Contemporary Mathematics
Volume **00**, 1998

Stein Small Deformations of Strictly Pseudoconvex Surfaces

Fedor A. Bogomolov; Bruno de Oliveira

In Memory of Professor Wei-Liang-Chow.

Abstract. This article analyses the behaviour of analytic cycles on deformations of strictly pseudoconvex surfaces. As a preliminary result we show that a relatively compact strictly pseudoconvex surface is the union of two Stein open subsets. The main result of the article is that there is a small deformation of a minimal relatively compact strictly pseudoconvex surface that has no positive dimensional analytic cycles, hence is Stein. We also prove that a strictly pseudoconvex surface contains a semiregular 1-dimensional cycle if it contains one 1-dim cycle. In the last section the main result is applied to the study of contact structures of three dimensional manifolds.

Introduction

In this article we are going to show that we can make a small deformation of a relatively compact smooth strictly pseudoconvex surface, in such a way that our surface becomes Stein. The article is divided into two parts. The first section, called Static, characterizes the various objects we are going to use, and proves:

Theorem (1). *A relatively compact strictly pseudoconvex smooth surface is the union of two open Stein.*

The second section, called Dynamic, describes the deformations of relatively compact strictly pseudoconcex smooth surfaces and the effect of deformations on compact analytic curves, and proves:

Theorem (2). *There is a small deformation of a relatively compact strictly pseudoconvex smooth minimal surface with no compact analytic curves.*

1991 *Mathematics Subject Classification.* Primary 14C30, 32F30; Secondary 14J25, 32G10, 53C15.

The second author was supported in part by a FLAD and JNIT Grant.

The end result is:

THEOREM (2'). *There is a small deformation of a relatively compact strictly pseudoconvex smooth minimal surface that is a Stein surface.*

To see why Theorem (2) implies Theorem(2'): notice that a Stein space is a holomorphically convex space where all compact analytic subspaces are points. The holomorphic convex property is preserved under small deformations of relatively compact stricly pseudoconvex manifolds and Theorem (2) explains the nonexistence of compact analytic subspaces that are positive dimensional, so we get our Stein surface. We also apply Theorem (2') to a problem proposed by Eliashberg in contact structures of three dimensional compact manifolds, getting:

THEOREM (5). *Let M be a compact three dimensional manifold. Let S (S') be the set of isotopy classes of contact structures on M that come from seeing M as a level set of a s.p.s.h. function on a smooth Stein surface (on a Stein surface with isolated singularities not in M). The sets S and S' are the same.*

Notation

r.s.p.c. = relatively compact strictly pseudoconvex
(s)p.s.h. = (strictly) plurisubharmonic
e.c.f.k = exceptional curve of the first kind

THE STATIC PART

The following results characterize r.s.p.c. smooth surfaces and tells us what is required from the deformation procedure to transform these surfaces into Stein surfaces. For the next results and Theorem (E) see [GR].

DEFINITION (A). *A C^2 function φ on a complex manifold U of dimension n is called plurisubharmonic iff the hermitian form:*

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} dz_i d\overline{z}_j$$

is positive semidefinite. It will be called strictly plurisubharmonic if the form is positive definite.

Definition (B). *A relatively compact strictly pseudoconvex domain D on a complex manifold U, is a relative compact domain of U such that there is a neighborhood W of ∂D and a s.p.s.h. function φ defined in W with:*

$$D \cap W = \{x \in W, \varphi(x) < 0\}$$

In particular, a r.s.p.c. surface is a r.s.p.c. domain of a smooth complex surface.

Definition (C). *A complex space U is said to be holomorphically convex if for every discrete infinite subset of U, there is a holomorphic function which is unbounded in this subset.*

Proposition (A). *A complex space with countable topology U is said to be Stein iff it is holomorphically convex and its compact analytic subspaces are points.*

Theorem (A). *Suppose D is a r.s.p.c. domain in a complex manifold U. Then D is holomorphically convex.*

With Proposition (A) and Theorem (A), we see that what separates a r.s.p.c. surface from a smooth Stein surface is just the existence of analytic compact curves, in fact we have:

Theorem (B). *[G] If V is a r.s.p.c. surface, then V is a modification of a Stein space at a finite number of points.*

Theorem (B) implies that a r.s.p.c. surface V has a maximal compact analytic subspace A, which is an exceptional set. In other words, A is the union of a finite number of connected components, A_α:

$$A_\alpha = \bigcup_{i=1}^{\alpha_i} A_{\alpha_i}$$

where all A_{α_i} are compact irreducible analytic curves and their intersection matrix is negative definite.

To prove Theorem (1) it is necessary that a r.s.p.c. surface is a domain of an algebraic surface. We will first give an important approximation of this result:

Theorem (C). *[VVT] Any r.s.p.c. surface V, is embeddable, i.e. it can be realized as a closed analytic subvariety of $\mathbf{C}^n \times \mathbf{P}^m$.*

Remark: In the proof of this result Vo Van Tan says that $H^3(V, A, \mathbf{Z})$ is torsion and then goes to prove that every line bundle $\mathcal{L}$ over A has a multiple $\mathcal{L}^{\otimes n}$ that is the restriction of some line bundle of V. In fact, one has more as Proposition (1) and Corollary (1) will show.

Proposition (1). *Let V be a r.s.p.c. surface and A its exceptional set. Then:*

$$H^3(V, A, \mathbf{Z}) = 0$$

Proof. Theorem (B) says that V is a modification of a Stein surface X at a finite number of points, $S = \{p_1, ..., p_m\}$. One can prove that the morphism of the pair, $(V, A) \to (X, S)$ induces an isomorphism in the relative homology,

$$H_k(V, A, \mathbf{Z}) \cong H_k(X, S, \mathbf{Z})$$

Since, S is 0-dimensional:

$$H_i(X, S, \mathbf{Z}) \cong H_i(X, \mathbf{Z}),\ i = 2, 3, 4$$

but X is a Stein surface with isolated singularities therefore $H_3(V, A, \mathbf{Z}) = 0$ and $H_2(V, A, \mathbf{Z})$ is free.

The result follows from the Universal Coefficient Theorem for cohomology:

$$0 \to Ext(H_{p-1}(V, A, \mathbf{Z}), \mathbf{Z}) \to H^p(V, A, \mathbf{Z}) \to Hom(H_p(V, A, \mathbf{Z}), \mathbf{Z}) \to 0$$

Corollary (1). *Let V be a r.s.p.c. surface and A its exceptional set. Then:*

$$Pic\ V \twoheadrightarrow Pic\ A$$

Proof. Using the cohomological results on r.s.p.c surfaces, stated below in Corollary (3), one has $H^2(V, \mathcal{I}) = 0$ and $H^2(V, \mathcal{O}) = 0$, where $\mathcal{I}$ is the ideal sheaf of A. One also has by Proposition (1), $H^3(V, A, \mathbf{Z}) = 0$. Therefore we get the following commutative diagram:

$$\begin{tikzcd}
H^1(V, \mathcal{O}_V) \arrow[r] \arrow[d] & H^1(V, \mathcal{O}_V^*) \arrow[r] \arrow[d, "\alpha"] & H^2(V, \mathbf{Z}) \arrow[r] \arrow[d] & 0 \\
H^1(A, \mathcal{O}_A) \arrow[r] \arrow[d] & H^1(A, \mathcal{O}_A^*) \arrow[r] \arrow[d] & H^2(A, \mathbf{Z}) & \\
0 & 0 & &
\end{tikzcd}$$

By diagram chasing, we can see that α is surjective.

Corollary (2). *With the notation of Proposition (1),* $H^3(V, \mathbf{Z}) = 0$.

Proof. The exact sequence for relative cohomology gives:

$$... \to H^3(V, A, \mathbf{Z}) \to H^3(V, \mathbf{Z}) \to H^3(A, \mathbf{Z}) \to ...$$

Since both extremes are zero, the result follows.

Remark: This result can be derived directly from Theorem (2'), because this theorem says that there exists a C^∞ trivial deformation of V, $\omega : \mathcal{V} \to Q$ with fibers $V_q = \omega^{-1}(q)$ that are Stein smooth surfaces. Hence, $H^3(V, \mathbf{Z}) = H^3(V_q, \mathbf{Z}) = 0$.

To get an algebraic embedding we need the theory of algebraic approximations, which gives:

Theorem (D). *[Le] Assume a reduced Stein X space has only isolated singularities, and $K \subset X$ is a compact subset. Then there is an affine variety V, and a neighborhood of K in X that is biholomorphic to an open set in V.*

This result implies the desired result for r.s.p.c surfaces since the resolution of singularities is closed in the algebraic category.

Now, we will prove Theorem (1):

PROOF OF THEOREM (1). To see V as the union of two Stein we need the following proposition:

PROPOSITION(B). *[B1] Let S be a projective surface and U_x be an arbitrary small analytic neighborhood of $x \in S$. Then there are two curves X_1 and X_2 such that*

$$X_1 \cap X_2 \subset U_x$$

$X_1 \in |\mathcal{O}(1)|$ and $X_2 \in |\mathcal{O}(N)|$ for $N \gg 0$ (N depends on the point x and on the radius of U_x).

From Theorem (D) follows that V is r.s.p.c. domain in a projective surface S. Consider S of Proposition (B) to be the same. Pick a point $x \in S$ and a analytic neighborhood U_x such that $U_x \cap V = \emptyset$, this is possibile because V is a r.s.p.c. domain in S. Then by Proposition (B) we find X_1 and X_2, s.t.

$$X_1 \cap X_2 \cap V = \emptyset \quad (*)$$

The complements $S \setminus X_1$ and $S \setminus X_2$ are affine, hence Stein. Since the intersection of a holomorphically convex with a Stein space is Stein, $U_1 = V \cap (S \setminus X_1)$ and $U_2 = V \cap (S \setminus X_2)$ are also Stein. By (*) U_1 and U_2 are our Stein cover of V.

To complement our knowledge of r.s.p.c. smooth surfaces, we describe its cohomology of coherent sheaves.

THEOREM (E). *[GR] Let D be a r.s.p.c. domain in a complex manifold U and $\mathcal{F}$ be a coherent sheaf defined in a neighborhood of $\overline{D}$. Then*

$$H^i(D, \mathcal{F}) < \infty$$

for $i \geq 1$.

Theorem (1) and Theorem (E) give:

COROLLARY (3). *Let V be a r.s.p.c. smooth surface in complex manifold U and let $\mathcal{F}$ be a coherent sheaf as in Theorem (E). Then:*

1. $H^1(V, \mathcal{F}) < \infty$
2. $H^2(V, \mathcal{F}) = 0$ *(just need $\mathcal{F}$ coherent on V)*

PROOF. 1) is a direct consequence of Theorem (E),

2) Proposition (1) says $V = U_1 \cup U_2$, where U_1 and U_2 are Stein, this implies $U_{1,2} = U_1 \cap U_2$ is also Stein. Since for all Stein complex spaces S, $\mathcal{F}$ coherent on S, we have $H^i(S, \mathcal{F}) = 0$, $i \geq 1$. Using Mayer-Vietoris exact sequence:

$$... \to H^1(U_{1,2}, \mathcal{F}) \to H^2(V, \mathcal{F}) \to H^2(U_1, \mathcal{F}) \oplus H^2(U_2, \mathcal{F}) \to ...$$

we see that $H^2(V, \mathcal{F})$ vanishes, because both ends are zero.

THE DYNAMIC PART

1. General Results on Deformations of r.s.p.c. Surfaces

First, we specify what we mean by a deformation of a r.s.p.c. surface.

Definition (1). *A deformation of a r.s.p.c. surface V is the set of the following data:*
1) A pointed complex manifold $(Q, 0)$.
2) A complex manifold $\mathcal{V}$.
3) Two holomorphic maps $\omega : \mathcal{V} \to Q$ and $i_0 : V \to \mathcal{V}$ satisfying:
i) The map i_0 is an isomorphism of V onto $\omega^{-1}(0)$.
ii) The map ω has maximal rank at any point of $\mathcal{V}$.

The deformation theory of r.p.s.c. smooth surfaces is very simple, since these surfaces are the union of two open Stein subsets (hence the cocycle condition for H^1 is trivially satisfied and $H^2 = 0$). Laufer in [La1] shows that r.s.p.c surfaces after the blow up of some points are the union of two Stein, and this is sufficient to get:

Theorem (F). *[La1] Let V be a r.s.p.c. surface. Then there exists a versal deformation $\omega : \mathcal{V} \to Q$ of $V = \omega^{-1}(0)$, where Q is a complex manifold and the Kodaira-Spencer map $\rho_0 : T_{0,Q} \to H^1(V, \Theta)$ is an isomorphism.*

We can make the versal deformation of V C^∞ trivial. To see this, we notice that we can find a r.s.p.c. neighborhood U of V, such that $U \subset V \cup W$ (notation of Definition (B)).

Lemma (1). *U and V have the same exceptional set and for any coherent sheaf, $\mathcal{F}$, the restriction map gives:*

$$r : H^i(U, \mathcal{F}) \xrightarrow{\sim} H^i(V, \mathcal{F}),\ i \geq 1$$

Proof. Let φ be the s.p.s.h. function of Definition (B). Suppose C is an exceptional curve of U that is not contained in V, then $C \cap \{\varphi \geq -\epsilon\} \neq \emptyset$. $C \cap \{\varphi \geq -\epsilon\}$ is compact, hence φ attains a maximum in it, by [Gunning and Rossi, p. 272] φ is constant in a neighborhood of a maximum, contradicting the s.p.s.h. condition.

Since both V and U are r.s.p.c. neighborhoods of the same exceptional set, by [La3] the restriction map $r : H^i(U, \mathcal{F}) \to H^i(V, \mathcal{F})$ gives the desired isomorphism.

Proposition (2). *For any r.s.p.c. surface V, there is a C^∞ trivial deformation $\omega : \mathcal{V} \to Q$, such that the Kodaira-Spencer map $\rho_0 : T_{0,Q} \to H^1(V, \Theta)$ is an isomorphism.*

Proof. Let V and U be as before and $\omega : \mathcal{U} \to Q$ be a versal deformation of U. By [Andreotti, A.; Vessentini, E.] there is a neighborhood ,$\mathcal{V}$, of V in $\mathcal{U}$, with $\mathcal{V} \cap U = V$, which is differentiably isomorphic to $V \times Q$, , we might have to shrink Q, (this isomorphism $I : V \times Q \to \mathcal{V}$ is fiber preserving and is the identity on $V \times 0$). Our deformation will be exactly the restriction of ω to $\mathcal{V}$, since if we denote the Kodaira-Spencer map for the versal deformation of U by ρ_0', we have $\rho_0 = r \circ \rho_0'$ and the result follows from Lemma (1) since $r : H^1(U, \Theta_U) \to H^1(V, \Theta_V)$ is an isomorphism.

Using the result from [R] that for any relatively compact open subset $U' \subset U$ there is an open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of U' s.t. if we restrict ω to $\mathcal{U}'$, we get a 1-convex deformation $\omega|_{\mathcal{U}'} : \mathcal{U}' \to \omega(\mathcal{U}')$, we finish this section, noticing that this result implies that our versal C^∞ trivial deformation of V is inside a versal 1-convex deformation of a r.s.p.c. surface having the same exceptional set in all fibers.

2. Curves under Deformations

2.1. Deformation Theory.

To see the effect of the variation of the complex structure of the surface V on the existence of curves on V. Let us recall relevant general facts of deformation theory.

Definition. *A small variation of the complex structure of V is a section $\varphi(q)$ of $A^{0,1}(V)(\Theta_V)$, that is a (0,1)-form with coefficients in the tangent bunle of V.*

Let the base of the deformation $\omega : \mathcal{V} \to Q$ be a simply connected smooth manifold. Then Kuranishi theory says that there exists a diffeomorphism between $V \times Q$ and $\mathcal{V}$ such that $\varphi(q) = \sum_i^\infty \varphi_i$ is a convergent power series with its coefficients in $A^{0,1}(V)(\Theta_V)$, where $\varphi_i = \sum_{j_1+\ldots+j_n=i} \varphi_{j_1,\ldots,j_n} q_1^{j_1} \ldots q_n^{j_n}$, $\varphi_{j_1,\ldots,j_n} \in A^{0,1}(V)(\Theta_V)$ and $q_1, \ldots, q_n$ are local coordinates of Q at 0.

The small variation of the complex structure gives a new almost complex structure:

$$T_\varphi^{0,1} = \{\mu + \varphi(\mu),\ \mu \in T^{0,1}\}$$

where φ is seen as an element of $Hom(T^{0,1}, T^{1,0})$. In particular if:

$$\varphi = \sum_{i,j} \varphi_{j,i} d\overline{z}_j \otimes \frac{\partial}{\partial z_i}$$

then $(T_\varphi^{0,1})^\vee$ is the span of $\{dz^{\varphi_i} = dz_i + \sum_j \varphi_{j,i} d\overline{z}_j\}$

The new almost complex structure will be integrable if and only if:

$$\overline{\partial}\varphi - \frac{1}{2}[\varphi,\varphi] = 0 \tag{0}$$

The connection of the previous description of a deformation and the Kodaira-Spencer map, appears from examining the 1-st order term in the change of complex structure, ϕ. The 1-st term has the special property that is a cocycle, this is a consequence of (0) and $\varphi(0) = 0$. Let $\gamma : (\Delta, 0) \to (Q, 0)$, Δ is the one dimensional unit disc, be such that $d\gamma(\frac{\partial}{\partial t}) = v \in T_{0,Q}$, giving:

$$\begin{array}{ccc} \gamma^{-1}\mathcal{V} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \omega \\ (\Delta, 0) & \xrightarrow{\gamma} & (Q, 0) \end{array} \tag{1}$$

Then we know that the 1st-order change in the complex structure for the family $\gamma^{-1}\mathcal{V}$ over Δ induced by γ is given by $\alpha.t \in A^{0,1}(V)(\Theta_V)$, with $[\alpha] \in H^1(V, \Theta)$ (Dolbeaut's isomorphism) such that $[\alpha] = \rho_0(v)$.

We will use this in section 2.2

2.2. Integral Obstructions to Lift Algebraic Curves.

Let C be an irreducible and reduced curve in V. Then we have the following exact sequences:

$$0 \to \Theta_V \otimes \mathcal{O}(-C) \to \Theta_V \to \Theta_V|_C \to 0 \tag{2}$$

$$0 \to \Theta_C \to \Theta_V|_C \to \Phi_C \to \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) \to 0 \tag{3}$$

where $\Phi_C = \mathcal{O}(C)_C$ is the generalized normal bundle. Inducing the map:

$$\sigma_C : H^1(V, \Theta_V) \to H^1(C, \Phi_C) \tag{4}$$

There is also the duality for embedded compact curves,

$$H^1(C, \Phi_C) \times H^0(C, \omega_V|_C) \longrightarrow H^1(C, \omega_C) \overset{Tr_C}{\simeq} \mathbf{C}$$

from which, we obtain:

$$\begin{aligned} \sigma'_C : H^1(V, \Theta_V) &\longrightarrow H^0(C, \omega_V|_C)^* \\ [\alpha] &\longrightarrow Tr_C\{\sigma_C([\alpha] \times -)\} \end{aligned} \tag{5}$$

We need to generalize the trace map, to the case of a reduced curve C on V. Let $\nu : C' \to C$ be the normalization of C. From the exact sequence:

$$0 \to \nu_* \omega_{C'} \to \omega_C \to \mathcal{S} \to 0$$

we get the natural isomorphism, $n : H^1(C', \omega_{C'}) \cong H^1(C, \omega_C)$. So, we define:

$$Tr_C = Tr_{C'} \circ n^{-1}$$

where $Tr_{C'}$ is defined using the Dolbeault's isomorphism. We can represent the elements in $H^1(C', \omega_{C'})$ by $\phi \in A^{1,1}(C')$ and $Tr_{C'}([\phi]) = \int_{C'} \phi$.

Proposition (3). *[de O] Let C be a reduced compact analytic curve on a surface V, possibly noncompact. Let $\omega : \mathcal{V} \to \Delta$ be a C^∞ trivial deformation of $V = \omega^{-1}(0)$ over the 1-dimensional disc Δ such that Kodaira-Spencer map satisfies: $\rho_0(\dfrac{\partial}{\partial t}) = [\alpha]$. Then if $\mu \in H^0(\mathcal{V}, \omega_{\mathcal{V}/\Delta})$ the following holds:*

$$\sigma'_C([\alpha])(\mu_0|_C) = \frac{d}{dt}|_{t=0} \int_{I(C \times t)} \mu_t$$

where $\mu_t = \mu|_{V_t}$ and I as in Proposition (2).

The previous proposition gives a method to annihilate curves on a surface, as the next result shows:

Theorem (3). *[de O] Let C be a reduced compact analytic curve on V and $\omega : \mathcal{V} \to \Delta$ be a C^∞ trivial deformation over the 1-dimensional disc Δ such that Kodaira-Spencer map satisfies: $\rho_0(\dfrac{\partial}{\partial t}) = [\alpha]$. Let $\mu \in H^0(\mathcal{V}, \omega_{\mathcal{V}/Q})$ and $\mu_t = \mu|_{V_t}$.*

If $\sigma'_C([\alpha])(\mu_0|_C) \neq 0$ then for $t \neq 0$ sufficiently small no 2-cycle homologous to C is an analytic cycle.

Proof. Proposition (3) implies that for $t \neq 0$:

$$\int_{I(C \times t)} \mu_t \neq 0$$

Since the integral of the (2,0)-form μ_t over an analytic cycle must vanish, no 2-cycle homologous to C can be analytic in V_t.

Remark: Actually, this theorem is about turning algebraic homology classes in $H_2(V, \mathbf{Z})$ into nonalgebraic classes.

3. Proof of Theorem (2)

In order to use the techniques of the previous section to carry over the program of annihilating the algebraic curves on a r.s.p.c surface V in a deformation $\omega : \mathcal{V} \to Q$. We need first to prove the existence of an element of $\mu \in H^0(\mathcal{V}, \omega_{\mathcal{V}/Q})$ such that $\mu|_{V_{q_0}}$ is not trivial on some exceptional curve. Secondly we need to prove the existence of a deformation of V inducing a functional on $H^0(V, \omega_V|_C)$, as in section

2.2, which is nontrivial on the form $\mu|_{V_{q_0}}$ restricted to the curve, and thirdly to describe topologically the exceptional set along the deformation.

3.1. Existence of Holomorphic Forms.

The following assertion says that if a r.s.p.c surface contains a 1-dimensional analytic cycle then it contains a 1-dimensional semiregular cycle.

PROPOSITION (4). *Let V be a r.s.p.c. surface and $A = \bigcup_{i=1}^{r} A_i$ its reduced exceptional set and A_i its irreducible components. Then there exists an irreducible component, w.l.o.g. A_1, such that:*

$$\nu : H^0(V, \omega_V) \twoheadrightarrow H^0(A_1, \omega_V|_{A_1})$$

PROOF. The map ν comes from the long exact cohomology sequence associated to:

$$0 \to \omega_V \otimes \mathcal{O}(-A_1) \to \omega_V \to \omega_V|_{A_1} \to 0$$

so its surjectivity can be assured by the vanishing of $H^1(V, \omega_V \otimes \mathcal{O}(-A_1))$. To get this, we will use, in the proof, the singular version, see [K], of the vanishing theorem of [La3] for r.s.p.c. surfaces. This theorem says: if $\mathcal{L}$ is a line bundle on V and satisfies the condition $\mathcal{L}.A_i \geq K_V.A_i$ for all curves A_i on V then $H^1(V, \mathcal{L}) = 0$.

The first step is to prove that there is an effective divisor, which is a sum of A_i, Z_1, such that:

$$H^1(V, \omega_V \otimes \mathcal{O}(-Z_1)) = 0$$

The vanishing theorem implies this if $Z_1.A_i \leq 0$ for all A_i, since $\mathcal{L} = \omega_V \otimes \mathcal{O}(-Z_1)$. Therefore a natural candidate for Z_1 is the numerical fundamental cycle of the resolution, and the first step is complete.

The next step and final step is to show that there is a decreasing sequence of effective divisors Z_j, with $Z_{j+1} = Z_j - A_k$, such that:

$$0 \to \omega_V \otimes \mathcal{O}(-Z_j) \to \omega_V \otimes \mathcal{O}(-Z_{j+1}) \to \omega_V \otimes \mathcal{O}(-Z_{j+1})|_{A_k} \to 0$$

gives for the 1st cohomology piece of the long cohomology exact sequence:

$$H^1(V, \omega_V \otimes \mathcal{O}(-Z_j)) \twoheadrightarrow H^1(V, \omega_V \otimes \mathcal{O}(-Z_{j+1}))$$

To show this we only need to garantee that $H^1(A_k, \omega_V \otimes \mathcal{O}(-Z_{j+1})) = 0$, by duality we know that:

$$H^1(A_k, \omega_V \otimes \mathcal{O}(-Z_{j+1})) \cong H^0(A_k, \omega_V^{-1} \otimes \mathcal{O}(Z_{j+1}) \otimes \omega_{A_k})$$

but recall that the dualizing sheaf for an embedded curve is $\omega_V \otimes \mathcal{O}_{A_k}(A_k)$, therefore we only have to prove:

$$H^0(A_k, \mathcal{O}_{A_k}(Z_j)) = 0$$

This would be true if $A_k.Z_j < 0$, but we know that the decreasing sequence, Z_i, can be constructed in such a way that the previous condition is satisfied, since the intersection matrix is negative definite.

Now, the last Z_j in this sequence is irreducible exceptional curve, w.l.o.g. A_1, and has the property:

$$0 = H^1(V, \omega_V \otimes \mathcal{O}(-Z_1)) \twoheadrightarrow H^1(V, \omega_V \otimes \mathcal{O}(-A_1))$$

and we are done.

The following theorem states that the holomorphic (2,0)-forms on the central fiber of a 1-convex deformation of r.s.p.c. surfaces always extend to sections of $\omega_{\mathcal{V}/Q}$ for some open neighborhood of the central fiber.

Theorem (4). *Let $\omega : \mathcal{V} \to Q$ be a 1-convex deformation of a r.s.p.c. surface $V = \omega^{-1}(q_0)$ with nonsingular base Q. Then all sections $\mu_0 \in H^0(V, \omega_V)$ can be extended to sections $\mu \in H^0(\omega^{-1}(Q'), \omega_{\mathcal{V}/Q})$, where Q' is a Stein neighborhood of q_0 in Q.*

Proof. The following piece of the long exact sequence for the cohomology:

$$H^0(\mathcal{V}, \omega_{\mathcal{V}/Q}) \to H^0(V, \omega_{\mathcal{V}/Q}|_V) \to H^1(\mathcal{V}, \omega_{\mathcal{V}/Q} \otimes \mathcal{O}(-V))$$

tells us that if:

$$(*) \qquad H^1(\mathcal{V}, \omega_{\mathcal{V}/Q} \otimes \mathcal{O}(-V)) = 0$$

the result follows, because $\omega_{\mathcal{V}/Q}|_V \sim \omega_V$.

The 1-convex generalization of Grauert's direct image theorem, see [S1], says:

$$H^1(\omega^{-1}(Q'), \omega_{\mathcal{V}/Q} \otimes \mathcal{O}(-V)) = H^0(Q', \omega_{*1}(\omega_{\mathcal{V}/Q} \otimes \mathcal{O}(-V)))$$

since Q' is Stein and $\omega_{\mathcal{V}/Q} \otimes O(-V)$ is a coherent sheaf.

To calculate $\omega_{*1}(\omega_{\mathcal{V}/Q} \otimes \mathcal{O}(-V))$ notice that $\omega_{\mathcal{V}/Q} \otimes \mathcal{O}(-V)|_{V_q} \sim \omega_{V_q}$, hence by Kato's vanishing theorem:

$$H^1(V_q, \omega_{\mathcal{V}/Q} \otimes \mathcal{O}(-V)|_{V_q}) = 0$$

and using the 1-convex generalization of the Semicontinuity and Base Change theorem, see [S1]:

$$\omega_{*1}(\omega_{\mathcal{V}/Q} \otimes \mathcal{O}(-V)) = 0$$

Therefore, we have (*).

The next lemma shows that all compact curves C on a minimal r.s.p.c. surface V have a nontrivial section of $\omega_V|_C$.

Lemma (2). *Let V be a minimal r.s.p.c. surface, and A_i be as in Proposition (4). Then $H^0(A_i, \omega_V|_{A_i}) \neq 0$.*

Proof. The degree of the canonical line bundle is given by:

$$deg_{A_i}(\omega_V|_{A_i}) = K_V.A_i = 2p_a(A_i) - 2 - A_i^2$$

Therefore $deg_{A_i}(\omega_V|_{A_i}) \geq p_a(A_i)$ if $p_a(A_i) > 0$ since $A_i^2 < 0$ (A_i are exceptional) and if A_i is a nonsingular rational curve with $A_i^2 < -2$. The only remaining case is the (-2) nonsingular rational curves, for which $deg_{A_i}(\omega_V|_{A_i}) = 0$, hence $\omega_V|_{A_i}$ is the trivial bundle and we are done.

The conclusion of this section is:

Corollary (4). *Let V be a minimal r.s.p.c. surface and $\omega : \mathcal{V} \to Q$ be the C^∞ trivial deformation of Proposition (2) induced from a 1-convex versal deformation $\overline{\omega} : \mathcal{U} \to Q$. Then there exists a compact curve C on V and a $\mu \in H^0(\omega^{-1}(Q'), \omega_{\mathcal{V}/Q})$, where Q' is a Stein neighborhood of q_0 in Q, such that $\mu|_C \neq 0$.*

Proof. Proposition (4), Theorem (4) and Lemma (2) imply the existence of $\mu \in H^0(\overline{\omega}^{-1}(Q'), \omega_{\mathcal{U}/Q})$ with $\mu|_{A_1} \neq 0$, now restrict μ to $\omega^{-1}(Q')$ (recall that $\mathcal{V} \subset \mathcal{U}$) and we are done.

3.2. Existence of Sufficient Deformations.

Corollary (4) shows that there exists an element $\mu \in H^0(\omega^{-1}(Q'), \omega_{\mathcal{V}/Q})$ and a curve C on V such that $\mu|_C \neq 0$. The next lemma shows, in particular, that there exists a deformation of V inducing a linear functional on $H^0(C, \omega_V|_C)$, as in section 2.2, that acts nontrivially on $\mu|_C$.

Lemma (3). *Let C be a reduced curve on a r.s.p.c. surface V. Then the map induced from (2) and (9):*

$$\sigma_C : H^1(V, \Theta_V) \to H^1(C, \Phi_C)$$

is surjective.

Proof. By the exact sequence (2), we have:

$$H^1(V, \Theta_V) \twoheadrightarrow H^1(C, \Theta|_C)$$

because $H^2(V, \mathcal{F}) = 0$, for $\mathcal{F}$ coherent sheaf.

Since C is a local complete intersection, we have:

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_V|_C \to \Omega^1_C \to 0 \tag{6}$$

Applying $hom_{\mathcal{O}_C}(-, \mathcal{O}_C)$ to the sequence, gives:

$$0 \to \Theta_C \to \Theta|_C \to \Phi \to \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) \to 0 \tag{7}$$

Due to C being reduced, at nonsingular points Ω^1_C is locally free. Therefore $\mathcal{E}xt^1_{O_C}(\Omega^1_C, O_C)$ is a sheaf with support only at the singular points and:

$$H^1(C, \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)) = 0$$

Break (7) in two:

$$0 \to \Theta_C \to \Theta|_C \to \mathcal{N} \to 0$$

$$0 \to \mathcal{N} \to \Phi \to \mathcal{E}xt^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)) \to 0$$

From the long exact sequence for cohomology of both sequences, we have:

$$H^1(C, \Theta|_C) \twoheadrightarrow H^1(C, \Phi)$$

and therefore the surjectivity of σ.

3.3 Topological Characterization of the Exceptional Set.

The results presented below describe the relationship between the homology classes of the exceptional curves along a deformation.

Lemma (4). *The homology classes $[A_{q,i}]$ of the irreducible components of the exceptional set A_q are linearly independent in $H_2(V_q, \mathbf{Q})$.*

Proof. Suppose they were not linearly independent then w.l.o.g. we would have:

$$\sum_{i=1}^{p} a_i [A_{q,i}] = \sum_{i=p+1}^{r} a_i [A_{q,i}] \qquad a_i > 0$$

Let $\pi : V_q \to V'_q$ be the blow down of the curve $A_{q,1} + ... + A_{q,p}$ and $\pi_* : H_2(V_q, \mathbf{Q}) \to H_2(V_q, \mathbf{Q})$ be the induced map in homology.

Then $\pi_*(\sum_{i=1}^{p} a_i [A_{q,i}]) = 0$ but $\pi_*(\sum_{i=p+1}^{r} a_i [A_{q,i}]) \neq 0$ since it is the homology class of a positive linear combination of subvarieties of a projective manifold (V'_q is a domain in a projective surface) and these can not be trivial, contradiction.

Let V be a r.s.p.c. surface and $\omega : \mathcal{V} \to Q$ be a C^∞ trivial deformation of $V = \omega^{-1}(q_0)$. Since we are only interested in the exceptional curves in V_q for $q \in Q$ close to q_0, we may assume that Q is such that the canonical map $i_{q*} : H_2(V_q, \mathbf{Z}) \to H_2(\mathcal{V}, \mathbf{Z})$ is an isomorphism for all $q \in Q$, where i_q is the inclusion $i_q : V_q \hookrightarrow \mathcal{V}$.

Proposition (5). *Using the hypothesis describred above. Let F_q be the subgroup of $H_2(\mathcal{V}, \mathbf{Z})$ generated by the homology classes $i_{q*}[A_{q,i}]$, where the $A_{q,i}$ are the irreducible components of the exceptional set A_q. Then $F_q \subset F_{q_0}$, for q sufficiently close to q_0.*

Proof. To see this, we notice that since V is a r.s.p.c. surface there exists a C^∞ function ϕ such that $\{\phi = 0\} = A_{q_0}$ and ϕ is a s.p.s.h. function on $\phi > 0$.

The function ϕ "extends" to $\overline{\phi}$ on $V \times Q \simeq \mathcal{V}$ such that $\overline{\phi}|_{\{x \in V | \phi(x) > \epsilon\} \times q}$ is s.p.s.h., by shrinking Q we can make ϵ arbitrarily small. This implies that all A_q are contained in a set $B \subset \mathcal{V}$ that is diffeomorphic to $\{x \in V | \phi(x) < \epsilon\} \times Q$. If we make ϵ sufficiently small B has a homological basis given by the irreducible components of A_{q_0}, we are done.

3.4. The Proof.

Proof of Theorem (2). We assume that the exceptional set has only one connected component, the general case follows from this one. To get Theorem (2) we do sucessive small deformations each one reducing the number of irreducible components of the exceptional set until there are none.

Let $\omega : \mathcal{V} \to Q$ be a versal C^∞ trivial deformation as in Corollary (4) and Q is such that the conditions described before Proposition (5) are satisfied.

Let $A_q = \bigcup_{i=1}^r A_{q,i}$ be the reduced exceptional set of $V_q = \omega^{-1}(q)$ and $A_{q,i}$ its irreducible components. We will call $A_{q_0,1}$ the semiregular curve A_1 of V_{q_0} described in Proposition (4).

From Corollary (4) follows the existence of an element $\mu \in H^0(\mathcal{V}, \omega_{\mathcal{V}/Q})$ with $\mu|_{A_{q_0,1}} \neq 0$. Lemma (3) asserts that there exists an $\alpha \in H^1(V, \Theta_V)$ such that:

$$\sigma'_{A_{q_0 1}}(\alpha)(\mu|_{A_{q_0,1}}) \neq 0$$

As in 2.1 pick an embedding $\gamma : \Delta \to Q$ such that the Kodaira-Spencer map for the induced family satisfies: $\rho_0(\dfrac{\partial}{\partial t}) = \alpha$, (this is possible since Q is the base of a versal deformation described in Theorem (F)). Let $I : V \times \Delta \to \mathcal{V}$ be a diffeomorphism preserving the fibering. Then by Proposition (3) for $t \neq 0$:

$$\int_{I(\mathbf{A}_{q_0,1} \times t)} \mu_t \neq 0 \tag{8}$$

where $\mu_t = \mu|_{V_t}$.

Proposition (5) and Lemma (4) states that:

$$F_t \subset F_0$$

Hence, $dim F_t \leq dim F_0 = r$. Suppose we have equality for $t \neq 0$ sufficiently small then $[I(A_{q_0,1} \times t)] = \sum_{i=1}^r a_i [A_{t,i}]$, $a_i \in \mathbf{Q}$, but since:

$$\int_{\mathbf{A}_{t,i}} \mu_t = 0$$

that would contradict (8). From this follows:

$$dimF_t < dimF_0 \tag{9}$$

Pick V_t for any sufficiently small $t \neq 0$, by (9) and Lemma (4) the number of irreducible components of its exceptional set is strictly less than for V, completing the inductive step.

4. An Application to Contact Structures

First we recall what are contact structures on a three dimensional manifold M, these are plane-fields on M that are completely nonintegrable. Therefore a potential class of examples are the compact smooth level sets of s.p.s.h. functions of a Stein surfaces X, where the plane-field is given by the J-invariant 2-planes in T_M, where J is the almost complex structure of X. This plane-field is not integrable because no analytic curve can have an open subset of it contained in a level set of a s.p.s.h function. We also recall:

Definition. *Two contact structures on M are diffeomorphic if there is a diffeomorphism that sends one hyperplane-field to the other hyperplane-field. Two contact structures on M are isotopic if they are connected by a one-parameter family of diffeomorphic contact structures.*

Second we will reply to a question formulated by Eliashberg to the first author. Let M be a compact three dimensional manifold. Let S (S') be the set of isotopy classes of contact structures on M that come from seeing M as a level set of a s.p.s.h. function on a smooth Stein surface (on a Stein surface with isolated singularities not in M). The question was: are the sets S and S' distinct? The answer is no, as the following theorem shows:

Theorem (5). *Let M be a compact three dimensional manifold. Let S (S') be the set of isotopy classes of contact structures on M that come from seeing M as a level set of a s.p.s.h. function on a smooth Stein surface (on a Stein surface with isolated singularities not in M). The sets S and S' are the same.*

Proof. The set S' is the same as the set of isotopy classes of contact structures on M that come from the canonical contact structures of level sets of s.p.s.h functions of r.s.p.c surfaces since the resolution of the singularities preserves M and the complex structure near it (we suppose M not passing through the singularities of the surface).

Theorem (2') says that a small deformation of a r.s.p.c. surface is Stein. Therefore we always have a one parameter family of contact structures connecting an element of S' to an element of S, since in the deformation the level set defining M remains a level set of a s.p.s.h function. The result then follows from the theorem of Gray, [Gr], which says that if two contact structures are connected by a one parameter family of contact structures then they are isotopic.

For this and some questions on CR-structures see [B1].

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Bogomolov, F., COURANT INSTITUTE MATHEMATICAL SCIENCES, New York, NY 10012

E-mail address: bogomolo@cims.nyu.edu

de Oliveira, B., DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, New York, NY 10027

E-mail address: oliveira@math.columbia.edu