VANISHING THEOREMS OF NEGATIVE VECTOR BUNDLES ON PROJECTIVE VARIETIES AND THE CONVEXITY OF COVERINGS.

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Abstract. We give a new proof of the vanishing of $H^1(X,V)$ for negative vector bundles V on normal projective varieties X satisfying rank $V <$ dim X. Our proof is geometric, it uses a topological characterization of the affine bundles associated with non-trivial cocycles $\alpha \in H^1(X,V)$ of negative vector bundles. Following the same circle of ideas, we use the analytic characteristics of affine bundles to obtain convexity properties of coverings of projective varieties. We suggest a weakened version of the Shafarevich conjecture: the universal covering $\tilde{X}$ of a projective manifold X is holomorphically convex modulo the pre-image $\rho^{-1}(Z)$ of a subvariety $Z \subset X$. We prove this conjecture for projective varieties X whose pullback map ρ^* identifies a nontrivial extension of a negative vector bundle V by $\mathcal{O}$ with the trivial extension.

1. Introduction

In this article we give a new proof a vanishing theorem for negative bundles and obtain convexity properties of covering of projective varieties. We use the properties of affine bundles that can be naturally associated with these questions. In particular, the geometric realization of cocycles $\alpha \in H^1(X,V)$ as affine bundles on X modelled on V is explored. It is well known that there is a 1-1 correspondence between cocycles $\alpha \in H^1(X,V)$ and isomorphism classes of extensions $0 \to V \to V_\alpha \to \mathcal{O}_X \to 0$. The pre-image in V_α of a nonzero constant section of $\mathcal{O}_X$ is an affine bundle $A_\alpha \subset V_\alpha$ (which independent of choice of the section). If the vector bundle V is negative then the affine bundle has precise analytic and topological properties. We use these properties to give a geometric proof of a vanishing theorem (section 2) and obtain holomorphic convexity properties of a given class of projective varieties (section 3). Below we describe the sections in detail.

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In section 2, we give a geometric proof of the vanishing of $H^1(X, V)$ for negative vector bundles V on normal projective varieties X satisfying rank $V <$ dim X. Another appealing characteristic of our proof is that on top of being geometric it also well suited to deal with the singular case. As mentioned above, we see a cocycle $\alpha \in H^1(X, V)$ realized geometrically as an affine bundle A_α. The affine bundle A_α is a vector bundle V if and only if $\alpha = 0$. The negativity of V implies that A_s has a proper birational morphism onto a Stein space $St(A_s)$. We will see that if rank $V <$ dim X then the homological properties of the Stein space $St(A_\alpha)$ imply that A_α must have a section. This is equivalent to $A_\alpha \cong V$, that is α has to be trivial.

In section 3, we use the same circle of ideas to approach the problem of finding the holomorphic convexity properties of universal covers of projective varieties. In the early 70's I. Shafarevich proposed the following conjecture on the function theory of universal covers: The universal cover $\tilde{X}$ of a projective variety X is holomorphically convex, i.e every discrete sequence of points of $\tilde{X}$ has a holomorphic function on $\tilde{X}$ that is unbounded on it. This conjecture has been proved in some cases (see [Ka95], [Ey04] and [EKPR] for the strongest results), but the general case has remained unreachable. For the general case, there is the work of Kollar [Ko93] on the existence of Shafarevich maps (Campana [Ca94] has dealt with Kahler case). The existence and properties of the Shafarevich maps do not give information on the existence of holomorphic functions on $\tilde{X}$. But they are an essential tool for dealing with and understanding the conjecture. There are two main reasons for the difficulty in proving the Shafarevich conjecture. The first reason is that the conjecture proposes that noncompact universal covers $\tilde{X}$ have many holomorphic functions. But, on the other hand, there is a lack of methods to construct holomorphic functions on $\tilde{X}$. The second reason comes from the main geometric obstruction to holomorphic convexity. A holomorphic convex analytic space can not have an infinite chain of compact subvarieties. The existence of these infinite chains on universal covers of projective varieties has not been ruled out. In fact, the first author and L. Katzarkov produced some examples of algebraic surfaces that possibly contain infinite chains [BoKa98].

The possible existence of infinite chains on universal covers asks for possible reformulations of the Shafarevich conjecture. We make a suggestion based on the results of section 3.

Conjecture 3.2. *The universal covering $\tilde{X}$ of a projective variety X is holomorphically convex modulo the pre-image of a subvariety $Z \subset X$.*

This means that for every infinite discrete sequence $\{x_i\}_{i \in \mathbb{N}}$ $x_i \in \tilde{X}$ such that $\{\rho(x_i)\}$ has no accumulation points on Z, there exists a holomorphic function f on $\tilde{X}$ which is unbounded on the sequence. This conjecture is still very strong but does not exclude the existence of infinite chains of compact subvarieties. A sign of the strength is that the conjecture still separates universal covers of projective varieties from some universal covers of compact non-kahler manifolds with many holomorphic functions. The example to have in mind is the case of the universal cover of an Hopf surface which is $\mathbb{C}^2 \setminus$

$\{(0,0)\}$. The complex manifold $\mathbb{C}^2 \setminus \{(0,0)\}$ has many holomorphic functions, but it is not holomorphic convex modulo of the pre-image of any subvariety of the Hopf surface.

An explicit motivation for the weakened conjecture can be found in theorem 3.3. This theorem proves the conjecture for projective varieties X satisfying: X has a negative bundle V such that the pullback map identifies a nontrivial extension of $\mathcal{O}$ by V the trivial extension. This condition (negativity might have to be replaced by semi-negativity, see corollary 3.4 and the preceding paragraph) is not arbitrary, it is necessary for the Shafarevich conjecture to hold.

The method to produce holomorphic functions used in the proof of theorem 3.3 also motivated us to suggest the conjecture. The method gives a subvariety Z of X for which the holomorphic functions on the full $\tilde{X}$ give very strong and precise holomorphic convexity properties for $\tilde{X} \setminus \rho^{-1}(Z)$. The natural appearance of the subvariety Z is not one of the method's shortfalls, but rather one of its strengths, since it is the existence of Z that permits the possible existence of infinite chains.

We show in corollary 3.4 that the conditions of theorem 3.3 imply that X has a generically large fundamental group, i.e the general fiber Shafarevich map is zero dimensional (see section 2.3). We note that the varieties with generically large fundamental group form a natural class of manifolds to consider when studying the Shafarevich conjecture [Ko93]. In particular, all the difficulties of the conjecture are present for this class of manifolds.

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2.1 Affine bundles and the negativity of vector bundles.

We recall a construction of affine bundles associated with extensions of a given vector bundle V. We also describe how the negativity properties of the vector bundle V influence the function theory of the affine bundle.

Let X be a projective variety and V a vector bundle of rank r on X. We use the common abuse of notation where V also denotes the sheaf of sections of V. An extension of $\mathcal{O}$ by a vector bundle V is an exact sequence:

$$0 \to V \to V_\alpha \to \mathcal{O}_X \to 0 \tag{2.1}$$

There is a 1-1 natural correspondence between cocycles $\alpha \in H^1(X,V)$ and isomorphism classes of extensions of $\mathcal{O}_X$ by V. The extension (2.1) gives a family of affine bundles, consisting of the pre-images in V_α of the nonzero constant sections of the trivial line bundle $\mathcal{O}_X$. All these affine bundles are isomorphic, hence one can associated one affine bundle to the extension (2.1) which is denoted by A_α.

A cocycle $\alpha \in H^1(X,V)$ cohomologous to zero corresponds to the trivial extension $V_\alpha = V \oplus \mathcal{O}_X$. Additionally, the affine bundle A_α is a vector bundle if and only if (2.1) splits or equivalently α is cohomologous to zero. Recall that an affine bundle is a vector bundle if and only if the affine bundle has a section.

The affine bundle A_α can be described in an alternative way. Let E be a vector bundle of rank r over X, $p : \mathbb{P}(E) \to X$ be the $\mathbb{P}^{r-1}$-bundle over X, whose points in the fiber $\mathbb{P}(E)_x$ are the hyperplanes in the vector space E_x, $x \in X$. Associated to a surjection $E \to F \to 0$ of vector bundles there is an inclusion $\mathbb{P}(F) \subset \mathbb{P}(E)$ of projective bundles. The affine bundle A_α can also be described by:

$$A_\alpha = \mathbb{P}(V_\alpha^*) \setminus \mathbb{P}(V^*)$$

where the inclusion $\mathbb{P}(V^*) \subset \mathbb{P}(V_\alpha^*)$ comes from (2.1) dualized.

Let V be a vector bundle over a projective variety X. We recall Grauert's characterization of negativity for vector bundles. The projective bundle $\mathbb{P}(V)$ has a naturally defined line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ on it. Let $F \subset p^*V$ be the tautological hyperplane bundle over $\mathbb{P}(V)$. The line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ is the quotient p^*V/F. The vector bundle V is *Grauert negative* if the line bundle $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ is ample (which is the same as V^* being ample). The negativity properties of a vector bundle imply complex analytic properties of the total space of the vector bundle and of the associated affine bundles (see the lemma 2.1).

Recall that a complex space X is *q-convex* in the sense of Andreotti-Grauert if there is a C^∞ function $\varphi : X \to \mathbb{R}$ such that outside a compact subset $K \subset X$:

(i) The subset $\{x \in X : \varphi(x) < c \text{ with } c < \sup_X \varphi\}$ is relative compact.
(ii) $\varphi|_{X\setminus K}$ is q-convex.

A function $f : X \to \mathbb{R}$ is q-convex if $\forall x \in X$ have an open neighborhood U_x with a biholomorphism $\phi_x : U_x \to Y$, Y a subvariety of an open subset $\Omega \subset \mathbb{C}^n$, such that $f|_{U_x} \equiv g \circ \phi_x$ where g is q-convex on Ω. A function $g : \Omega \subset \mathbb{C}^n \to \mathbb{R}$ is q-convex if the Levi form $L(g) = \sum_{i,j} \frac{\partial^2 g}{\partial z_i \partial \bar{z}_i} dz_i \otimes d\bar{z}_j$ has at most $q-1$ non-positive eigenvalues at all $x \in \Omega$.

The following lemma will be used in the proofs of theorem 2.4 and 3.3.

Lemma 2.1. *Let V be a negative vector bundle on a normal projective variety X and A_α be an affine bundle associated with an extension (2.1). Then A_α is 1-convex and has a proper birational morphism onto a Stein space $St(A_\alpha)$, $r : A_\alpha \to St(A_\alpha)$.*

Proof. If A_α is 1-convex then it follows that there is a proper bimeromorphic morphism onto a Stein space $St(A_\alpha)$, $r : A_\alpha \to St(A_\alpha)$ (see for example [Si71]). We proceed to show that A_α is 1-convex.

The exact sequence obtained by dualizing (2.1) gives a surjection of vector bundles, $q : V_\alpha^* \to V^*$. The surjection $q : V_\alpha^* \to V^*$ induces naturally an inclusion $\mathbb{P}(V^*) \subset \mathbb{P}(V_\alpha^*)$ of the respective projective bundles. The affine bundle A_α is the complement of $\mathbb{P}(V^*)$ in $\mathbb{P}(V_\alpha^*)$. To derive complex analytic properties of A_α we need to determine the properties of the normal bundle $N_{\mathbb{P}(V^*)/\mathbb{P}(V_\alpha^*)}$.

Let $E \to F$ be a surjection of vector bundles over a projective variety Y and $\mathbb{P}(F) \subset \mathbb{P}(E)$ be the respective inclusion of projective bundles. The expression for the normal bundle $N_{\mathbb{P}(F)/\mathbb{P}(E)}$ is ($p : \mathbb{P}(F) \to Y$ is the projection):

$$N_{\mathbb{P}(F)/\mathbb{P}(E)} \cong p^*(E^*/F^*) \otimes \mathcal{O}_{\mathbb{P}(F)}(1) \tag{2.2}$$

In our case, we get that $N_{\mathbb{P}(V^*)/\mathbb{P}(V_\alpha^*)} \cong \mathcal{O}_{\mathbb{P}(V^*)}(1)$.

Hence, by the definition of negativity of V it follows that the normal bundle $N_{\mathbb{P}(V^*)/\mathbb{P}(V_\alpha^*)}$ is positive. The conclusion follows from theorem 2.1 of [Fr77] which states, in particular, that if the normal bundle of an hypersurface Y of complex space X is positive then the complement $X \setminus Y$ is 1-convex. □

We recall that the 1-convexity property of A_α implies that A_α is holomorphic convex. This fact will play a role in section 3. The following special case should be singled out, suppose the extension (2.1) giving V_α is trivial, then $V_\alpha^* = V^* \oplus \mathcal{O}$,

$$\mathbb{P}(V^* \oplus \mathcal{O}) \setminus \mathbb{P}(V^*) \simeq V \tag{2.4}$$

So, in particular, the total space of V is 1-convex if V is negative.

The following is a method to construct many negative bundles V with rank $V \geq \dim X$ and nontrivial first cohomology. Let L be a very ample line bundle on X which gives an embedding $X \subset \mathbb{P}^n$. There is a surjective map $h : \mathcal{O}_X^{\oplus n+1} \to L$ which defines a rank n subbundle $\ker h = F \subset \mathcal{O}_X^{\oplus n+1}$. The extension

$$0 \to F \otimes L^{-1} \to \bigoplus^{n+1} L^{-1} \to \mathcal{O} \to 0 \tag{2.5}$$

is the pullback of the Euler exact sequence of $\mathbb{P}^n$ to X. The vector bundle $F \otimes L^{-1}$ is a negative bundle, $F \otimes L^{-1} \cong \Omega^1_{\mathbb{P}^n|X}$, and $H^1(X, F \otimes L^{-1}) \neq 0$. Namely there is a nontrivial element $s \in H^1(X, F \otimes L^{-1}) \neq 0$ which corresponds to the above nontrivial extension.

The results presented in section 3 spring from the existence of nontrivial cocycles $\alpha \in H^1(X, V)$ that become trivial when pulled back to the universal cover. The following standard result (see [La04], lemma 4.1.14) shows that this is only possible for infinite covers.

Lemma 2.2. *Let $f : Y \to X$ a finite morphism between irreducible normal varieties X and Y and V a vector bundle over X. If $s \in H^1(X, V)$ is nontrivial then $f^*s \in H^1(Y, f^*V)$ is also nontrivial.*

Below we give another application of this lemma that will appear in the proof of theorem 3.3.

Proposition 2.3. *Let X be a projective normal variety with a vector bundle V and V_s be the extension associated with a nontrivial cocycle $s \in H^1(X, V)$. Then any compact subvariety M of the affine bundle $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ satisfies* $\dim M < \dim X$.

Proof. It is clear that if $M \subset A_s$ is a compact subvariety then $\dim M \leq \dim X$ (the intersection of M with any fiber of the projection map $p : A_s \to X$ will be at most 0-dimensional). We need to rule out $\dim M = \dim X$. Below we show that if $\dim M = \dim X$ then $p^*s|_M$ must be nontrivial (lemma 2.2) but that contradicts $p^*s|_M \in H^1(M, p^*V|_M)$ must be trivial.

The triviality of $p^*s|_M$ follows from the triviality of $p^*s \in H^1(A_s, p^*V)$. The equality $p^*s = 0$ holds if and only if the pullback of the extension (3.1) to A_s splits. The affine bundle A_s is $\mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ hence there is a canonical 1-1 association between the points $y \in A_s$ with $p(y) = x$ and the hyperplanes $(V_s^*)_x$ surjecting to $(V^*)_x$. Let W_y denote the hyperplane of $(V_s^*)_x$ corresponding to $y \in A_s$. This association gives the canonical hyperplane subbundle W of $p^*V_s^*$ on A_s. The vector bundle W is such that the restriction of the surjection $q : V_s^* \to V^*$ to W is actually an isomorphism. The splitting of $0 \to \mathcal{O} \to p^*V_s^* \to p^*V \to 0$ is obtained by inverting $q : W \to V^*$.

If $\dim M = \dim X$ then M has an irreducible component M' such that the map $p|'_M : M' \to X$ is finite map. Let $n : \hat{M}' \to M'$ be the normalization map, then $n \circ p|_{M'} : \hat{M}' \to X$ is finite map between normal varieties. Hence by the lemma 2.2 $(n \circ p|_{M'})^*s \neq 0$ which contradicts $(p|_{M'})^*s = 0$. $\square$

2.2 The vanishing theorem.

We give an alternative proof of the vanishing theorem for a negative vector bundle V over a normal projective variety X whose $\operatorname{rk} V < \dim X$. See [Mu67] for the case of line bundles over normal varieties and [Kb95] for the higher rank case over smooth complex manifolds (see also [ShSo86], [EsVi92]).

Theorem 2.4. *If V is a negative vector bundle on a normal projective variety X with $\operatorname{rk} V < \dim X$, then $H^1(X, V) = 0$.*

Proof:. Suppose there exists a nontrivial cocycle $s \in H^1(X, V)$ and let:

$$0 \to V \to V_s \to \mathcal{O} \to 0$$

be the associated extension. Consider the dual exact sequence and the associated affine bundle $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$. By the lemma 2.1 the negativity of V implies that $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ is 1-convex and has a birational morphism onto a subvariety of some $\mathbb{C}^l$, $r : A_s \to St(A_s)$ (which is the Remmert reduction of A_s). The morphism r is proper and contracts $M = \cup_{i=1}^k M_i$ and $St(A_s)$ is a Stein space with isolated singularities.

The aim is to obtain a contradiction from topological conditions. The Stein space $St(A_s)$ has $\dim_{\mathbb{C}} St(A_s) = \dim_{\mathbb{C}} X + r$ and hence it has the homotopy type of a simplicial complex of real dimension at most equal to $\dim_{\mathbb{C}} X + r$. On the other hand, $St(A_s) = A_s/(\coprod_i M_i)$ as a topological space and so for the reduced singular homology of A_s $\widetilde{H}_i(St(A_s), \mathbb{C}) = H_i((A_s, \coprod_i M_i), \mathbb{C})$. Now the long exact homology sequence of the pair $(A_s, \coprod_i M_i)$ together with the fact that $\coprod_i M_i$ is compact of complex dimension strictly less than $\dim_{\mathbb{C}} X = n$ (by proposition 2.5) gives that $H_{2n}(A_s, \mathbb{C}) \cong \widetilde{H}_{2n}(St(A_s), \mathbb{C}) = H_{2n}(St(A_s), \mathbb{C})$.

In conclusion, the Stein space $St(A_s)$ is such that $\dim_{\mathbb{C}} St(A_s) = n + r < 2n$ and hence $H_{2n}(St(A_s), \mathbb{C}) = 0$. The previous argument gives $H_{2n}(St(A_s), \mathbb{C}) \cong H_{2n}(A_s, \mathbb{C})$. The contradiction follows since A_s as an affine bundle over X is homotopicaly equivalent to X and therefore $H_{2n}(A_s, \mathbb{C}) \cong H_{2n}(X, \mathbb{C}) \neq 0$. □

The proof of the above theorem uses the normality condition on X only when it uses proposition 2.3 to guarantee the non-existence of subvarieties of A_α with the same dimension of X. There are examples of non-normal projective varieties where theorem 2.4 fails. An example of a non-normal surface X with a negative line bundle L such that $H^1(X, L) \neq 0$ can be found in [ArJa89].

The topological argument used in the proof of the theorem is reminiscent of the sort of considerations Fulton and Lazarsfeld used in their work on the degeneracy loci. See for example section 7.2 of [La04].

3. Convexity properties of universal covers

In this section we use the same line of thought used in the proof of the vanishing theorem 2.3 to take some conclusions on the Shafarevich conjecture.

3.1 Holomorphic convexity of universal covers.

A complex manifold X is *holomorphic convex* if for every infinite discrete sequence $\{x_i\}_{i\in\mathbb{N}}$ of points in X there exists a holomorphic function f on X which is unbounded on the sequence. Shafarevich proposed the following:

Conjecture. *(Shafarevich) The universal cover of a projective variety is holomorphic convex.*

The Shafarevich Conjecture predicts that noncompact universal covers of projective manifolds have many holomorphic functions. The holomorphic convexity of the universal cover implies that there is a proper map of $\tilde{X}$ into $\mathbb{C}^n$. In particular, holomorphic convexity implies that there are enough holomorphic functions to separate points that are not connected by a chain of compact analytic subvarieties. This pointwise holomorphic separability property is the strongest possible for a complex manifold.

We propose a weakened version of holomorphic convexity, that will appear in subsection 3.2 to generalize the Shafarevich conjecture.

Definition 3.1. *Let X be a complex manifold and $\rho : \tilde{X} \to X$ the universal covering of X. The universal cover X is holomorphic convex modulo an analytic subset $\rho^{-1}(Z)$, $Z \subset X$, if for every infinite discrete sequence $\{x_i\}_{i\in\mathbb{N}}$ $x_i \in \tilde{X}$ such that $\{\rho(x_i)\}$ has no accumulation points on Z, there exists a holomorphic function f on $\tilde{X}$ which is unbounded on the sequence.*

Remark: The holomorphic convexity modulo $\rho^{-1}(Z)$ is not the same as saying that $\tilde{X} \setminus \rho^{-1}(Z)$ is holomorphic convex. Projective varieties always have subvarieties Z such that $\tilde{X} \setminus \rho^{-1}(Z)$ is holomorphic convex, e.g Z an hyperplane section. On another direction, we note that holomorphic convexity modulo an analytic subset of an universal cover also implies the abundance of holomorphic functions on $\tilde{X}$.

There is also a generalization of the notion of holomorphic convexity to line bundles other than $\mathcal{O}$. Let X be a complex manifold and L a line bundle on X with an Hermitian metric h. X is *holomorphic convex with respect to* (L, h) if for every infinite discrete sequence $\{x_i\}_{i\in\mathbb{N}}$ of points in X there exists a section $s \in H^0(X, L)$ such that the function $|s|_h$ is unbounded on the sequence (holomorphic convexity is the special case of holomorphic convexity with respect to the trivial line bundle equipped with the trivial metric). In subsection 3.2 we use the following result from [Na90] concerning holomorphic convexity with respect to positive line bundles. Let X be a smooth projective variety and L a positive line bundle on X. If $p \gg 0$ then the universal cover $\tilde{X}$, $\rho : \tilde{X} \to X$, is holomorphic convex with respect to $(\rho^* L^p, h)$, h is any continuous Hermitian metric on L^p.

3.2 The weakened Shafarevich conjecture.

The Shafarevich conjecture claims that a noncompact universal cover of a projective manifold has many holomorphic functions but it also claims the non-existence of infinite chains. As mentioned in the introduction, the second claim may not hold. In this subsection we describe an approach to obtain information on the the algebra of holomorphic functions of universal covers which has natural place for infinite chains. The main result of this section, theorem 3.2, inspired us to suggest the following weakened conjecture:

Conjecture 3.2. *The universal covering $\tilde{X}$ of a projective manifold X is holomorphically convex modulo the pre-image of a subvariety $Z \subset X$.*

As the Shafarevich conjecture, our conjecture also claims a rich algebra of holomorphic functions but it allows the existence of infinite chains. The infinite chains of compact analytic subvarieties would lie in the pre-image of the subvariety $Z \subset X$ described in the conjecture.

We describe briefly the methodology our approach. Let X be a projective manifold, $\rho : \tilde{X} \to X$ be the universal covering and $\mathcal{O}_{\tilde{X}}(\tilde{X})$ the algebra of global holomorphic functions of $\tilde{X}$. We derive properties of $\mathcal{O}_{\tilde{X}}(\tilde{X})$ from the existence of nontrivial extensions of $\mathcal{O}$ by negative vector bundles V which become trivial once pulled back to $\tilde{X}$. From such a special nontrivial extension we construct a map from the universal cover $\tilde{X}$ to the associated affine bundle. This map is a local embedding. From subsection 2.1, it follows that the negativity of V implies strong analytic geometric properties of the algebra of global holomorphic functions of the affine bundle. We use these analytic geometric properties and the local embedding of $\tilde{X}$ in the associated affine bundle to obtain information on $\mathcal{O}_{\tilde{X}}(\tilde{X})$.

Our approach suggests the conjecture on two levels. First, theorem 3.3 is an explicit confirmation of the conjecture for projective manifolds X having a nontrivial extension of $\mathcal{O}$ by a negative vector bundle V whose pullback to $\tilde{X}$ is trivial. Second, it is the nature of the approach to give very strong holomorphic convexity properties for $\tilde{X}$ but also to give a subvariety Z of X for which all the holomorphic functions on $\tilde{X}$ created by the method must be constant over $\rho^{-1}(Z)$. The subvariety Z is the projection into X of the maximal compact analytic subset of the affine bundle associated with the extension.

Theorem 3.3. *Let X be a projective manifold with a negative vector bundle V and $\rho : \tilde{X} \to X$ its universal covering. If there exists a nontrivial cocycle $s \in H^1(X, V)$ such that $\rho^* s = 0$ then $\tilde{X}$ is holomorphic convex modulo $\rho^{-1}(Z)$, Z is a subvariety of X.*

Proof. First, we will identify the subvariety $Z \subset X$ described in the theorem. The nontrivial cocycle $s \in H^1(X, V)$ has an associated strongly pseudoconvex affine bundle $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ originating from the nonsplit exact sequence:

$$0 \to \mathcal{O}_X \to V_s^* \to V^* \to 0 \tag{3.1}$$

The 1-convex manifold A_s (hence holomorphic convex) has a proper holomorphic map onto a Stein space, $r : A_s \to St(A_s)$ (the Remmert reduction). Moreover, A_s has a subset M called the maximal compact analytic subset of A_s such that the map $r|_{A_s \setminus M} : A_s \setminus M \to r(A_s \setminus M)$, is a biholomorphism. The subvariety $Z \subset X$ is $Z = p(M)$.

We proceed to construct a holomorphic map $g : \tilde{X} \to A_s$ that is locally an embedding and such that $g(\tilde{X}) \not\subset M$. The pullback of the exact sequence (3.1) to the universal covering $\tilde{X}$ splits into:

$$0 \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}} \oplus \rho^* V^* \to \rho^* V^* \to 0$$

since it is associated with the trivial cocycle $\rho^* s \in H^1(\tilde{X}, \rho^* V)$. As observed in (2.4) $A_{\rho^* s} \equiv \mathbb{P}(\mathcal{O}_{\tilde{X}} \oplus \rho^* V^*) \backslash \mathbb{P}(\rho^* V^*) \simeq \rho^* V$, hence $\tilde{X}$ embeddeds in $A_{\rho^* s}$ as the zero section of $\rho^* V$. The affine bundle $A_{\rho^* s}$ is the fiber product $A_{\rho_s^*} = \tilde{X} \times_X A_s$. Denote the projection of $A_{\rho_s^*} = \tilde{X} \times_X A_s$ onto the second factor by $\rho' : A_{\rho^* s} \to A_s$ and the zero section embedding of $\tilde{X}$ in $A_{\rho^* s} \cong \rho^* V^*$ by $s : \tilde{X} \to A_{\rho^* s}$. The holomorphic map $g : \tilde{X} \to A_s$ will be the composition $g = \rho' \circ s : \tilde{X} \to A_s$. The map g is a local biholomorphism between $\tilde{X}$ and $g(\tilde{X})$ hence the condition $g(\tilde{X}) \not\subset M$ will hold if $\dim M < dim X$.

The maximal compact analytic subset of A_s is of the form $M = \cup_{i=1}^k M_i$, where the M_i are the compact irreducible positive dimensional subvarieties of A_s. The proposition 2.3 showed that $\dim M_i < dim X$, hence $g(\tilde{X}) \not\subset M$.

We proceed to verify that $\tilde{X}$ is holomorphic convex modulo $\rho^{-1}(Z)$. Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence of points in $\tilde{X} \setminus \rho^{-1}(Z)$ such that $\{\rho(x_i)\}_{i \in \mathbb{N}}$ has no accumulation points on Z. The sequence $\{x_i\}_{i \in \mathbb{N}}$ has a subsequence $\{y_i\}_{i \in \mathbb{N}}$ satisfying that $\{\rho(y_i)\}_{i \in \mathbb{N}}$ converges to $a \in X \setminus Z$. Consider the sequence $\{g(y_i)\}_{i \in \mathbb{N}}$ of points in A_s, we have two cases: 1) $\{g(y_i)\}_{i \in \mathbb{N}}$ is a discrete sequence of points of A_s; 2) $\{g(y_i)\}_{i \in \mathbb{N}}$ has a subsequence converging to a point in $a' \in p^{-1}(a)$ (we denote the corresponding subsequence of $\{y_i\}_{i \in \mathbb{N}}$ again by $\{y_i\}_{i \in \mathbb{N}}$).

If case 1) holds, the holomorphic convexity of A_s (lemma 2.1) implies that there is a function $f' \in \mathcal{O}_{A_s}(A_s)$ that is unbounded on $\{g(y_i)\}_{i \in \mathbb{N}}$. Hence $f' \circ g$ is the desired unbounded function on $\{x_i\}_{i \in \mathbb{N}}$.

We procced to deal with case 2). Let L be a positive line bundle on X, Napier's result [Na90] states that for $p \gg 0$:

$$\exists s \in H^0(\tilde{X}, \rho^* L^p), \text{ with } \{|s(y_i)|_{\rho^* h^p}\}_{i \in \mathbb{N}} \uparrow \infty$$

(h is an C^∞ hermitian metric on L). Let $s' \in H^0(X, L^p)$ be such that $a \notin (s')_0 \equiv D$. The meromorphic function $h = \frac{s}{\rho^* s'}$ is holomorphic outside $\rho^{-1}(D)$ and unbounded on $\{y_i\}_{i \in \mathbb{N}}$. Assume the existence of a holomorphic function $q \in \mathcal{O}_{\tilde{X}}(\tilde{X})$ satisfying

$\inf_{z\in g^{-1}(a')}\{|q(z)|\} \neq 0$ and vanishing on $\rho^{-1}(D)$. Then for l sufficiently large the function $f = hq^l$ would be the desired unbounded holomorphic function on $\{y_i\}_{i\in\mathbb{N}}$.

An holomorphic function $q \in \mathcal{O}_{\tilde{X}}(\tilde{X})$, as desired above, can be obtained by pulling back, using g, a holomorphic function $q' \in \mathcal{O}_{A_s}(A_s)$ that satisfies $q'(p^{-1}(D)) = 0$ and $q'(a') = 1$. The existence of such q' follows from $r : A_s \to St(A_s)$ being a proper map, $r(a') \cap r(p^{-1}(D \cup Z)) = \emptyset$ and $St(A_s)$ being Stein. □

It is important to complement theorem 3.3 with an example that shows that the hypothesis of the theorem do not imply that the universal cover $\tilde{X}$ is Stein. The Steiness of $\tilde{X}$ holds if the affine bundle A_s is Stein (corollary 3.5), but otherwise $\tilde{X}$ need not be Stein.

Example: We give an example of a projective variety and a vector bundle satisfying the hypothesis of theorem 3.3 but whose universal cover is not Stein. Let X be a nonsingular projective variety whose universal cover $\tilde{X}$ is Stein. Let $\sigma : Y \to X$ be the blow up of X at a point $p \in X$ $E = \sigma^{-1}(p) = \mathbb{P}^{n-1}$, $n = \dim X$. If we pullback the exact sequence (2.5) to Y and tensor it with $\mathcal{O}(E)$ we obtain:

$$0 \to \sigma^*(F \otimes L^{-1}) \otimes \mathcal{O}(E) \to \bigoplus^{n+1} \sigma^* L^{-1} \otimes \mathcal{O}(E) \to \mathcal{O}(E) \to 0 \qquad (3.2)$$

The vector bundle $\sigma^*(F \otimes L^{-1}) \otimes \mathcal{O}(E)$ on Y is negative since $\sigma^* L^{-1} \otimes \mathcal{O}(E)$ is a negative line bundle on Y. Tensoring a negative vector bundle with a globally generated vector bundle gives a negative vector bundle [Ha66]. The pair Y and $\sigma^*(F\otimes L^{-1})\otimes\mathcal{O}(E)$ satisfies the conditions of theorem 3.3 but $\tilde{Y}$ is clearly not Stein (it contains $\pi_1(X)$ copies of $\mathbb{P}^{n-1}$).

In the work of Kollar [Ko93] and Campana [Ca94] on the Shafarevich conjecture, it was shown that every projective (Kahler) manifold X has a dominant connected rational (meromorphic) map to a normal variety (analytic space) $Sh(X)$, $sh : X \dashrightarrow Sh(X)$ such that:

- There are countably many closed proper subvarieties $D_i \subset X$ such that for every irreducible $Z \subset X$ with $Z \not\subset \bigcup D_i$, one has: $sh(Z) =$ point if and only if $\mathrm{im}[\pi_1(\bar{Z}) \to \pi_1(X)]$ is finite, $\bar{Z}$ is the normalization of Z.

The map $sh : X \dashrightarrow Sh(X)$ is called the *Shafarevich map* and $Sh(X)$ is called the *Shafarevich variety of* X. If the Shafarevich conjecture would hold, then Shafarevich map would be a morphism with the property:

- For every subvariety $Z \subset X$, $sh(Z) =$ point iff $\mathrm{im}[\pi_1(\bar{Z}) \to \pi_1(X)]$ is finite, $\bar{Z}$ is the normalization of Z.

There is a class of projective varieties X that is particularly important with respect to the Shafarevich conjecture [Ko93]. This class consists of the projective varieties X satisfying $\dim Sh(X) = \dim X$, its elements are said to have a *generically large fundamental group*.

The next result defines the natural setting for theorem 3.3. As mentioned in the introduction, projective varieties with generically large fundamental group form a natural class to test and explore the Shafarevich conjecture. Theorem 3.3 asks for a negative vector bundle V on X having a nontrival cocycle $\alpha \in H^1(X,V)$ with trivial pullback $\rho^*\alpha = 0$. We show that the existence of such V imposes that X has a generically large fundamental group. A similar result to theorem 3.3 for general X has to replace the negativity of V by semi-negativity. The right condition of semi-negativity has to be related in particularly with the $\dim Sh(X)$ (this problem is addressed in [BDeO04]).

Proposition 3.4. *Let X be a projective manifold with a negative vector bundle V and $\rho : \tilde{X} \to X$ its universal covering. If it exists a nontrivial cocycle $s \in H^1(X,V)$ such that $\rho^* s = 0$ then:*

a) X has a generically large fundamental group.

b) The holomorphic functions on $\tilde{X}$ separate points on $\tilde{X} \setminus \rho^{-1}(Z)$, Z a subvariety of X. There is a finite collection of holomorphic functions such that $(f_1, ..., f_l) : \tilde{X} \to \mathbb{C}^l$ is a local embedding of $\tilde{X}$ at every point in $\tilde{X} \setminus \rho^{-1}(Z)$.

Proof. In the proof of theorem 3.3 a holomorphic map $g : \tilde{X} \to A_s$ giving a local embedding of $\tilde{X}$ into A_s was constructed. The affine bundle A_s is 1-convex, hence it has the Remmert reduction map $r : A_s \to St(A_s)$ which is a bimeromorphic morphism. The maximal compact subset M of A_s consists of the union of the positive dimensional fibers of the Remmert reduction. Let Z be the projection of $M \subset A_s$ into X. Let $(f'_1, ..., f'_l) : St(A_s) \to \mathbb{C}^l$ be an embedding of the Stein space $St(A_s)$ into an complex Euclidean space. The collection of the functions $f_i = f'_i \circ r \circ g$ give the local embedding in described b) since $g(\tilde{X}) \subsetneq M$.

To completely establish b), we still need to show that if a and b are two different points in $\tilde{X} \setminus Z$ then there is an $f \in \mathcal{O}_{\tilde{X}}(\tilde{X})$ such that $f(a) \neq f(b)$. The $\bar{\partial}$-method gives that for any positive line bundle L on X the line bundle $\rho^* L^p$ on $\tilde{X}$ for $p \gg 0$ has a section s with $s(a) = 0$ and $s(b) \neq 0$ [Na90]. Let $s' \in H^0(X, L^p)$ be such that $\rho(b) \notin D = (s')_0$. The meromorphic function $h = \frac{s}{\rho^* s'}$ is holomorphic outside $\rho^{-1}(D)$ and $h(b) \neq 0$. As shown in the proof of theorem 3.3 there is a $q \in \mathcal{O}(\tilde{X})$ vanishing on $\rho^{-1}(D)$ and not vanishing at $\rho^{-1}(b)$. Then for l sufficiently large $f = hq^l$ would be the desired holomorphic function.

a) follows from b). The pre-image $\rho^{-1}(F)$ of the general fiber of the Shafarevich map, constructed by Kollar, is a countable union of compact analytic subvarieties of $\tilde{X}$. The projective variety X has a generically large fundamental group since no positive dimensional compact subvariety of $\tilde{X}$ passes through any point in $\tilde{X} \setminus \rho^{-1}(Z)$.

□

Let us consider the special case where the affine bundle A_s over X is a Stein manifold. The condition that A_s is a Stein manifold can be easily fulfilled in the following examples.

Over $\mathbb{P}^n$ we have that the the affine bundle A_ω associated with the Euler sequence $0 \to \Omega^1_{\mathbb{P}^n} \to \bigoplus^{n+1} \mathcal{O}(-1) \to \mathcal{O} \to 0$ is isomorphic to the affine variety:

$$F = \mathbb{P}^n \times \mathbb{P}^{n\vee} \setminus \{(x,h) \in \mathbb{P}^n \times \mathbb{P}^{n\vee} | x \in h\}$$

Let X be a projective variety embedded in $\mathbb{P}^n$ and $A_\omega|_X$ the affine bundle associated with pullback to X of the above extension. The affine bundle $A_\omega|_X$ is a Stein manifold since it is a closed subvariety of F. The following is a corollary of the theorem for the case where A_s is Stein.

Corollary 3.5. *Let X be a projective manifold, V a vector bundle and $s \in H^1(X,V)$. Assume furthermore that $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ is a Stein variety. Let $f : Y \to X$ be any infinite unramified covering s.t. $f^*s = 0$. Then Y is Stein.*

Proof. Since any non-ramified covering of a Stein space is Stein [5] the assumption that A_s is affine yields that $A_s \times_X Y$ is Stein. On the other hand in the proof of Theorem A we saw that $Y \subset A_s \times_X Y$ is a closed analytic subset and so Y is Stein. □

Corollary 3.5 suggests that the theorem may be extendable to orbicoverings of X. Let us first describe precisely the notion of orbicovering in the case of a complex variety. Let X be a complex variety and $S \subset X$ be a proper analytic subset. Consider for any point $q \in S$ the local fundamental group $\pi_q = \pi_1(U(q) \setminus S)$ where $U(q)$ is a small ball in X centered at q. Let $L \subset \pi_1(X \setminus S)$ be a subgroup with the property that $L \cap \pi_q$ is of finite index in π_q for all $q \in S$. Then the nonramified covering of $X \setminus S$ corresponding to L can be naturally completed into a normal complex variety Y_L with a locally finite and locally compact surjective map $f_L : Y_L \to X$. The map $f_L : Y_L \to X$ is called an *orbicovering* of X with a ramification set S. The following holds:

Corollary 3.6. *Let X be a projective manifold, V a vector bundle and $s \in H^1(X,V)$. Assume furthermore that $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ is an affine variety. Let $f : Y \to X$ be any orbicovering s.t. $f^*s = 0$. Then Y is Stein.*

Proof. Since every orbicovering of a Stein space is also Stein (see Theorem 4.6 of [On86]) the proof is exactly the same as the proof of Corollary 3.5. □

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