Homework

May 3, 2010

Problem 1: Consider the evaluation map $ev_{\mathbf{a}} : \mathbf{C}[X_1, ..., X_n] \to \mathbf{C}$, sending $f \in \mathbf{C}[X_1, ..., X_n]$ to $f(a_1, ..., a_n)$. Let $I \subset \mathbf{C}[X_1, ..., X_n]$ be an ideal. Show that there must exist an $\mathbf{a} \in \mathbf{C^n}$ such that $ev_{\mathbf{a}}(I) \neq \mathbf{C}$.

Problem 2 a) Let k be an infinite field and $f \in k[X_1, ..., X_n]$ suppose that f is nonconstant, that is, $f \notin k$. Show that $V((f)) \neq \mathbf{A}_k^n$.

b) Let k be algebraically closed and f as in a). Show that $V(f)$ is infinite if $n \geq 2$.

c) Show that if f_1 and f_2 are irreducible elements of $k[X_1, ..., X_n]$, k algebraically closed, that are distinct and not multiples of each other, then $V((f_1)) \neq V((f_2))$.

Problem 3: Prove that if X is an algebraic subset of $\mathbf{A}_k^n$, then X is irreducible iff $I(X)$ is prime. Do we have also I is prime iff $V(I)$ is irreducible?

Problem 3: Show that the radical of an ideal is the intersection of all prime ideals containing I.

Problem 5: Let A be a ring and $A \subset B$ a finite A-algebra. Prove that if m is a maximal ideal of A, then $mB \neq B$ (Hint: go by contradiction, suppose $B = mB$; use the fact that for each i, $b_i = \sum a_{ij} b_j$, where $a_{ij} \in m$. Use determinants in a fashion similar to the proof done in class for the statement that if B is a finite A-algebra then every element of B satisfies a monic equation over A).

Problem 6: Let k be an infinite field, $A = k[a_1, ..., a_n]$ be a finite generated k-algebra, $I = Ker(k[X_1, ..., X_n] \to k[a_1, ..., a_n])$ $(X_i \to a_i)$ and $V = V(I)$. Assume I is prime.

Let $Y_1, ..., Y_m$ be general linear forms in $X_1, ..., X_m$, and write $\pi : \mathbf{A}_k^n \to \mathbf{A}_k^m$ for the linear projection defined by $Y_1, ..., Y_m$, set $p = \pi|_V : V \to \mathbf{A}_k^m$.

Show that the two conclusions of Noether normalization lemma imply that above every point $P \in \mathbf{A}_k^m$, $p^{-1}(P)$ is a finite set and nonempty if k is algebraically closed. (Hint: for non-emptiness use the previous exercise to show that for any $P = (b_1, ..., b_m) \in \mathbf{A}_k^m$, the ideal $J_P = I + (Y_1 - b_1, ..., Y_m - b_m) \neq k[X_1, ..., X_m]$)).

(Recall: the two conclusion of NNL are: 1) there exists $y_1, ..., y_m$, $m < n$, $y_i \in A$ that are algebraically independent over k and 2) A is a finite $k[y_1, ..., y_m]$-algebra.)