}(m) = f^*\mathcal{O}_{\mathbb{P}^N}(m)$ and $\pi_*(\mathcal{O}_{\mathbb{P}(\widetilde{\Omega^1_X}(1))}(m) \otimes \pi^* L) \cong S^m[\widetilde{\Omega^1_X}(1)] \otimes L$. These isomorphisms give:

$$H^0(X, S^m[\widetilde{\Omega^1_X}(1)] \otimes L) \cong H^0(\mathbb{P}(\widetilde{\Omega^1_X}(1)), f^*\mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^* L)$$

The vanishing of the last group follows from the negativity of the line bundle $f^*\mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^* L$ along each fiber of the map f. More precisely, $f^*\mathcal{O}_{\mathbb{P}^N}(m)$ is trivial on the fibers and $\pi^* L$ is negative on the fibers since L is negative on X the map π is injective on each fiber of f.

We need the fibers of the map f to be positive dimensional. Since is only in this case that the negativity of the line bundle $f^*\mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^* L$, $l < 0$, makes sense. This negativity implies that all sections of $H^0(\mathbb{P}(\widetilde{\Omega^1_X}(1)), f^*\mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^* L)$ vanish along all fibers of f and hence vanish on all $\mathbb{P}(\widetilde{\Omega^1_X}(1))$, which completes the proof. $\square$

As an important case of theorem A one has another proof to the result first proved by Schneider [Sc92].

Corollary 1.2. *Let X be a smooth projective subvariety of $\mathbb{P}^N$ whose dimension $n > N/2$. Then:*

$$H^0(X, S^m\Omega^1_X \otimes \mathcal{O}(k)) = 0$$

if $k < m$.

Proof. The dimensional hypothesis $n > N/2$ guarantee that all fibers of the tangent map f for X are positive dimensional. The condition $k < m$ gives that $S^m\Omega^1_X \otimes \mathcal{O}(k) = S^m[\Omega^1_X(1)] \otimes \mathcal{O}_X(l)$, with $l < 0$. The theorem then follows from theorem A for the negative line bundle $L = \mathcal{O}_X(l)$, $l < 0$. □

What happens in the key case $k = m$? The results just mentioned use the negativity $f^*\mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^*L$, along the fibers of the map f, which no longer holds if $k = m$. Indeed, one has $H^0(X, S^m\widetilde{\Omega^1_X} \otimes \mathcal{O}_X(m)) = H^0(\mathbb{P}(\widetilde{\Omega^1_X}(1)), f^*\mathcal{O}_{\mathbb{P}^N}(m))$ which is no longer trivial. The analysis of the nonexistence of twisted symmetric differentials $\omega \in H^0(X, S^m[\Omega^1_X(1)])$ on X is more delicate. One has to describe the sections $H^0(X, S^m\widetilde{\Omega^1_X} \otimes \mathcal{O}_X(m))$ and characterize which ones are in $H^0(X, S^m\Omega^1_X \otimes \mathcal{O}_X(m))$. The answers will depend on geometric properties involving the variety of tangent lines to the subvariety X.

To describe the twisted symmetric extended differentials in $H^0(X, S^m[\widetilde{\Omega^1_X}(1)])$ one needs to use the properties of the tangent map for $X \subset \mathbb{P}^N$. The lemma 1.1 gives a good description of $H^0(X, S^m[\widetilde{\Omega^1_X}(1)])$ if the tangent map f is a connected surjection. The next lemma shows that this is the case when $\dim X > 2/3(N-1)$.

This paragraph about the tautological $\mathbb{P}^1$-bundle over the grassmanian is used in the lemma below. Let $q : I \to G(1, \mathbb{P}^n)$ be the tautological $\mathbb{P}^1$-bundle over the grassmanian and $p : I \to \mathbb{P}^n$ the natural map. For any point $x \in \mathbb{P}^n$ there is a $\mathbb{P}^{n-1} \subset G(1, \mathbb{P}^n)$ consisting of all lines passing through x. The restriction (or the pullback) of the tautological $\mathbb{P}^1$-bundle to $\mathbb{P}^{n-1}$ is $q : \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}) \to \mathbb{P}^{n-1}$.

Theorem 1.3. *Let X be a n-dimensional submanifold of $\mathbb{P}^N$ with $n > 2/3(N-1)$ then the natural map $f : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to \mathbb{P}^N$ associated with the Gauss map is a surjective and connected morphism.*

Proof. The description of the map f in (1.2.7) implies that:

$$f(\mathbb{P}(\widetilde{\Omega^1_X}(1))) = Tan(X)$$

where $Tan(X)$ is the tangent variety of X, in other words $Tan(X) = \bigcup_{x \in X} T_xX \subset \mathbb{P}^N$. Denote by $Sec(X) \subset \mathbb{P}^N$ the secant variety of X. Zak's results [Za81] about tangencies state that one of the following must hold: i) $\dim Tan(X) = 2n$ and $\dim Sec(X) = 2n+1$; ii) $Tan(X) = Sec(X)$. It follows immediately that if $\dim X \geq N/2$ then $Tan(X) = Sec(X)$.

It is also a result of Zak, coming from applying the results on tangencies plus the Terracini's lemma on the tangent spaces of secant varieties, that $Sec(X) \neq \mathbb{P}^N$ can hold only if $n \leq 2/3(N-2)$. Hence surjectivity of f is guaranteed if $n > 2/3(N-2)$, which is the case.

It remains to show the connectedness of the fibers. Denote the fibers of f by $Y_x = f^{-1}(x)$ for $x \in \mathbb{P}^N$ and $\pi : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to X$ be the projection map. The injectivity of f restricted to the fibers of π implies that Y_x is connected if and only if $\pi(Y_x)$ is connected. The subvariety $R_x = \pi(Y_x)$ is the locus of X consisting of all the points in X having a tangent line passing through x. The Stein factorization implies that f is connected if its general fiber is connected, i.e. if for the general $x \in \mathbb{P}^N$ the locus R_x is connected.

In the following arguments we always assume that $x \in \mathbb{P}^N$ is general. The first observation to make is that $R_x \subset Z_x$, where Z_x is the double locus of the projection $p_x : X \to \mathbb{P}^{N-1}$ (i.e. the locus of points in X belonging to lines passing through x and meeting X at least twice). By dimensional arguments one has that R_x is a Weyl divisor of Z_x. A key element in our argument is the result of [RaLo03] stating that the double locus Z_x is irreducible if $n > 2/3(N-1)$.

Let $S \subset \mathbb{P}^{N-1}$ be the image of Z_x by the projection p_x. The irreducible variety S can be seen as a subvariety of the $\mathbb{P}^{N-1} \subset G(1, \mathbb{P}^N)$ of lines passing through x. We can pullback the tautological $\mathbb{P}^1$-bundle on $G(1, \mathbb{P}^N)$ to S and obtain $q_S : \mathbb{P}(\mathcal{O}(1)|_S \oplus \mathcal{O}) \to S$. The natural map $p : I \to \mathbb{P}^N$, see the paragraph before the lemma, induces a map $p : \mathbb{P}(\mathcal{O}(1)|_S \oplus \mathcal{O}) \to \mathbb{P}^N$, whose image is the cone with vertex x and base S. The map p is a biregular morphism of the complement of $p^{-1}(x)$ onto the cone without the vertex.

The $\mathbb{P}^1$-bundle $q_S : \mathbb{P}(\mathcal{O}(1)|_S \oplus \mathcal{O}) \to S$ comes with two natural sections (one for each surjection onto the line bundles $\mathcal{O}$ and $\mathcal{O}(1)$). Geometrically these two sections come from the pre-image of x and the pre-image of S via the map p. The subvariety $M = p^{-1}(Z_x)$ is biregular to Z_x and is a divisor in the total space of the line bundle $\mathcal{O}_S(1)$, $\mathbb{P}(\mathcal{O}_S(1) \oplus \mathcal{O}) \setminus \mathbb{P}(\mathcal{O})$. The points in $p^{-1}(R_x)$ are the points $y \in M$ for which the fibers of q_S meet M at y with multiplicity ≥ 2. The generality of x implies by the classical trisecant lemma that the general fiber of q_S meets M only twice counting with multiplicity. This makes the projection $q_S|_M : M \to S$ a generically 2 to 1 map.

Consider the pullback $L = q_S|_M^* \mathcal{O}_S(1)$ which is an ample line bundle on M. The line bundle L comes naturally with a nontrivial section denote the corresponding divisor of the total space of L, $Tot(L)$, by D_1. Denote the natural map between the total spaces of L and $\mathcal{O}_S(1)$ by $g : Tot(L) \to Tot(\mathcal{O}_S(1))$. The divisorial component of $g^{-1}(M)$ is decomposed in two irreducible components D_1 and D_2. Let $h : Tot(L) \to M$ be the natural projection, then $h(D_1 \cap D_2) \subset R_x$. If D_2 is also a section of L, then $h(D_1 \cap D_2)$ is connected since it is the zero locus $(s)_0$ of a section s of the ample line bundle L. The result would follow since any other possible component of R_x has to meet $(s)_0$. If D_2 is not a section the result still follows from the same argument after base change (pulling back L to D_2 using h).

$\square$

In conjunction with lemma 1.1 one obtains the following description of the space of twisted extended symmetric differentials on X:

Corollary 1.4. *Let X be a n-dimensional submanifold of $\mathbb{P}^N$ with $n > 2/3(N-1)$ then:*

$$H^0(X, S^m[\widetilde{\Omega^1_X}(1)]) = S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N].$$

The characterization of the subset $H^0(X, S^m[\Omega^1_X(1)]) \subset H^0(X, S^m[\widetilde{\Omega^1_X}(1)])$, within the dimensional range $\dim X > 2/3(N-1)$, is given by the following proposition:

Proposition 1.5. *Let X be a n-dimensional submanifold of $\mathbb{P}^N$ with $n > 2/3(N-1)$ then:*

$$H^0(X, S^m[\Omega^1_X(1)]) = \{\Omega \in S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N] |\ Z(\Omega) \cap T_xX \textit{ is a cone with vertex at } x,\ \forall x \in X\}$$

Proof. The inclusion $H^0(X, S^m[\Omega^1_X(1)]) \subset H^0(X, S^m[\widetilde{\Omega^1_X}(1)])$ and corollary 1.4 imply that all the sections of $H^0(X, S^m[\Omega^1_X(1)])$ are induced from the symmetric m-differentials $S^m[\mathbb{C}dz_0 + ... + \mathbb{C}dz_N]$ on $\mathbb{C}^{N+1}$.

Let $\hat{X} \subset \mathbb{C}^{N+1}$ be the affine cone over $X \subset \mathbb{P}^N$, $T\hat{X}$ the sheaf on X associated with the tangent bundle of $\hat{X}$ and $T_xX \subset \mathbb{P}^N$ the embedded tangent space to X at x. Consider the rational map $p : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \dashrightarrow \mathbb{P}(\Omega^1_X(1))$, which is fiberwise geometrically described by the projections from the point $x \in T_xX$ $p_x : T_xX = \mathbb{P}_l(T\hat{X})_x \dashrightarrow \mathbb{P}_l(TX)_x$, ($\mathbb{P}_l(E)$ is the projective bundle of lines in the vector bundle E, $\mathbb{P}_l(E) = \mathbb{P}(E^*)$). The map p gives an explicit inclusion $H^0(X, S^m[\Omega^1_X(1)]) = p^*H^0(\mathbb{P}(\Omega^1_X(1)), \mathcal{O}_{\mathbb{P}(\Omega^1_X(1))}(m)) \subset H^0(\mathbb{P}(\widetilde{\Omega^1_X}(1)), \mathcal{O}_{\mathbb{P}(\widetilde{\Omega^1_X}(1))}(m)) = S^m[\mathbb{C}dz_0 + ... + \mathbb{C}dz_N]$.

Recall that if $0 \to V \to \widetilde{V} \to \mathbb{C} \to 0$ is a sequence of vector spaces, then one gets a projection from $[V] \in \mathbb{P}(\widetilde{V})$, $p : \mathbb{P}(\widetilde{V}) \dashrightarrow \mathbb{P}(V)$. The sections in $H^0(\mathbb{P}(\widetilde{V}), \mathcal{O}_{\mathbb{P}(\tilde{V})}(m))$ which are in $p^*H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(m))$ are the ones corresponding to homogeneous polynomials whose zero locus is a cone with vertex at $[V]$.

An element $\Omega \in S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N]$ corresponds in a natural way to an homogeneous polynomial in $\mathbb{P}^N$ which we still denote by Ω. From the last two paragraphs it follows that Ω induces an element in $\omega \in H^0(X, S^m[\Omega^1_X(1)])$ if and only if $\forall x \in X$ the zero locus $Z(\Omega) \cap T_xX$ is a cone with vertex x. $\square$

We proceed to extract from proposition 1.5 the geometric conditions required for the existence of twisted symmetric differentials on smooth subvarieties $X \subset \mathbb{P}^N$. First, we need to introduce some objects and notation.

Let X be an irreducible subvariety and Y be any subvariety of $\mathbb{P}^N$. Consider the incidence relation:

$$\mathcal{C}_X Y := \overline{\{(x,z) \in X_{sm} \times \mathbb{P}^N \mid z \in \overline{xy},\ y \neq x \text{ and } y \in Y \cap T_x X\}} \subset X \times \mathbb{P}^N$$

where X_{sm} denotes the smooth locus of X. The variety $\mathcal{C}_X Y$ comes with two projections. Denote by $C_X Y := p_2(\mathcal{C}_X Y)$. Equivalently, $\forall x \in X_{sm}$ denote by $C_x Y \subset T_x X$ the cone with vertex at x consisting of the closure of the union of all chords joining x to $y \neq x$ with $y \in Y \cap T_x X$, where $T_x X$ is the projective embedded tangent space to X at x. Then $C_X Y = \overline{\bigcup_{x \in X_{sm}} C_x Y} \subset \mathbb{P}^N$.

Definition 1.6. *Let X be an irreducible subvariety of $\mathbb{P}^N$. The union of all tangent trisecant lines to X is called the tangent trisecant variety of X and denoted by $Trisec^t(X)$. The subvariety $C_X X \subset \mathbb{P}^N$, described above, lies inside $Trisec^t(X)$ and is called the tangent 2-trisecant variety of X.*

We can rewrite proposition 1.5 in a form that will be useful later:

Proposition 1.5'. *Let X be a n-dimensional submanifold of $\mathbb{P}^N$ with $n > 2/3(N-1)$ then:*

$$\mathbb{P}(H^0(X, S^m[\Omega^1_X(1)])) = \{H \subset \mathbb{P}^N \mid \, dim H_{\text{red}} = N-1, C_X H_{\text{red}} = H_{\text{red}}\}$$

Proof. Let H be an hypersurface in $\mathbb{P}^N$ (possibly non-reduced). Proposition 1.5 states that $H = Z(\omega)$ with $\omega \in H^0(X, S^m[\Omega^1_X(1)]) \subset S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N]$ if and only if $C_x H_{\text{red}} = T_x X \cap H_{\text{red}}$ $\forall x \in X$ (where $H = Z(\omega)$ follows the notation of proposition 1.5). Since $Tan(X) = \mathbb{P}^N$, $C_x H_{\text{red}} = T_x X \cap H_{\text{red}}$ $\forall x \in X$ is the same as $C_X H_{\text{red}} = H_{\text{red}}$.

□

For hypersurfaces H, including singular, one has the following useful result.

Proposition 1.7. *Let $Y \subset \mathbb{P}^N$, $N > 2$, be a nondegenerate irreducible and reduced hypersurface. Then $C_Y Y$ is either $\mathbb{P}^N$ or Y and in this last case Y is a quadric.*

Proof. Let Y be a quadric and $x \in Y$ a smooth point. The lines $l \subset C_x Y$ passing through x must touch Y at least 3 times (counting with multiplicity) hence $l \subset Y$. This implies that $C_x Y \subset Y$, for all $x \in Y$, therefore $C_Y Y = Y$.

Let Y be of degree greater than 2. The result follows from the trivial case of curves in $\mathbb{P}^2$. For a general 2-plane L in $\mathbb{P}^N$ the intersection $Y \cap L = D$ is an irreducible and reduced curve of the same degree as Y. The irreducible and reduced curve $D = Y \cap L$ of degree ≥ 3 in $L = \mathbb{P}^2$ satisfies $C_D D = L$ (clear but read the remark below). The result follows since the following inclusion holds $C_{(Y \cap L)}(Y \cap L) \subset (C_Y Y) \cap L$ and hence $C_Y Y$ contains the general 2-plane.

□

Remark. *Let $x \in Y$ be a general point of an hypersurface in $\mathbb{P}^N$ and $\mathbb{C}^N \subset P^N$ be an affine chart containing x, where w.l.o.g. $x = 0$. Let the hypersurface $Y \cap \mathbb{C}^N$ be given by $f = 0$. The quadratic part f_2 of the Taylor expansion of f at x can not be trivial on T_xY, otherwise there would be an open subset of Y on which the second fundamental form of Y is trivial which would force Y to be an hyperplane. Denote by Q_x the quadric defined by $f_2|_{T_xH}$. In the case $Y \subset \mathbb{P}^2$ of degree $d \geq 3$, then Q_x not being trivial implies that tangent line $l = T_xY$ is such that $(l \cap H)_x = 2$ and thus l must meet Y away from x. Hence $C_Y Y = Tan(Y)$.*

Combining the last two results one obtains a description of the space of twisted symmetric differentials on smooth hypersurfaces in $\mathbb{P}^N$.

Theorem B. *Let X be a smooth projective hypersurface in $\mathbb{P}^N$. Then:*

$$H^0(X, S^m[\Omega^1_X(1)]) = 0$$

if and only if the $C_X X = \mathbb{P}^N$, i.e. X is a not quadric.

Proof. By the proposition 1.5' $H^0(X, S^m[\Omega^1_X(1)]) \neq 0$ iff there is an hypersurface H such that $C_X H = H$. One notes that if $C_X H = H$ then $X \subset H$ and hence $C_X X \subset C_X H$. The proposition 1.7 states that there are two possibilities for $C_X X$ either $C_X X = \mathbb{P}^N$ (X is not a quadric) and hence there are no hypersurfaces H with $C_X H = H$; or $C_X X = X$ and hence all the multiples H of X are hypersurfaces with $C_X H_{\text{red}} = H_{\text{red}}$ ($H_{\text{red}} = X$ and X is a quadric). $\square$

Remark. It follows from the arguments given above that for a smooth quadric $Q \in \mathbb{P}^N$, $H^0(Q, S^m[\Omega^1_Q(1)]) = \mathbb{C}$ if m is even and $H^0(Q, S^m[\Omega^1_Q(1)]) = 0$ if m is odd. There is another way to obtain this result for the smooth quadric $X \subset \mathbb{P}^3$. The surface X is $\mathbb{P}^1 \times \mathbb{P}^1$, $\Omega^1_X = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2, 0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0, -2)$ and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 1)$. Hence $S^m[\Omega^1_X(1)] = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-m, m) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-m + 2, m - 2) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(m, -m)$ which gives $H^0(X, S^2[\Omega^1_X(1)]) = \mathbb{C}$ if m is even and $= 0$ if m is odd.

We proceed to analyse what happens in the higher codimensions. In the codimension 1 case the knowledge about $C_X X$ is sufficient to obtain the complete answer, as appears in theorem B. But in higher codimension, one should also consider iterations of the construction $C_X Y$. Define $C^2_X Y = C_X(C_X Y)$ (note $C^2_X Y \neq C_{(C_X Y)}(C_X Y)$) and proceed inductively to obtain $C^k_X Y$.

Theorem C. *Let X be a non degenerated smooth projective subvariety of $\mathbb{P}^N$ of dimension $n > 2/3(N - 1)$. If $C^k_X X = \mathbb{P}^N$ for some k, Then:*

$$H^0(X, S^m[\Omega^1_X(1)]) = 0$$

Proof. It follows from the proposition 1.5 that in the dimensional range $n > 2/3(N-1)$ the differentials $\omega \in H^0(X, S^m[\Omega^1_X(1)])$ are induced from symmetric m-differentials $\Omega \in S^m[\mathbb{C}dz_0 + ... + \mathbb{C}dz_N]$ on $\mathbb{C}^{N+1}$. Moreover, Proposition 1.5 also says that the zero locus $Z(\Omega)$ must be such that $Z(\Omega) \cap T_x X$ is a cone with vertex x, $\forall x \in X$. In the proof of theorem B, it was shown that this implies that $C_X X \subset Z(\Omega)$.

Following the same reasoning, since $Z(\Omega) \cap T_x X$ is a cone with vertex x and $C_X X \subset Z(\Omega)$ then $C^2_X X \subset Z(\Omega)$. Repeating the argument one gets $C^l_X X \subset Z(\Omega)$ for all $l \geq 1$. If $C^k_X X = \mathbb{P}^N$ for some k, then clearly every symmetric differential $\Omega \in S^m[\mathbb{C}dz_0 + ... + \mathbb{C}dz_N]$ inducing $\omega \in H^0(X, S^m[\Omega^1_X(1)])$ must be trivial. $\square$

Theorem B and C reveal the importance of the characterization of the subvarieties $X \subset \mathbb{P}^N$ with $\dim X > 2/3(N-1)$ with $C^k_X X \neq \mathbb{P}^N$ for any k. They must be special and, as in the hypersurface case, subjectable to a description. The quadrics containing X play a key role in this description. First, we observe:

Proposition 1.8. *Let X be a subvariety of $\mathbb{P}^N$ such that $X \subset Q_1 \cap ... \cap Q_l$, where Q_1,..., Q_l are quadrics. Then $C^k_X X \subset Q_1 \cap ... \cap Q_l$ for all $k \geq 1$.*

Proof. There is the inclusion of the t-trisecant varieties $C_X X \subset C_{Q_i} Q_i$ for all quadrics Q_i $i = 1, ..., l$. The equality $C_{Q_i} Q_i = Q_i$ proved in proposition 1.7 gives $C_X X \subset Q_1 \cap ... \cap Q_l$. In an equal fashion one sees that $C^2_X X = C_X(C_X X) \subset C_{Q_i} Q_i$ for all $i = 1, ..., l$. Induction then gives the result. $\square$

Remark. *One should investigate the generality of the assertion that $C^k_X X$ for k sufficiently large is the intersection of all quadrics containing X .*

We show below that the assertion in the remark is true for the case where X is of codimension 2, $X^n \subset \mathbb{P}^{n+2}$ (if X is not contained in any quadric then the intersection of all quadrics containing X should be considered to be the full $\mathbb{P}^{(n+2)}$). The answer follows from establishing that $C_X X$ is the usual trisecant variety of X, $Tr(X)$, if $n \geq 3$ and the use of the known results on trisecant varieties of varieties of codimension 2 of Ziv Ran [Ra83], $n \geq 4$, and Kwak [Kw02] for the threefold case.

Let $X \subset \mathbb{P}^N$ be a subvariety and $l \subset \mathbb{P}^N$ a line meeting X at k points, x_i $i = 1, ..., k$, the line l is said to is of type $(n_1, ..., n_k)$ if $n_i = length_{x_i}(X \cap l)$. A line l is a trisecant line if $\sum n_i \geq 3$ and a tangent trisecant line if additionally one of the $n_i \geq 2$.

Lemma 1.9. *Let X be a subvariety of $\mathbb{P}^N$ and $L \subset \mathbb{P}^N \times T$ be a family of lines in $\mathbb{P}^N$ over an irreducible projective curve T such that all lines pass through a fixed $z \notin X$ and whose union is not a line. If the general line meets X at least twice, then one of the lines must meet X with multiplicity at least 2 at some point.*

Proof. Let H be an hyperplane not containing z and $f : T \to H$ be the map which sends t to $L_t \cap H$. Denote by C the image of map f. Let $C(z, C)$ the cone over C with vertex z. Let D be the curve which consists of the divisorial part of $X \cap C(z, C)$. The possibly nonreduced curve D is such that any line l_c joining z to $c \in C$ meets D at least twice (counting with multiplicity). We can assume that the lines l_c meet X with at most multiplicity one at any $x \in X$, otherwise the result would immediately follow. Hence the the curve $D \subset C(z, C)$ is reduced and clearly does not pass through z.

Resolve the cone $C(z, C)$ by normalizing C, $\bar{C}$, and blowing up the singularity at the vertex. The resulting surface Y is a ruled surface over $\bar{C}$, which comes with two maps $\sigma : Y \to C(z, C)$ and $f : Y \to \bar{C}$. Let $\bar{D}$ be the pre-image of D by σ. If $\bar{D}$ meets any of the fibers of $f : Y \to \bar{C}$ with multiplicity ≥ 2 then we are done. Hence $\bar{D}$ is smooth moreover it must be a multi-section. This is impossible since by base change we would obtain a ruled surface which would have at least two disjoint positive sections not intersecting (the unique negative section lies over the the pre-image of p). $\square$

We proceed by giving an alternative proof of Zak's theorem on the equality of the secant and tangent variety for smooth subvarieties X whose secant variety does not have the expected dimension.

Corollary 1.10. *(Zak's Theorem) Let X be a smooth subvariety of $\mathbb{P}^N$. If $\dim Sec(X) < 2n + 1$ then $Tan(X) = Sec(X)$.*

Proof. Assume $Sec(X) \neq X$ since if the equality holds then clearly $Tan(X) = Sec(X)$. Let z be a point of $Sec(X) \setminus X$. Since $Sec(X)$ has less than the expected dimension there is a positive dimensional family $\pi : L \to T$ of secant lines passing through z. Apply lemma 1.10 to a 1-dimensional subfamily of $\pi : L \to T$ and obtain that one of this lines L_{t_0} must meet X with multiplicity at least 2 at some point $x \in X$ hence L_{t_0} is tangent to X at x and $z \in Tan(X)$. $\square$

The lemma 1.9 gives an important case when the trisecant variety is equal to the tangent trisecant variety.

Corollary 1.11. *Let X be a smooth subvariety of $\mathbb{P}^N$. If the family of trisecant lines of X through a general point of $Trisec(X)$ is at least 1-dimensional, then $Trisec^t(X) = Trisec(X)$.*

Proof. The same argument after replacing $Sec(X)$ by $Trisec(X)$ and $Tan(X)$ by $Trisec^t(X)$. $\square$

In order to establish that $C_X X = Trisec(X)$ if X has codimension 2 and dimension $n \geq 3$ we need to make a few observations about the subvariety $Trisec^{tt}(X) \subset Trisec^t(X)$ which is the closure of the union of all tangent trisecant lines that meet X at only one point. This subvariety does not need to be contained in $C_X X$ in general (but it will be in the case at study) and one has that $Trisec(X) = C_X X \cup Trisec^{tt}(X)$.

There is a stratification of X in 3 strata according to the dimension of the linear family of quadrics $|II|_x$ in $\mathbb{P}_l(T_x X)$ with associated with the 2nd fundamental form of X

at x. Let $x \in X$ if $dim|II|_x = 2$, $dim|II|_x = 1$ and $dim|II|_x = 0$ then x is called general, special and very special respectively. The lines that touch X at x with multiplicity at least 3 are the lines l whose corresponding points in $[l] \in \mathbb{P}_l(T_xX)$ belong to the base locus of the family of quadrics $|II|_x$.

If x is very general then the union of all the lines that touch X at x with multiplicity at least 3, $Trisec_x(X)$, forms a cone of codimension 2 with vertex at x which is an intersection of two quadrics in T_xX (coming from two generating quadrics in $|II|_x$). In this case $Trisec_x(X) = TC_x(X \cap T_xX)$ and hence $Trisec_x(X) \subset C_xX$ since $TC_x(X \cap T_xX) \subset C_xX$.

If x is special then $Trisec_x(X)$ is a quadric in T_xX, the dimension of the strata of special points is at most 1 dimensional by Zak's theorem on tangencies [Za83]. If x is very special then $Trisec_x(X) = T_xX$ and the dimension of the strata of very special points is at most 0-dimensional again by Zak's theorem on tangencies.

Proposition 1.12. *Let X be a smooth subvariety of codimension 2 in $\mathbb{P}^{n+2}$. If $n \geq 3$ then:*

1) $C_XX = Trisec(X)$.

2) $C_XX = \mathbb{P}^{n+2}$ or C_XX is the intersection of the quadrics containing X.

Proof. First we establish that under the dimensional and codimensional hypothesis of the theorem one has $Trisec^t(X) = Trisec(X)$. Recall that the trisecant variety of a subvariety of codimension 2 with dimension $n \geq 2$ is irreducible, so it is enough to show that through the general point of $Trisec(X)$ passes a tangent trisecant line. Let z be a general point of the trisecant variety $Trisec(X)$. Let l be a trisecant line passing through z, assume it is not tangent to X since otherwise there is nothing to prove. Consider the projection $p_z : X \to \mathbb{P}^{n+1}$ from the point z to an hyperplane $\mathbb{P}^{n+1} \subset \mathbb{P}^{n+2}$. Denote 3 of the points in $l \cap X$ by x_1, x_2 and x_3 and $p = p_z(x_i) = l \cap \mathbb{P}^{n+1}$. The hypersurface $p_z(X) \subset \mathbb{P}^{n+1}$ has at p a decomposition into local irreducible components $p_z(X) \cap U_p = H_1 \cup ... \cup H_k$, where U_p is a sufficiently small neighborhood of p. The points x_i $i = 1, ..., 3$ have neighborhoods U_i such that $p_z : U_i \to p_z(U_i)$ is finite and $p_z(U_i)$ contains one of H_j. Consider the case where the local irreducible components H_j contained by $p_z(U_i)$ are all distinct, w.l.o.g. denote them by H_1, H_2 and H_3 (the other cases will follow by the same argument and are more favorable to our purposes). In this case $H_1 \cap H_2 \cap H_3$ will be of dimension $n-2$. It follows that there is at least a 1-dimensional family of trisecant lines passing though z and hence $z \in Trisec^t(X)$ by corollary 1.11.

The irreducibility of trisecant variety gives that $C_XX = Trisec(X)$ if $\dim Trisec(X) = \dim C_XX$. This follows from the equalities $Trisec(X) = Trisec^t(X) = C_XX \cup Trisec^{tt}(X)$ plus the fact the components of $Trisec^{tt}(X)$ not contained in C_XX must come from the special and very special strata and therefore are of dimension at most $n \leq \dim C_XX$ (it follows from the remarks about $Trisec^{tt}(X)$ done above).

The part 2) follows from known facts about the trisecant varieties of smooth varieties X of codimension 2 in projective space $\mathbb{P}^N$. Ziv Ran [Ra83] showed that if $n \geq 4$ and

$Tr(X) \neq \mathbb{P}^{n+2}$ then X must be contained in a quadric (this result is not explicitly stated but clearly follows from the article). Later Kwak [Ka02] showed that the same holds for $n = 3$. Ran also showed that if the degree of X is less or equal to its dimension then X, $d \leq n$, then X is a complete intersection.

We assume X is nondegenerate in $\mathbb{P}^N$ (the degenerate case follows the from hypersurface case). It follows from the last paragraph that if $\dim Tr(X) = n + 1$ then $C_X X = Tr(X)$ is the quadric containing X. The case $\dim Tr(X) = n$ or equivalently $Tr(X) = X$ is settled by slicing and the case $n = 3$. What is known for the case $n = 3$, see for example remark 3.6 of [Ka02], is that if X and $Tr(X) = X$ then X is a complete intersection of two quadrics or the Segree variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ which is the intersection of 3 quadrics. If $n \geq 4$ consider a general 5-plane $L \subset \mathbb{P}^N$, then $X \cap L$ is a smooth 3-fold in $L = \mathbb{P}^5$ for which $Tr(X \cap L) = X \cap L$, since $Tr(X \cap L) \subset Tr(X) \cap L$ and $X \cap L \subset Tr(X \cap L)$. Then $X \cap L$ is one of the two cases described above. Both cases have degree equal to 4 hence the degree X is also 4. It follows from the result of Ran at the end of the previous paragraph that X is a complete intersection of two quadrics. □

Theorem D. *Let X be a smooth subvariety of codimension 2 in $\mathbb{P}^{n+2}$. If $n \geq 3$ then:*

$$H^0(X, S^m[\Omega^1_X(1)]) = 0$$

if and only if X is not contained in a quadric.

Proof. If X is not contained in a quadric, then $C_X X = \mathbb{P}^{n+2}$ by proposition 1.12. The vanishing $H^0(X, S^m[\Omega^1_X(1)]) = 0$ follows theorem C.

To analyse the case where X is contained in a quadric Q recall that proposition 1.5 states that $H^0(X, S^m[\Omega^1_X(1)]) = \{\Omega \in S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N] | \ Z(\Omega|) \cap T_x X)$ is a cone with vertex at x, $\forall x \in X\}$. Consider the symmetric differential $\Omega_Q \in S^2[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N]$ associated with the quadric Q. For all $x \in X$ $Z(\Omega_Q) \cap T_x X) = Q \cap T_x X$ is a cone with vertex x since $T_x X \in T_x Q$. Hence Ω_Q defines an element of $H^0(X, S^m[\Omega^1_X(1)])$ and this element is nontrivial since $Tan(X) = \mathbb{P}^{n+2}$. □

The algebra $\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)])$ contains the the subalgebra $\mathbb{C}[Q_1, ..., Q_l]$ generated by quadrics $Q_1, ..., Q_l$ such that $C_X X = Q_1 \cap ... \cap Q_l$. These two algebras coincide in the case $C_X X = Q$ and are expected to coincide in the general case.

We do a short presentation without proofs of the results which are the analogue to theorem A and part of theorem C for subvarieties of abelian varieties. Again we are having in mind subvarieties with "low" codimension. Recently, Debarre [De06] using the same perspective tackled the problem of which subvarieties have an ample cotangent bundle, which are in the other end in terms of codimension.

Let X be a smooth subvariety of an abelian variety A^n. The surjection on the conormal exact sequence:

$$0 \to N^*_{X/A^n} \to \Omega^1_{A^n}|_X \to \Omega^1_X \to 0$$

induces the inclusion $j : \mathbb{P}(\Omega^1_X) \to \mathbb{P}(\Omega^1_{A^n}|_X)$ of projectivized cotangent bundles. The projectivized cotangent bundle of A^n is trivial, i.e. $\mathbb{P}(\Omega^1_{A^n}) \simeq A^n \times \mathbb{P}^{n-1}$. Let $p_2 : \mathbb{P}(\Omega^1_{A^n}) \to \mathbb{P}^{n-1}$ denote the projection onto the second factor. Then $\mathcal{O}_{\mathbb{P}(\Omega^1_{A^n})}(m) \simeq p_2^*\mathcal{O}_{\mathbb{P}^{n-1}}(m)$. The composed map $f = p_2 \circ j$:

$$f : \mathbb{P}(\Omega^1_X) \to \mathbb{P}^{n-1}$$

is called the tangent map for X in A^n.

Theorem F. *Let X be a smooth subvariety of an abelian variety A^n. If the tangent map $f : \mathbb{P}(\Omega^1_X) \to \mathbb{P}^{n-1}$ is both surjective and connected then $\forall m \geq 0$:*

$$H^0(X, S^m\Omega^1_X) = H^0(A^n, S^m\Omega^1_{A^n})$$

.

Corollary 1.13. *Let X be a smooth hypersurface of an abelian variety A^n with $n > 2$ which does not contain any translate of an abelian subvariety of A^n. Then $\forall m \geq 0$:*

$$H^0(X, S^m\Omega^1_X) = H^0(A^n, S^m\Omega^1_{A^n})$$

As in the case of subvarieties of $\mathbb{P}^N$ we obtain a vanishing theorem.

Theorem G. *Let X be a smooth subvariety of an abelian variety A^n. If the general fiber of the tangent map $f : \mathbb{P}(\Omega^1_X) \to \mathbb{P}^{n-1}$ is positive dimensional then $\forall m \geq 0$:*

$$H^0(X, S^m\Omega^1_X \otimes L) = 0$$

if L is a negative line bundle on X.

1.3 The non-invariance of the twisted symmetric plurigenera.

The symmetric m-plurigenera $\dim H^0(X, S^m\Omega^1_X)$ if $m \geq 4$ is not preserved under deformation [Bo2-78], [BoDeO06]. This contrasts with the case $m = 1$ where the symmetric 1-genus is just the irregularity of X and by Hodge theory the topological invariant $1/2b_1(X)$ (hence it can not jump under deformations). It also contrasts with the amazing result of Siu about the invariance of all the plurigenera $P_m(X) = \dim H^0(X, (\bigwedge^n \Omega^1_X)^{\otimes m})$, [Si98].

The result of Siu motivated the following question posed by Paun: are the dimensions of $H^0(X_t, S^m\Omega^1_{X_t} \otimes K_{X_t})$ constant for a family of smooth projective varieties? We answer this question negatively.

Theorem I. *Let Y_t be a family of smooth projective varieties. The invariance of the dimension of $H^0(Y_t, S^m\Omega^1_{Y_t} \otimes K_{Y_t})$ does not necessarily hold along the family.*

Proof. Let X_t be a family over Δ of smooth hypersurfaces of degree d of $\mathbb{P}^3$ specializing to a nodal hypersurface X_0 with $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$ nodes. This is possible as long $d \geq 6$ [Mi83]. Denote by Y_t the family which is the simultaneous resolution of the family X_t, $t \in \Delta$. The general member of Y_t is a smooth hypersurface of $\mathbb{P}^3$ of degree d and Y_0 is the minimal resolution of X_0. We proved in [BoDeO06] that if $d \geq 6$ Y_0 has plenty of symmetric differentials, more precisely $H^0(Y_0, S^m\Omega^1_{Y_0}) \uparrow m^3$. This result plus the effectivity of the canonical divisor K_{Y_0}, in particular, implies that:

$$H^0(Y_0, S^m\Omega^1_{Y_0} \otimes K_{Y_0}) \neq 0, \quad m \gg 0 \tag{2.1.7}$$

Theorem B gives that $H^0(Y_t, S^m\Omega^1_{Y_t} \otimes \mathcal{O}_{Y_t}(m)) = 0$. The canonical divisor of the hypersurface Y_t is $K_{Y_t} = \mathcal{O}_{Y_t}(d-4)$, which implies that :

$$H^0(Y_t, S^m\Omega^1_{Y_t} \otimes K_{Y_t}) = 0, \quad m \geq d-4 \tag{2.1.8}$$

The result follows from (1.2.7) and (1.2.8). □

Paun's question and our answer motivates naturally the following question: Can one find for each n a $b(n)$ such that for $\alpha > b(n)$ $h^0(X_t, S^m(\Omega^1_{X_t} \otimes \alpha K_{X_t}))$ is invariant along all families of smooth projective varieties X_t of dimension n with big K_{X_t}?

The answer again is no. An example illustrating our negative answer is a family of surfaces of general type X_t whose general member has ample cotangent bundle and the special member has (-2)-curves. If $m \gg 0$ then for the general member X_{gen} the dimension $h^0(X_{gen}, S^m(\Omega^1_{X_{gen}} \otimes \alpha K_{X_{gen}})) = \chi(X_{gen}, S^m(\Omega^1_{X_{gen}} \otimes \alpha K_{X_{gen}}))$ by the ampleness of $\Omega^1_{X_{gen}}$. To understand what happens at the special member X_0 one needs to consider the cohomological long exact sequence associated with:

$$0 \to S^m(\Omega^1_{X_0} \otimes \alpha K_{X_0}) \otimes \mathcal{O}(-E) \to S^m(\Omega^1_{X_0} \otimes \alpha K_{X_0}) \to S^m(\Omega^1_{X_0} \otimes \alpha K_{X_0})|_E \to 0$$

where E is one of the (-2)-curves in X_0 (note that the last term in the sequence can be simplified since $K_{X_0}|_E = \mathcal{O}$). The special member X_0 has $h^2(X_0, S^m(\Omega^1_{X_0} \otimes \alpha K_{X_0}) \otimes \mathcal{O}(-E)) = h^2(X_0, S^m(\Omega^1_{X_0} \otimes \alpha K_{X_0})) = 0$ by the usual arguments about surfaces for general type (see for example [BoDeo06]). On the other hand $h^1(X_0, S^m(\Omega^1_{X_0} \otimes \alpha K_{X_0})) \neq 0$ due to the presence of the (-2)-curve E since $h^1(E, S^m(\Omega^1_{X_0}|_E)) \neq 0$. Therefore the invariance of Euler characteristic along the family implies a jump on $h^0(X_t, S^m(\Omega^1_{X_t} \otimes \alpha K_{X_t}))$. In order to obtain invariance of $h^0(X_t, S^m(\Omega^1_{X_t} \otimes \alpha K_{X_t}))$ one needs to impose extra conditions on the family X_t (it is plausible that ampleness of the canonical class will be the right condition).

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