Rational Parking Functions

Drew Armstrong *et al.*

University of Miami
www.math.miami.edu/∼armstrong

December 9, 2012

Rational Catalan Numbers

CONVENTION

Given $x \in \mathbb{Q} \setminus [-1, 0]$, there exist **unique coprime** $(a, b) \in \mathbb{N}^2$ such that

$$x = \frac{a}{b-a}.$$

We will always identify $x \leftrightarrow (a, b)$.

Definition

For each $x \in \mathbb{Q} \setminus [-1, 0]$ we define the **Catalan number**:

$$\mathrm{Cat}(x) = \mathrm{Cat}(a, b) := \frac{1}{a+b}\binom{a+b}{a, b} = \frac{(a+b-1)!}{a!\,b!}.$$

Rational Catalan Numbers

CONVENTION

Given $x \in \mathbb{Q} \setminus [-1, 0]$, there exist **unique coprime** $(a, b) \in \mathbb{N}^2$ such that

$$x = \frac{a}{b-a}.$$

We will always identify $x \leftrightarrow (a, b)$.

Definition

For each $x \in \mathbb{Q} \setminus [-1, 0]$ we define the **Catalan number**:

$$\mathrm{Cat}(x) = \mathrm{Cat}(a, b) := \frac{1}{a+b}\binom{a+b}{a, b} = \frac{(a+b-1)!}{a!b!}.$$

Special cases

When $b = 1 \bmod a \ldots$

- *Eugène Charles Catalan (1814-1894)*

  $(a, b) = (n, n+1)$ gives the **good old Catalan number**:

$$\mathrm{Cat}(n) = \mathrm{Cat}\left(\frac{n}{(n+1)-n}\right) = \frac{1}{2n+1}\binom{2n+1}{n}.$$

- *Nicolaus Fuss (1755-1826)*

  $(a, b) = (n, kn+1)$ gives the **Fuss-Catalan number**:

$$\mathrm{Cat}\left(\frac{n}{(kn+1)-n}\right) = \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n}.$$

Special cases

When $b = 1 \bmod a \ldots$

- *Eugène Charles Catalan (1814-1894)*

  $(a, b) = (n, n+1)$ gives the **good old Catalan number**:

$$\mathrm{Cat}(n) = \mathrm{Cat}\left(\frac{n}{(n+1)-n}\right) = \frac{1}{2n+1}\binom{2n+1}{n}.$$

- *Nicolaus Fuss (1755-1826)*

  $(a, b) = (n, kn+1)$ gives the **Fuss-Catalan number**:

$$\mathrm{Cat}\left(\frac{n}{(kn+1)-n}\right) = \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n}.$$

Special cases

When $b = 1 \bmod a \ldots$

- *Eugène Charles Catalan (1814-1894)*

  $(a, b) = (n, n+1)$ gives the **good old Catalan number**:

$$\mathrm{Cat}(n) = \mathrm{Cat}\left(\frac{n}{(n+1)-n}\right) = \frac{1}{2n+1}\binom{2n+1}{n}.$$

- *Nicolaus Fuss (1755-1826)*

  $(a, b) = (n, kn+1)$ gives the **Fuss-Catalan number**:

$$\mathrm{Cat}\left(\frac{n}{(kn+1)-n}\right) = \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n}.$$

Symmetry

Definition

By definition we have $\mathrm{Cat}(a, b) = \mathrm{Cat}(b, a)$, which translates to

$$\mathrm{Cat}(x) = \mathrm{Cat}(-x-1)$$

(i.e. symmetry about $x = -1/2$), which implies that

$$\mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

We call this the **derived Catalan number**:

$$\mathrm{Cat}'(x) := \mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

Symmetry

Definition

By definition we have $\mathrm{Cat}(a, b) = \mathrm{Cat}(b, a)$, which translates to

$$\mathrm{Cat}(x) = \mathrm{Cat}(-x-1)$$

(i.e. symmetry about $x = -1/2$), which implies that

$$\mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

We call this the **derived Catalan number**:

$$\mathrm{Cat}'(x) := \mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

Symmetry

Definition

By definition we have $\mathrm{Cat}(a, b) = \mathrm{Cat}(b, a)$, which translates to

$$\mathrm{Cat}(x) = \mathrm{Cat}(-x-1)$$

(i.e. symmetry about $x = -1/2$), which implies that

$$\mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

We call this the **derived Catalan number**:

$$\mathrm{Cat}'(x) := \mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

Symmetry

Definition

By definition we have $\mathrm{Cat}(a, b) = \mathrm{Cat}(b, a)$, which translates to

$$\mathrm{Cat}(x) = \mathrm{Cat}(-x - 1)$$

(i.e. symmetry about $x = -1/2$), which implies that

$$\mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

We call this the **derived Catalan number**:

$$\mathrm{Cat}'(x) := \mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

Euclidean Algorithm

Observation

The process $\mathrm{Cat}(x) \mapsto \mathrm{Cat}'(x) \mapsto \mathrm{Cat}''(x) \mapsto \cdots$ is a categorification of the Euclidean algorithm.

Example: $x = 5/3$ and $(a, b) = (5, 8)$

Subtract the smaller from the larger:

$$\begin{aligned} \mathrm{Cat}(5,8) &= 99, \\ \mathrm{Cat}'(5,8) &= \mathrm{Cat}(3,5) = 7, \\ \mathrm{Cat}''(5,8) &= \mathrm{Cat}'(3,5) = \mathrm{Cat}(2,3) = 2, \\ \mathrm{Cat}'''(5,8) &= \mathrm{Cat}''(3,5) = \mathrm{Cat}'(2,3) = \mathrm{Cat}(1,2) = 1 \quad \text{(STOP)} \end{aligned}$$

Euclidean Algorithm

Observation

The process $\mathrm{Cat}(x) \mapsto \mathrm{Cat}'(x) \mapsto \mathrm{Cat}''(x) \mapsto \cdots$ is a categorification of the Euclidean algorithm.

Example: $x = 5/3$ and $(a, b) = (5, 8)$

Subtract the smaller from the larger:

$$\begin{aligned} \mathrm{Cat}(5,8) &= 99, \\ \mathrm{Cat}'(5,8) &= \mathrm{Cat}(3,5) = 7, \\ \mathrm{Cat}''(5,8) &= \mathrm{Cat}'(3,5) = \mathrm{Cat}(2,3) = 2, \\ \mathrm{Cat}'''(5,8) &= \mathrm{Cat}''(3,5) = \mathrm{Cat}'(2,3) = \mathrm{Cat}(1,2) = 1 \quad \text{(STOP)} \end{aligned}$$

Euclidean Algorithm

Observation

The process $\mathrm{Cat}(x) \mapsto \mathrm{Cat}'(x) \mapsto \mathrm{Cat}''(x) \mapsto \cdots$ is a categorification of the Euclidean algorithm.

Example: $x = 5/3$ and $(a, b) = (5, 8)$

Subtract the smaller from the larger:

$$\begin{aligned}
\mathrm{Cat}(5,8) &= 99, \\
\mathrm{Cat}'(5,8) &= \mathrm{Cat}(3,5) = 7, \\
\mathrm{Cat}''(5,8) &= \mathrm{Cat}'(3,5) = \mathrm{Cat}(2,3) = 2, \\
\mathrm{Cat}'''(5,8) &= \mathrm{Cat}''(3,5) = \mathrm{Cat}'(2,3) = \mathrm{Cat}(1,2) = 1 \quad \text{(STOP)}
\end{aligned}$$

Euclidean Algorithm

Observation

The process $\mathrm{Cat}(x) \mapsto \mathrm{Cat}'(x) \mapsto \mathrm{Cat}''(x) \mapsto \cdots$ is a categorification of the Euclidean algorithm.

Example: $x = 5/3$ and $(a, b) = (5, 8)$

Subtract the smaller from the larger:

$$
\begin{aligned}
\mathrm{Cat}(5,8) &= 99, \\
\mathrm{Cat}'(5,8) &= \mathrm{Cat}(3,5) = 7, \\
\mathrm{Cat}''(5,8) &= \mathrm{Cat}'(3,5) = \mathrm{Cat}(2,3) = 2, \\
\mathrm{Cat}'''(5,8) &= \mathrm{Cat}''(3,5) = \mathrm{Cat}'(2,3) = \mathrm{Cat}(1,2) = 1 \quad \text{(STOP)}
\end{aligned}
$$

How to put it in Sloane's OEIS

Suggestion

The **Calkin-Wilf sequence** is defined by $q_1 = 1$ and

$$q_n := \frac{1}{2\lfloor q_{n-1} \rfloor - q_{n-1} + 1}.$$

Theorem: $(q_1, q_2, \ldots) = \mathbb{Q}_{>0}$.
Proof: See "Proofs from THE BOOK", Chapter 17.

Study the function $n \mapsto \mathrm{Cat}(q_n)$.

q	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{3}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{3}{1}$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{5}$	$\frac{5}{2}$	$\frac{2}{5}$	$\frac{5}{3}$	$\cdots$
$\mathrm{Cat}(q)$	1	1	2	1	7	3	5	1	30	15	66	4	99	$\cdots$

How to put it in Sloane's OEIS

Suggestion

The **Calkin-Wilf sequence** is defined by $q_1 = 1$ and

$$q_n := \frac{1}{2\lfloor q_{n-1} \rfloor - q_{n-1} + 1}.$$

Theorem: $(q_1, q_2, \ldots) = \mathbb{Q}_{>0}$.
Proof: See "Proofs from THE BOOK", Chapter 17.

Study the function $n \mapsto \mathrm{Cat}(q_n)$.

q	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{3}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{3}{1}$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{5}$	$\frac{5}{2}$	$\frac{2}{5}$	$\frac{5}{3}$	$\cdots$
$\mathrm{Cat}(q)$	1	1	2	1	7	3	5	1	30	15	66	4	99	$\cdots$

How to put it in Sloane's OEIS

Suggestion

The **Calkin-Wilf sequence** is defined by $q_1 = 1$ and

$$q_n := \frac{1}{2\lfloor q_{n-1} \rfloor - q_{n-1} + 1}.$$

Theorem: $(q_1, q_2, \ldots) = \mathbb{Q}_{>0}$.
Proof: See "Proofs from THE BOOK", Chapter 17.

Study the function $n \mapsto \mathrm{Cat}(q_n)$.

q	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{3}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{3}{1}$	$\frac{1}{4}$	$\frac{4}{3}$	$\frac{3}{5}$	$\frac{5}{2}$	$\frac{2}{5}$	$\frac{5}{3}$	$\cdots$
$\mathrm{Cat}(q)$	1	1	2	1	7	3	5	1	30	15	66	4	99	$\cdots$

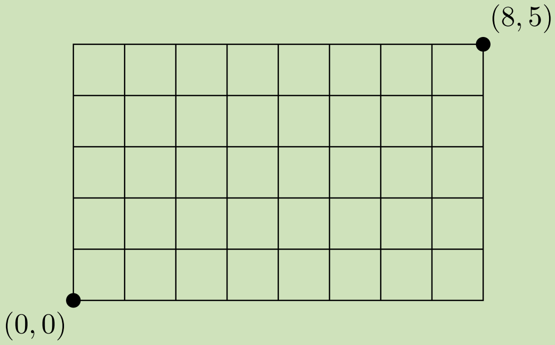
Pause

Well, that was fun.

The Prototype: Actuarial Science

- Consider the “Dyck paths” in an $a \times b$ rectangle.

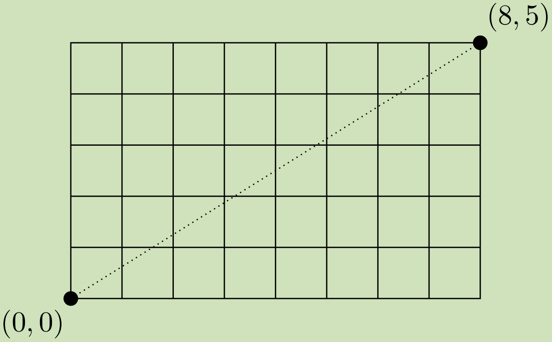
Example $(a, b) = (5, 8)$



The Prototype: Actuarial Science

- Again let $x = a/(b-a)$ with a, b positive and coprime.

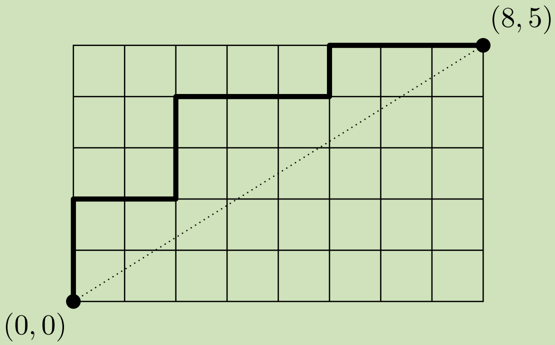
Example $(a, b) = (5, 8)$



The Prototype: Actuarial Science

- Let $\mathcal{D}(x) = \mathcal{D}(a, b)$ denote the set of Dyck paths.

Example $(a, b) = (5, 8)$



The Prototype: Actuarial Science

Theorem (Grossman 1950, Bizley 1954)

The number of Dyck paths is the Catalan number:

$$|\mathcal{D}(x)| = \mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a,b}.$$

- *Claimed by Grossman (1950), "Fun with lattice points, part 22".*
- *Proved by Bizley (1954), in Journal of the Institute of Actuaries.*
- *Proof: Break $\binom{a+b}{a,b}$ lattice paths into cyclic orbits of size $a+b$. Each orbit contains a unique Dyck path.*

The Prototype: Actuarial Science

Theorem (Grossman 1950, Bizley 1954)

The number of Dyck paths is the Catalan number:

$$|\mathcal{D}(x)| = \mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a,b}.$$

- *Claimed by Grossman (1950), "Fun with lattice points, part 22".*
- *Proved by Bizley (1954), in Journal of the Institute of Actuaries.*
- *Proof: Break $\binom{a+b}{a,b}$ lattice paths into cyclic orbits of size $a+b$. Each orbit contains a unique Dyck path.*

The Prototype: Actuarial Science

Theorem (Grossman 1950, Bizley 1954)

The number of Dyck paths is the Catalan number:

$$|\mathcal{D}(x)| = \mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a, b}.$$

- *Claimed by Grossman (1950), "Fun with lattice points, part 22".*
- *Proved by Bizley (1954), in Journal of the Institute of Actuaries.*
- *Proof: Break $\binom{a+b}{a,b}$ lattice paths into cyclic orbits of size $a+b$. Each orbit contains a unique Dyck path.*

The Prototype: Actuarial Science

Theorem (Grossman 1950, Bizley 1954)

The number of Dyck paths is the Catalan number:

$$|\mathcal{D}(x)| = \mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a, b}.$$

- *Claimed by Grossman (1950), "Fun with lattice points, part 22".*
- *Proved by Bizley (1954), in Journal of the Institute of Actuaries.*
- *Proof: Break $\binom{a+b}{a,b}$ lattice paths into cyclic orbits of size $a+b$. Each orbit contains a unique Dyck path.*

The Prototype: Actuarial Science

Theorem (Grossman 1950, Bizley 1954)

The number of Dyck paths is the Catalan number:

$$|\mathcal{D}(x)| = \mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a,b}.$$

- *Claimed by Grossman (1950), "Fun with lattice points, part 22".*
- *Proved by Bizley (1954), in Journal of the Institute of Actuaries.*
- *Proof: Break $\binom{a+b}{a,b}$ lattice paths into cyclic orbits of size $a+b$. Each orbit contains a unique Dyck path.*

The Prototype: Actuarial Science

Theorem (Armstrong 2010, Loehr 2010)

- *The number of Dyck paths with k vertical runs equals*

$$\mathrm{Nar}(x;k) := \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}.$$

Call these the **Narayana numbers**.

- *And the number with r_j vertical runs of length j equals*

$$\mathrm{Krew}(x;\mathbf{r}) := \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a} = \frac{(b-1)!}{r_0! r_1! \cdots r_a!}.$$

Call these the **Kreweras numbers**.

The Prototype: Actuarial Science

Theorem (Armstrong 2010, Loehr 2010)

- *The number of Dyck paths with k vertical runs equals*

$$\mathrm{Nar}(x;k) := \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}.$$

Call these the **Narayana numbers**.

- *And the number with r_j vertical runs of length j equals*

$$\mathrm{Krew}(x;\mathbf{r}) := \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a} = \frac{(b-1)!}{r_0! r_1! \cdots r_a!}.$$

Call these the **Kreweras numbers**.

The Prototype: Actuarial Science

Theorem (Armstrong 2010, Loehr 2010)

- *The number of Dyck paths with k vertical runs equals*

$$\mathrm{Nar}(x;k) := \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}.$$

Call these the **Narayana numbers***.*

- *And the number with r_j vertical runs of length j equals*

$$\mathrm{Krew}(x;\mathbf{r}) := \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a} = \frac{(b-1)!}{r_0! r_1! \cdots r_a!}.$$

Call these the **Kreweras numbers***.*

Motivation: Core Partitions

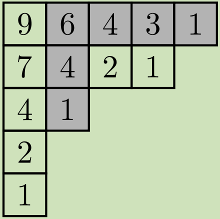
Definition

Let $\lambda \vdash n$ be an integer partition of "size" n.

- Say λ is a p-core if it has **no cell with hook length** p.
- Say λ is an (a, b)-core if it has **no cell with hook length** a **or** b.

Example

The partition $(5, 4, 2, 1, 1) \vdash 13$ is a $(5, 8)$-core.



Motivation: Core Partitions

Theorem (Anderson 2002)

The number of (a, b)-cores (of any size) is finite if and only if (a, b) are coprime, in which case they are counted by the Catalan number

$$\mathrm{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a, b}.$$

Theorem (Olsson-Stanton 2005, Vandehey 2008)

For (a, b) coprime $\exists$ unique largest (a, b)-core of size $\frac{(a^2-1)(b^2-1)}{24}$, which contains all others as subdiagrams.

Suggestion

Study Young's lattice restricted to (a, b)-cores.

Motivation: Core Partitions

Theorem (Anderson 2002)

The number of (a, b)-cores (of any size) is finite if and only if (a, b) are coprime, in which case they are counted by the Catalan number

$$\mathrm{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a, b}.$$

Theorem (Olsson-Stanton 2005, Vandehey 2008)

For (a, b) coprime $\exists$ unique largest (a, b)-core of size $\frac{(a^2-1)(b^2-1)}{24}$, which contains all others as subdiagrams.

Suggestion

Study Young's lattice restricted to (a, b)-cores.

Motivation: Core Partitions

Theorem (Anderson 2002)

The number of (a, b)-cores (of any size) is finite if and only if (a, b) are coprime, in which case they are counted by the Catalan number

$$\text{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a, b}.$$

Theorem (Olsson-Stanton 2005, Vandehey 2008)

For (a, b) coprime $\exists$ unique largest (a, b)-core of size $\frac{(a^2-1)(b^2-1)}{24}$, which contains all others as subdiagrams.

Suggestion

Study Young's lattice restricted to (a, b)-cores.

Motivation: Core Partitions

Theorem (Anderson 2002)

The number of (a,b)-cores (of any size) is finite if and only if (a,b) are coprime, in which case they are counted by the Catalan number

$$\mathrm{Cat}(a,b) = \frac{1}{a+b}\binom{a+b}{a,b}.$$

Theorem (Olsson-Stanton 2005, Vandehey 2008)

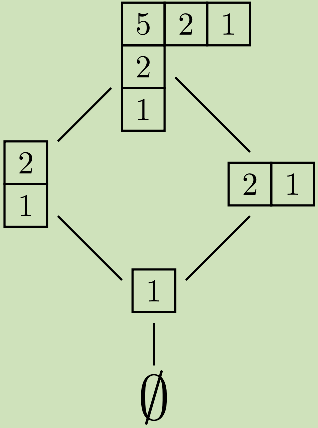
For (a,b) coprime $\exists$ unique largest (a,b)-core of size $\frac{(a^2-1)(b^2-1)}{24}$, which contains all others as subdiagrams.

Suggestion

Study Young's lattice restricted to (a,b)-cores.

Motivation: Core Partitions

Example: The poset of $(3,4)$-cores.



Motivation: Core Partitions

Theorem (Ford-Mai-Sze 2009)

For a, b coprime, the number of self-conjugate (a, b)-cores is $\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor}$.

Note: Beautiful bijective proof! (omitted)

Observation/Problem

$$\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor} = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q \Bigg|_{q=-1}$$

Conjecture (Armstrong 2011)

The average size of an (a, b)-core and the average size of a self-conjugate (a, b)-core are **both equal** *to* $\frac{(a+b+1)(a-1)(b-1)}{24}$.

Motivation: Core Partitions

Theorem (Ford-Mai-Sze 2009)

For a, b coprime, the number of self-conjugate (a, b)-cores is $\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor}$.

Note: Beautiful bijective proof! (omitted)

Observation/Problem

$$\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor} = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q \Bigg|_{q=-1}$$

Conjecture (Armstrong 2011)

The average size of an (a, b)-core and the average size of a self-conjugate (a, b)-core are **both equal** *to* $\frac{(a+b+1)(a-1)(b-1)}{24}$.

Motivation: Core Partitions

Theorem (Ford-Mai-Sze 2009)

For a, b coprime, the number of self-conjugate (a, b)-cores is $\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor}$.

Note: Beautiful bijective proof! (omitted)

Observation/Problem

$$\binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor, \lfloor \frac{b}{2} \rfloor} = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q \Bigg|_{q=-1}$$

Conjecture (Armstrong 2011)

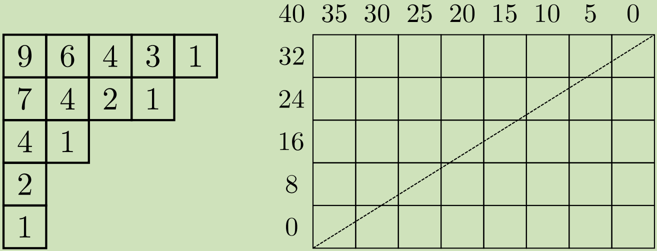
The average size of an (a, b)-core and the average size of a self-conjugate (a, b)-core are **both equal** *to* $\frac{(a+b+1)(a-1)(b-1)}{24}$.

Anderson's Beautiful Proof

Proof.

Bijection: (a, b)-cores $\leftrightarrow$ Dyck paths in $a \times b$ rectangle □

Example (The $(5, 8)$-core from earlier.)



Anderson's Beautiful Proof

Proof.

Bijection: (a, b)-cores $\leftrightarrow$ Dyck paths in $a \times b$ rectangle □

Example (Label the rectangle cells by "height".)

9	6	4	3	1
7	4	2	1	
4	1			
2				
1				

40	35	30	25	20	15	10	5	0
32	27	22	17	12	7	2		
24	19	14	9	4				
16	11	6	1					
8	3							
0								

Anderson's Beautiful Proof

Proof.

Bijection: (a, b)-cores $\leftrightarrow$ Dyck paths in $a \times b$ rectangle $\square$

Example (Label the first column hook lengths.)

9	6	4	3	1
7	4	2	1	
4	1			
2				
1				

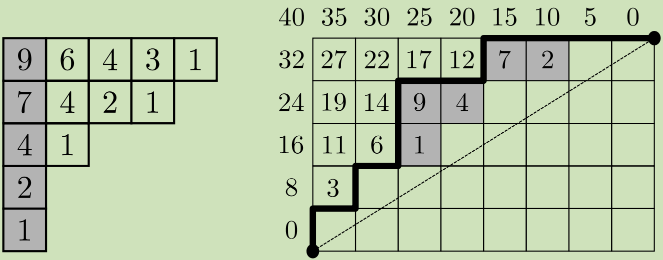
40	35	30	25	20	15	10	5	0
32	27	22	17	12	7	2		
24	19	14	9	4				
16	11	6	1					
8	3							
0								

Anderson's Beautiful Proof

Proof.

Bijection: (a, b)-cores $\leftrightarrow$ Dyck paths in $a \times b$ rectangle $\square$

Example (Voila!)

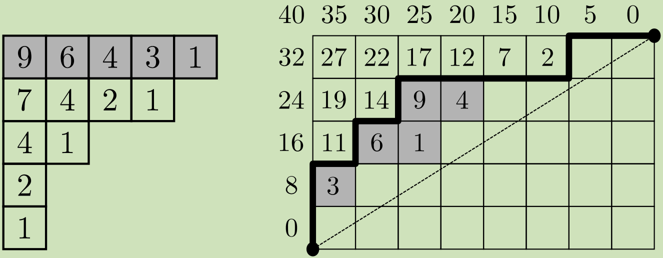


Anderson's Beautiful Proof

Proof.

Bijection: (a, b)-cores $\leftrightarrow$ Dyck paths in $a \times b$ rectangle $\square$

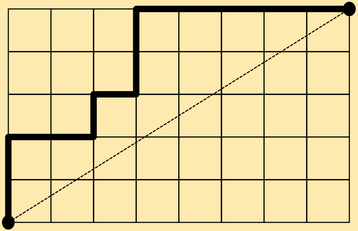
Example (Observe: Conjugation is a bit strange.)



Rational Parking Functions

Definition

- Label the up-steps by $\{1, 2, \ldots, a\}$, increasing up columns.

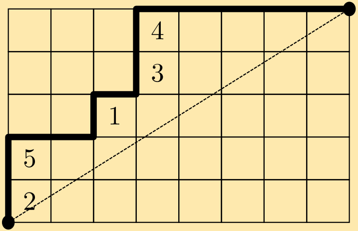


- Call this a **parking function**.
- Let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote the set of parking functions.
- **Classical form** $(z_1, z_2, \ldots, z_a)$ has label z_i in column i.
- Example: $(3, 1, 4, 4, 1)$

Rational Parking Functions

Definition

- Label the up-steps by $\{1, 2, \ldots, a\}$, increasing up columns.

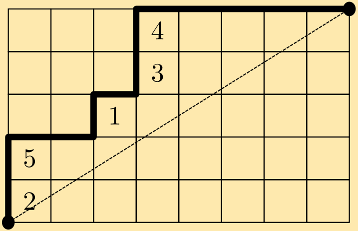


- Call this a **parking function**.
- Let $\text{PF}(x) = \text{PF}(a, b)$ denote the set of parking functions.
- **Classical form** $(z_1, z_2, \ldots, z_a)$ has label z_i in column i.
- Example: $(3, 1, 4, 4, 1)$

Rational Parking Functions

Definition

- Label the up-steps by $\{1, 2, \ldots, a\}$, increasing up columns.

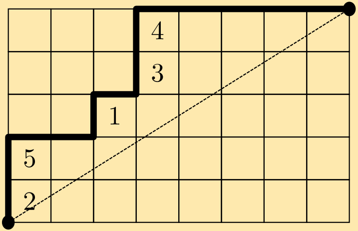


- Call this a **parking function**.
- Let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote the set of parking functions.
- **Classical form** $(z_1, z_2, \ldots, z_a)$ has label z_i in column i.
- Example: $(3, 1, 4, 4, 1)$

Rational Parking Functions

Definition

- Label the up-steps by $\{1, 2, \ldots, a\}$, increasing up columns.

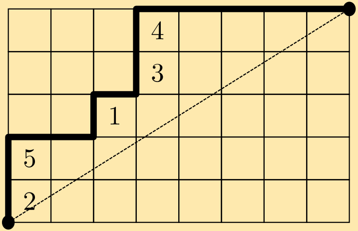


- Call this a **parking function**.
- Let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote the set of parking functions.
- **Classical form** $(z_1, z_2, \ldots, z_a)$ has label z_i in column i.
- Example: $(3, 1, 4, 4, 1)$

Rational Parking Functions

Definition

- Label the up-steps by $\{1, 2, \ldots, a\}$, increasing up columns.

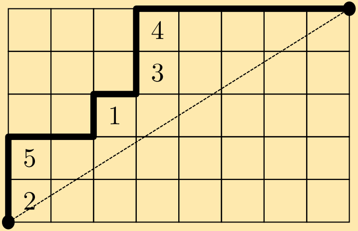


- Call this a **parking function**.
- Let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote the set of parking functions.
- **Classical form** $(z_1, z_2, \ldots, z_a)$ has label z_i in column i.
- Example: $(3, 1, 4, 4, 1)$

Rational Parking Functions

Definition

- The symmetric group $\mathfrak{S}_a$ acts on classical forms.

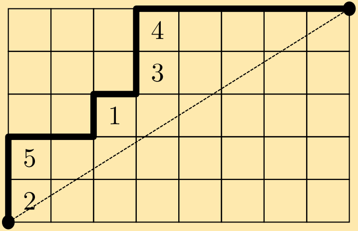


- Example: $(3, 1, 4, 4, 1)$ versus $(3, 1, 1, 4, 4)$
- By abuse, let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote this representation of $\mathfrak{S}_a$.
- Call it the **rational parking space**.

Rational Parking Functions

Definition

- The symmetric group $\mathfrak{S}_a$ acts on classical forms.

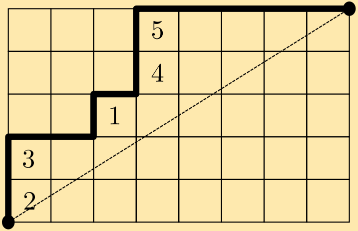


- Example: $(3, 1, 4, 4, 1)$ versus $(3, 1, 1, 4, 4)$
- By abuse, let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote this representation of $\mathfrak{S}_a$.
- Call it the **rational parking space**.

Rational Parking Functions

Definition

- The symmetric group $\mathfrak{S}_a$ acts on classical forms.

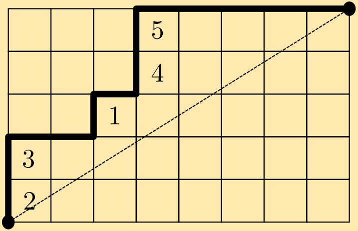


- Example: $(3, 1, 4, 4, 1)$ versus $(3, 1, 1, 4, 4)$
- By abuse, let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote this representation of $\mathfrak{S}_a$.
- Call it the **rational parking space**.

Rational Parking Functions

Definition

- The symmetric group $\mathfrak{S}_a$ acts on classical forms.

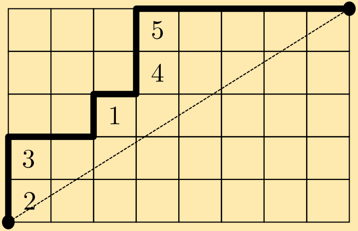


- Example: $(3, 1, {\color{red}4}, 4, {\color{red}1})$ versus $(3, 1, {\color{red}1}, 4, {\color{red}4})$
- By abuse, let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote this representation of $\mathfrak{S}_a$.
- Call it the **rational parking space**.

Rational Parking Functions

Definition

- The symmetric group $\mathfrak{S}_a$ acts on classical forms.



- Example: $(3, 1, {\color{red}4}, 4, {\color{red}1})$ versus $(3, 1, {\color{red}1}, 4, {\color{red}4})$
- By abuse, let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote this representation of $\mathfrak{S}_a$.
- Call it the **rational parking space**.

A Few Facts

Theorems (with N. Loehr and N. Williams)

- The dimension of $\mathrm{PF}(a,b)$ is b^{a-1}.
- The **complete homogeneous expansion** is

$$\mathrm{PF}(a,b) = \sum_{\mathbf{r} \vdash a} \frac{1}{b} \binom{b}{r_0, r_1, \ldots, r_a} h_{\mathbf{r}},$$

where the sum is over $\mathbf{r} = 0^{r_0} 1^{r_1} \cdots a^{r_a} \vdash a$ with $\sum_i r_i = b$.

- That is: $\mathrm{PF}(a,b)$ is the coefficient of t^a in $\frac{1}{b} H(t)^b$, where

$$H(t) = h_0 + h_1 t + h_2 t^2 + \cdots.$$

A Few Facts

Theorems (with N. Loehr and N. Williams)

- The dimension of $\mathrm{PF}(a, b)$ is b^{a-1}.
- The **complete homogeneous expansion** is

$$\mathrm{PF}(a, b) = \sum_{\mathbf{r} \vdash a} \frac{1}{b} \binom{b}{r_0, r_1, \ldots, r_a} h_{\mathbf{r}},$$

where the sum is over $\mathbf{r} = 0^{r_0} 1^{r_1} \cdots a^{r_a} \vdash a$ with $\sum_i r_i = b$.

- That is: $\mathrm{PF}(a, b)$ is the coefficient of t^a in $\frac{1}{b} H(t)^b$, where

$$H(t) = h_0 + h_1 t + h_2 t^2 + \cdots .$$

A Few Facts

Theorems (with N. Loehr and N. Williams)

- The dimension of $\mathrm{PF}(a, b)$ is b^{a-1}.
- The **complete homogeneous expansion** is

$$\mathrm{PF}(a, b) = \sum_{\mathbf{r} \vdash a} \frac{1}{b} \binom{b}{r_0, r_1, \ldots, r_a} h_{\mathbf{r}},$$

where the sum is over $\mathbf{r} = 0^{r_0} 1^{r_1} \cdots a^{r_a} \vdash a$ with $\sum_i r_i = b$.

- That is: $\mathrm{PF}(a, b)$ is the coefficient of t^a in $\frac{1}{b} H(t)^b$, where

$$H(t) = h_0 + h_1 t + h_2 t^2 + \cdots .$$

A Few Facts

Theorems (with N. Loehr and N. Williams)

- The dimension of $\mathrm{PF}(a,b)$ is b^{a-1}.
- The **complete homogeneous expansion** is

$$\mathrm{PF}(a,b) = \sum_{\mathbf{r} \vdash a} \frac{1}{b} \binom{b}{r_0, r_1, \ldots, r_a} h_{\mathbf{r}},$$

  where the sum is over $\mathbf{r} = 0^{r_0} 1^{r_1} \cdots a^{r_a} \vdash a$ with $\sum_i r_i = b$.

- That is: $\mathrm{PF}(a,b)$ is the coefficient of t^a in $\frac{1}{b} H(t)^b$, where

$$H(t) = h_0 + h_1 t + h_2 t^2 + \cdots .$$

A Few Facts

Theorems (with N. Loehr and N. Williams)

Then using the Cauchy product identity we get...

- The **power sum expansion** is

$$\mathrm{PF}(a,b) = \sum_{\mathbf{r} \vdash a} b^{\ell(\mathbf{r})-1} \frac{p_{\mathbf{r}}}{z_{\mathbf{r}}}$$

i.e. the # of parking functions fixed by $\sigma \in \mathfrak{S}_a$ is $b^{\#\mathrm{cycles}(\sigma)-1}$.

- The **Schur expansion** is

$$\mathrm{PF}(a,b) = \sum_{\mathbf{r} \vdash a} \frac{1}{b} s_{\mathbf{r}}(1^b)\, s_{\mathbf{r}}.$$

A Few Facts

Theorems (with N. Loehr and N. Williams)

Then using the Cauchy product identity we get...

- The **power sum expansion** is

$$\mathrm{PF}(a,b) = \sum_{\mathbf{r} \vdash a} b^{\ell(\mathbf{r})-1} \frac{p_{\mathbf{r}}}{z_{\mathbf{r}}}$$

  i.e. the # of parking functions fixed by $\sigma \in \mathfrak{S}_a$ is $b^{\#\text{cycles}(\sigma)-1}$.

- The **Schur expansion** is

$$\mathrm{PF}(a,b) = \sum_{\mathbf{r} \vdash a} \frac{1}{b} s_{\mathbf{r}}(1^b)\, s_{\mathbf{r}}.$$

A Few Facts

Theorems (with N. Loehr and N. Williams)

Then using the Cauchy product identity we get...

- The **power sum expansion** is

$$\mathrm{PF}(a,b) = \sum_{\mathbf{r} \vdash a} b^{\ell(\mathbf{r})-1} \frac{p_{\mathbf{r}}}{z_{\mathbf{r}}}$$

  i.e. the # of parking functions fixed by $\sigma \in \mathfrak{S}_a$ is $b^{\#\text{cycles}(\sigma)-1}$.

- The **Schur expansion** is

$$\mathrm{PF}(a,b) = \sum_{\mathbf{r} \vdash a} \frac{1}{b} s_{\mathbf{r}}(1^b)\, s_{\mathbf{r}}.$$

A Few Facts

Observation/Definition

The multiplicities of the **hook Schur functions** $s[k+1, 1^{a-k-1}]$ in $\mathrm{PF}(a, b)$ are given by the **Schröder numbers**

$$\mathrm{Schrö}(a, b; k) := \frac{1}{b} s_{[k+1,1^{a-k-1}]}(1^b) = \frac{1}{b}\binom{a-1}{k}\binom{b+k}{a}.$$

Special Cases:

- Trivial character: $\mathrm{Schrö}(a, b; a-1) = \mathrm{Cat}(a, b)$.
- Smallest k that occurs is $k = \max\{0, a-b\}$, in which case

$$\mathrm{Schrö}(a, b; k) = \mathrm{Cat}'(a, b).$$

- Hence $\mathrm{Schrö}(x; k)$ interpolates between $\mathrm{Cat}(x)$ and $\mathrm{Cat}'(x)$.

A Few Facts

Observation/Definition

The multiplicities of the **hook Schur functions** $s[k+1, 1^{a-k-1}]$ in $\mathrm{PF}(a, b)$ are given by the **Schröder numbers**

$$\mathrm{Schrö}(a, b; k) := \frac{1}{b} s_{[k+1,1^{a-k-1}]}(1^b) = \frac{1}{b}\binom{a-1}{k}\binom{b+k}{a}.$$

Special Cases:

- Trivial character: $\mathrm{Schrö}(a, b; a-1) = \mathrm{Cat}(a, b)$.
- Smallest k that occurs is $k = \max\{0, a-b\}$, in which case

$$\mathrm{Schrö}(a, b; k) = \mathrm{Cat}'(a, b).$$

- Hence $\mathrm{Schrö}(x; k)$ interpolates between $\mathrm{Cat}(x)$ and $\mathrm{Cat}'(x)$.

A Few Facts

Observation/Definition

The multiplicities of the **hook Schur functions** $s[k+1, 1^{a-k-1}]$ in $\mathrm{PF}(a,b)$ are given by the **Schröder numbers**

$$\mathrm{Schrö}(a,b;k) := \frac{1}{b} s_{[k+1,1^{a-k-1}]}(1^b) = \frac{1}{b}\binom{a-1}{k}\binom{b+k}{a}.$$

Special Cases:

- Trivial character: $\mathrm{Schrö}(a,b;a-1) = \mathrm{Cat}(a,b)$.
- Smallest k that occurs is $k = \max\{0, a-b\}$, in which case

$$\mathrm{Schrö}(a,b;k) = \mathrm{Cat}'(a,b).$$

- Hence $\mathrm{Schrö}(x;k)$ interpolates between $\mathrm{Cat}(x)$ and $\mathrm{Cat}'(x)$.

A Few Facts

Observation/Definition

The multiplicities of the **hook Schur functions** $s[k+1, 1^{a-k-1}]$ in $\mathrm{PF}(a, b)$ are given by the **Schröder numbers**

$$\mathrm{Schrö}(a, b; k) := \frac{1}{b} s_{[k+1,1^{a-k-1}]}(1^b) = \frac{1}{b}\binom{a-1}{k}\binom{b+k}{a}.$$

Special Cases:

- Trivial character: $\mathrm{Schrö}(a, b; a-1) = \mathrm{Cat}(a, b)$.
- Smallest k that occurs is $k = \max\{0, a-b\}$, in which case

$$\mathrm{Schrö}(a, b; k) = \mathrm{Cat}'(a, b).$$

- Hence $\mathrm{Schrö}(x; k)$ interpolates between $\mathrm{Cat}(x)$ and $\mathrm{Cat}'(x)$.

A Few Facts

Observation/Definition

The multiplicities of the **hook Schur functions** $s[k+1, 1^{a-k-1}]$ in $\mathrm{PF}(a, b)$ are given by the **Schröder numbers**

$$\mathrm{Schrö}(a, b; k) := \frac{1}{b} s_{[k+1,1^{a-k-1}]}(1^b) = \frac{1}{b}\binom{a-1}{k}\binom{b+k}{a}.$$

Special Cases:

- Trivial character: $\mathrm{Schrö}(a, b; a-1) = \mathrm{Cat}(a, b)$.
- Smallest k that occurs is $k = \max\{0, a-b\}$, in which case

$$\mathrm{Schrö}(a, b; k) = \mathrm{Cat}'(a, b).$$

- Hence $\mathrm{Schrö}(x; k)$ interpolates between $\mathrm{Cat}(x)$ and $\mathrm{Cat}'(x)$.

A Few Facts

Problem

Given a, b coprime we have an $\mathfrak{S}_a$-module $\mathrm{PF}(a, b)$ of dimension b^{a-1} and an $\mathfrak{S}_b$-module $\mathrm{PF}(b, a)$ of dimension a^{b-1}.

- What is the relationship between $\mathrm{PF}(a, b)$ and $\mathrm{PF}(b, a)$?
- Note that hook multiplicities are the same:

$$\mathrm{Schrö}(a, b; k) = \mathrm{Schrö}(b, a; k + b - a).$$

- See Eugene Gorsky, *Arc spaces and DAHA representations*, 2011.

A Few Facts

Problem

Given a, b coprime we have an $\mathfrak{S}_a$-module $\mathrm{PF}(a, b)$ of dimension b^{a-1} and an $\mathfrak{S}_b$-module $\mathrm{PF}(b, a)$ of dimension a^{b-1}.

- What is the relationship between $\mathrm{PF}(a, b)$ and $\mathrm{PF}(b, a)$?
- Note that hook multiplicities are the same:

$$\mathrm{Schrö}(a, b; k) = \mathrm{Schrö}(b, a; k + b - a).$$

- See Eugene Gorsky, *Arc spaces and DAHA representations*, 2011.

A Few Facts

Problem

Given a, b coprime we have an $\mathfrak{S}_a$-module $\mathrm{PF}(a, b)$ of dimension b^{a-1} and an $\mathfrak{S}_b$-module $\mathrm{PF}(b, a)$ of dimension a^{b-1}.

- What is the relationship between $\mathrm{PF}(a, b)$ and $\mathrm{PF}(b, a)$?
- Note that hook multiplicities are the same:

$$\mathrm{Schrö}(a, b; k) = \mathrm{Schrö}(b, a; k + b - a).$$

- See Eugene Gorsky, *Arc spaces and DAHA representations*, 2011.

A Few Facts

Problem

Given a, b coprime we have an $\mathfrak{S}_a$-module $\mathrm{PF}(a, b)$ of dimension b^{a-1} and an $\mathfrak{S}_b$-module $\mathrm{PF}(b, a)$ of dimension a^{b-1}.

- What is the relationship between $\mathrm{PF}(a, b)$ and $\mathrm{PF}(b, a)$?
- Note that hook multiplicities are the same:

$$\mathrm{Schrö}(a, b; k) = \mathrm{Schrö}(b, a; k + b - a).$$

- See Eugene Gorsky, *Arc spaces and DAHA representations*, 2011.

How about q and t?

We want a "Shuffle Conjecture"

Define a quasisymmetric function with coefficients in $\mathbb{N}[q, t]$ by

$$\mathrm{PF}_{q,t}(a, b) := \sum_{P} q^{\mathrm{qstat(P)}} t^{\mathrm{tstat(P)}} F_{\mathrm{iDes(P)}}.$$

- Sum over (a, b)-parking functions P.
- F is a fundamental (Gessel) quasisymmetric function.
  — *natural refinement of Schur functions*
- We require $\mathrm{PF}_{1,1}(a, b) = \mathrm{PF}(a, b)$.
- Must define qstat, tstat, iDes for (a, b)-parking function P.

How about q and t?

We want a "Shuffle Conjecture"

Define a quasisymmetric function with coefficients in $\mathbb{N}[q,t]$ by

$$\mathrm{PF}_{q,t}(a,b) := \sum_{P} q^{\mathrm{qstat(P)}} t^{\mathrm{tstat(P)}} F_{\mathrm{iDes(P)}}.$$

- Sum over (a,b)-parking functions P.
- F is a fundamental (Gessel) quasisymmetric function.
  — *natural refinement of Schur functions*
- We require $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- Must define qstat, tstat, iDes for (a,b)-parking function P.

How about q and t?

We want a "Shuffle Conjecture"

Define a quasisymmetric function with coefficients in $\mathbb{N}[q,t]$ by

$$\mathrm{PF}_{q,t}(a,b) := \sum_P q^{\mathrm{qstat(P)}} t^{\mathrm{tstat(P)}} F_{\mathrm{iDes(P)}}.$$

- Sum over (a,b)-parking functions P.
- F is a fundamental (Gessel) quasisymmetric function. — *natural refinement of Schur functions*
- We require $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- Must define qstat, tstat, iDes for (a,b)-parking function P.

How about q and t?

We want a "Shuffle Conjecture"

Define a quasisymmetric function with coefficients in $\mathbb{N}[q,t]$ by

$$\mathrm{PF}_{q,t}(a,b) := \sum_P q^{\mathrm{qstat(P)}} t^{\mathrm{tstat(P)}} F_{\mathrm{iDes(P)}}.$$

- Sum over (a,b)-parking functions P.
- F is a fundamental (Gessel) quasisymmetric function.
  — *natural refinement of Schur functions*
- We require $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- Must define qstat, tstat, iDes for (a,b)-parking function P.

How about q and t?

We want a "Shuffle Conjecture"

Define a quasisymmetric function with coefficients in $\mathbb{N}[q,t]$ by

$$\mathrm{PF}_{q,t}(a,b) := \sum_{P} q^{\mathrm{qstat(P)}} t^{\mathrm{tstat(P)}} F_{\mathrm{iDes(P)}}.$$

- Sum over (a,b)-parking functions P.
- F is a fundamental (Gessel) quasisymmetric function.
  — *natural refinement of Schur functions*
- We require $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- Must define qstat, tstat, iDes for (a,b)-parking function P.

How about q and t?

We want a "Shuffle Conjecture"

Define a quasisymmetric function with coefficients in $\mathbb{N}[q,t]$ by

$$\mathrm{PF}_{q,t}(a,b) := \sum_{P} q^{\mathrm{qstat(P)}} t^{\mathrm{tstat(P)}} F_{\mathrm{iDes(P)}}.$$

- Sum over (a,b)-parking functions P.
- F is a fundamental (Gessel) quasisymmetric function.
  — *natural refinement of Schur functions*
- We require $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- Must define qstat, tstat, iDes for (a,b)-parking function P.

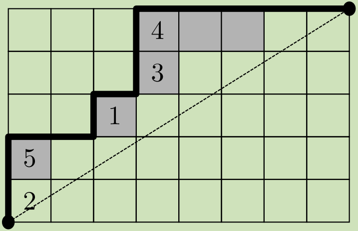
qstat is easy

Definition

- Let qstat := area := # boxes between the path and diagonal.
- Note: Maximum value of area is $(a-1)(b-1)/2$. (Frobenius) — *see Beck and Robins, Chapter 1*

Example

- This $(5, 8)$-parking function has area $= 6$.



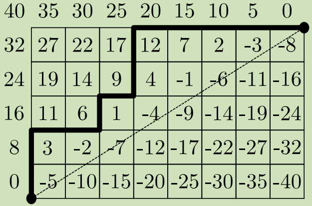
iDes is reasonable

Definition

- Read labels by increasing "height" to get permutation $\sigma \in \mathfrak{S}_a$.
- iDes := the descent set of σ^{-1}.

Example

- Remember the "height"?



- iDes $= \{1, 4\}$

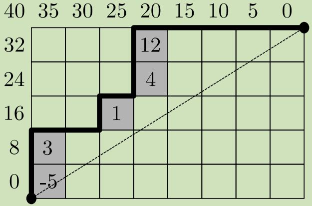
iDes is reasonable

Definition

- Read labels by increasing "height" to get permutation $\sigma \in \mathfrak{S}_a$.
- iDes := the descent set of σ^{-1}.

Example

- Look at the heights of the vertical step boxes.



- iDes $= \{1, 4\}$

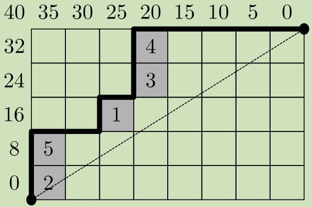
iDes is reasonable

Definition

- Read labels by increasing “height” to get permutation $\sigma \in \mathfrak{S}_a$.
- iDes $:=$ the descent set of σ^{-1}.

Example

- Remember the labels we had before.



- $\text{iDes} = \{1, 4\}$

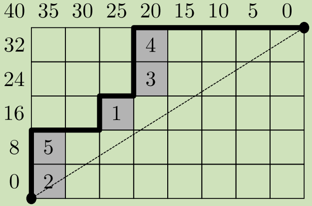
iDes is reasonable

Definition

- Read labels by increasing "height" to get permutation $\sigma \in \mathfrak{S}_a$.
- iDes := the descent set of σ^{-1}.

Example

- Read them by increasing height to get $\sigma = 2\bar{1}53\bar{4} \in \mathfrak{S}_5$.



- iDes $= \{1, 4\}$

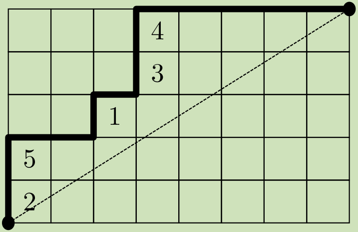
tstat is hard (as usual)

Definition

- "Blow up" the (a, b)-parking function.
- Compute "dinv" of the blowup.

Example

- Recall our favorite the $(5, 8)$-parking function.



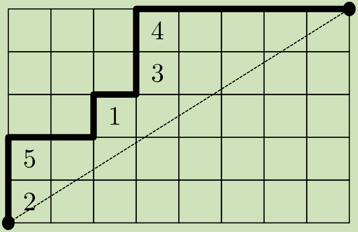
tstat is hard (as usual)

Definition

- "Blow up" the (a, b)-parking function.
- Compute "dinv" of the blowup.

Example

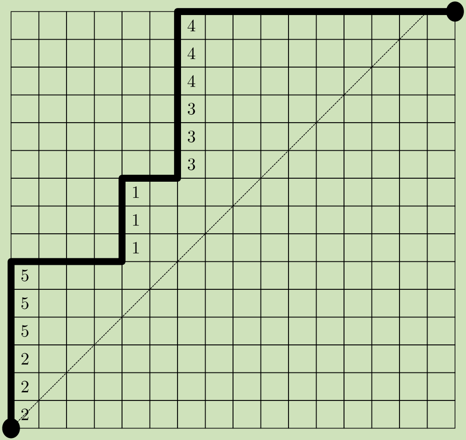
- Since $2 \cdot 8 - 3 \cdot 5 = 1$ we "blow up" by 2 horiz. and 3 vert....



tstat is hard (as usual)

Example

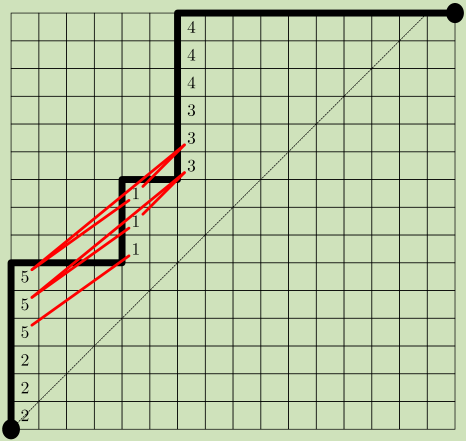
- To get this!



tstat is hard (as usual)

Example

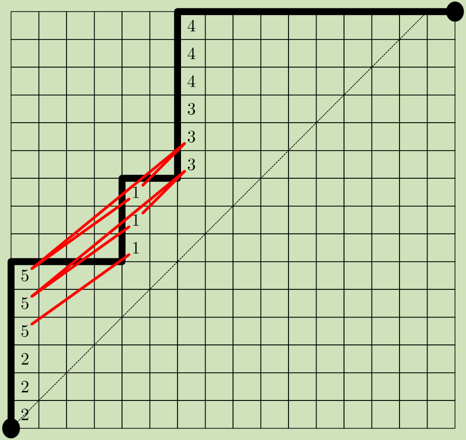
- To get this! Now compute dinv = 7.



tstat is hard (as usual)

Example

- (There's a scaling factor *depending on the path*, so tstat = 3.)



All Together

Example

- So our favorite $(5,8)$-parking function contributes $q^6 t^3 F_{\{1,4\}}$.
- Proof of Concept: The coefficient of $s[2,2,1]$ in $\mathrm{PF}_{q,t}(5,8)$ is

$$\begin{pmatrix}
 & & & & & & 1 & 1 & 1 & 1 & 1 \\
 & & & & 1 & 3 & 4 & 3 & 2 & 1 & \\
 & & & 2 & 6 & 6 & 4 & 2 & 1 & & \\
 & & 2 & 7 & 7 & 4 & 2 & 1 & & & \\
 & 1 & 6 & 7 & 4 & 2 & 1 & & & & \\
 & 3 & 6 & 4 & 2 & 1 & & & & & \\
1 & 4 & 4 & 2 & 1 & & & & & & \\
1 & 3 & 2 & 1 & & & & & & & \\
1 & 2 & 1 & & & & & & & & \\
1 & 1 & & & & & & & & & \\
1 & & & & & & & & & &
\end{pmatrix}$$

All Together

Example

- So our favorite $(5,8)$-parking function contributes $q^6 t^3 F_{\{1,4\}}$.
- Proof of Concept: The coefficient of $s[2,2,1]$ in $\mathrm{PF}_{q,t}(5,8)$ is

$$\begin{pmatrix}
 & & & & & & 1 & 1 & 1 & 1 & 1 \\
 & & & & 1 & 3 & 4 & 3 & 2 & 1 & \\
 & & & 2 & 6 & 6 & 4 & 2 & 1 & & \\
 & & 2 & 7 & 7 & 4 & 2 & 1 & & & \\
 & 1 & 6 & 7 & 4 & 2 & 1 & & & & \\
 & 3 & 6 & 4 & 2 & 1 & & & & & \\
1 & 4 & 4 & 2 & 1 & & & & & & \\
1 & 3 & 2 & 1 & & & & & & & \\
1 & 2 & 1 & & & & & & & & \\
1 & 1 & & & & & & & & & \\
1 & & & & & & & & & &
\end{pmatrix}$$

A Few Facts

Facts

- $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- $\mathrm{PF}_{q,t}(a,b)$ is symmetric and Schur-positive with coeffs $\in \mathbb{N}[q,t]$.
  — *via LLT polynomials (HHLRU Lemma 6.4.1)*
- **Experimentally:** $\mathrm{PF}_{q,t}(a,b) = \mathrm{PF}_{t,q}(a,b)$.
  — *this will be "impossible" to prove (see Loehr's Maxim)*
- **Definition:** The coefficient of the hook $s[k+1, 1^{a-k-1}]$ is the q,t-Schröder number $\mathrm{Schrö}_{q,t}(a,b;k)$.
- **Experimentally:** Specialization $t = 1/q$ gives

$$\mathrm{Schrö}_{q,\frac{1}{q}}(a,b;k) = \frac{1}{[b]_q} \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \begin{bmatrix} b+k \\ a \end{bmatrix}_q \qquad \text{(centered)}$$

A Few Facts

Facts

- $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- $\mathrm{PF}_{q,t}(a,b)$ is symmetric and Schur-positive with coeffs $\in \mathbb{N}[q,t]$.
  — *via LLT polynomials (HHLRU Lemma 6.4.1)*
- **Experimentally:** $\mathrm{PF}_{q,t}(a,b) = \mathrm{PF}_{t,q}(a,b)$.
  — *this will be "impossible" to prove (see Loehr's Maxim)*
- **Definition:** The coefficient of the hook $s[k+1, 1^{a-k-1}]$ is the q,t-Schröder number $\mathrm{Schrö}_{q,t}(a,b;k)$.
- **Experimentally:** Specialization $t = 1/q$ gives

$$\mathrm{Schrö}_{q,\frac{1}{q}}(a,b;k) = \frac{1}{[b]_q} \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \begin{bmatrix} b+k \\ a \end{bmatrix}_q \qquad \text{(centered)}$$

A Few Facts

Facts

- $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- $\mathrm{PF}_{q,t}(a,b)$ is symmetric and Schur-positive with coeffs $\in \mathbb{N}[q,t]$.
  — *via LLT polynomials (HHLRU Lemma 6.4.1)*
- **Experimentally:** $\mathrm{PF}_{q,t}(a,b) = \mathrm{PF}_{t,q}(a,b)$.
  — *this will be "impossible" to prove (see Loehr's Maxim)*
- **Definition:** The coefficient of the hook $s[k+1, 1^{a-k-1}]$ is the q,t-Schröder number $\mathrm{Schrö}_{q,t}(a,b;k)$.
- **Experimentally:** Specialization $t = 1/q$ gives

$$\mathrm{Schrö}_{q,\frac{1}{q}}(a,b;k) = \frac{1}{[b]_q} \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \begin{bmatrix} b+k \\ a \end{bmatrix}_q \qquad \text{(centered)}$$

A Few Facts

Facts

- $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- $\mathrm{PF}_{q,t}(a,b)$ is symmetric and Schur-positive with coeffs $\in \mathbb{N}[q,t]$.
  — *via LLT polynomials (HHLRU Lemma 6.4.1)*
- **Experimentally:** $\mathrm{PF}_{q,t}(a,b) = \mathrm{PF}_{t,q}(a,b)$.
  — *this will be "impossible" to prove (see Loehr's Maxim)*
- **Definition:** The coefficient of the hook $s[k+1, 1^{a-k-1}]$ is the q,t-Schröder number $\mathrm{Schrö}_{q,t}(a,b;k)$.
- **Experimentally:** Specialization $t = 1/q$ gives

$$\mathrm{Schrö}_{q,\frac{1}{q}}(a,b;k) = \frac{1}{[b]_q} \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \begin{bmatrix} b+k \\ a \end{bmatrix}_q \qquad \text{(centered)}$$

A Few Facts

Facts

- $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- $\mathrm{PF}_{q,t}(a,b)$ is symmetric and Schur-positive with coeffs $\in \mathbb{N}[q,t]$.
  — *via LLT polynomials (HHLRU Lemma 6.4.1)*
- **Experimentally:** $\mathrm{PF}_{q,t}(a,b) = \mathrm{PF}_{t,q}(a,b)$.
  — *this will be "impossible" to prove (see Loehr's Maxim)*
- **Definition:** The coefficient of the hook $s[k+1, 1^{a-k-1}]$ is the q,t-Schröder number $\mathrm{Schrö}_{q,t}(a,b;k)$.
- **Experimentally:** Specialization $t = 1/q$ gives

$$\mathrm{Schrö}_{q,\frac{1}{q}}(a,b;k) = \frac{1}{[b]_q}\begin{bmatrix} a-1 \\ k \end{bmatrix}_q \begin{bmatrix} b+k \\ a \end{bmatrix}_q \qquad \text{(centered)}$$

A Few Facts

Facts

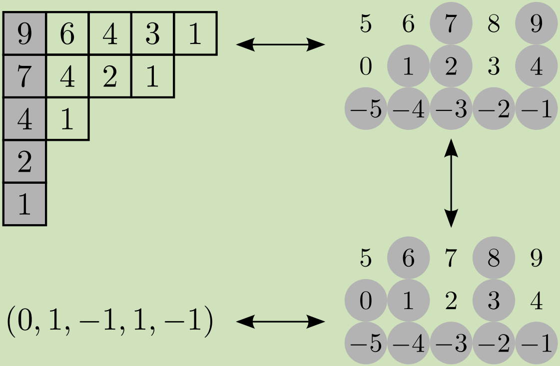
- $\mathrm{PF}_{1,1}(a,b) = \mathrm{PF}(a,b)$.
- $\mathrm{PF}_{q,t}(a,b)$ is symmetric and Schur-positive with coeffs $\in \mathbb{N}[q,t]$.
  — *via LLT polynomials (HHLRU Lemma 6.4.1)*
- **Experimentally:** $\mathrm{PF}_{q,t}(a,b) = \mathrm{PF}_{t,q}(a,b)$.
  — *this will be "impossible" to prove (see Loehr's Maxim)*
- **Definition:** The coefficient of the hook $s[k+1, 1^{a-k-1}]$ is the q,t-Schröder number $\mathrm{Schrö}_{q,t}(a,b;k)$.
- **Experimentally:** Specialization $t = 1/q$ gives

$$\mathrm{Schrö}_{q,\frac{1}{q}}(a,b;k) = \frac{1}{[b]_q} \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \begin{bmatrix} b+k \\ a \end{bmatrix}_q \qquad \text{(centered)}$$

Motivation: Lie Theory

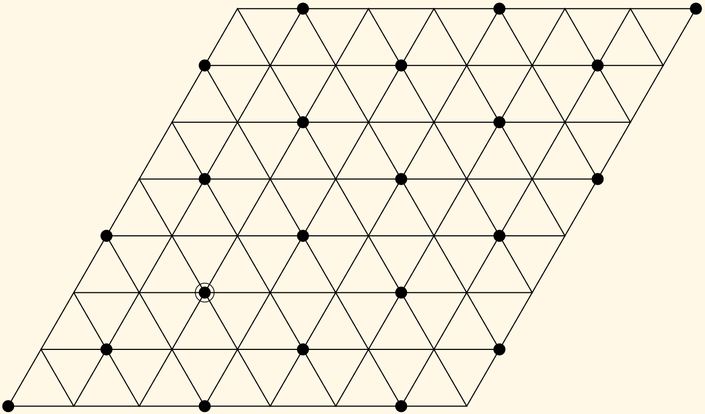
The James-Kerber Bijection

- between a-cores and the root lattice of the Weyl group $\mathfrak{S}_a$



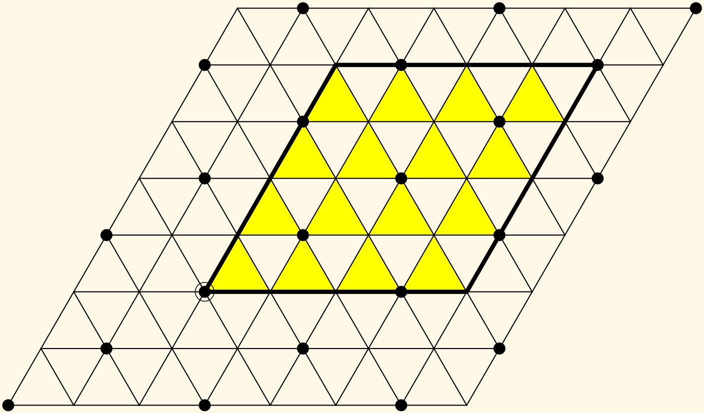
Here's The Picture

- These are the root and weight lattices $Q \subseteq \Lambda$ of $\mathfrak{S}_a$.



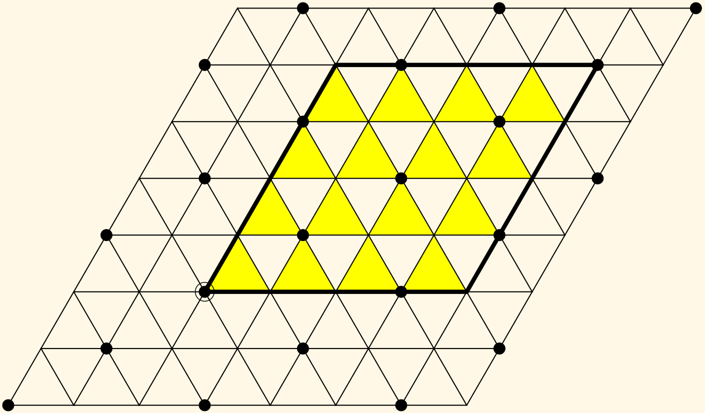
Here's The Picture

- Here is a fundamental parallelepiped for $\Lambda / b\Lambda$.



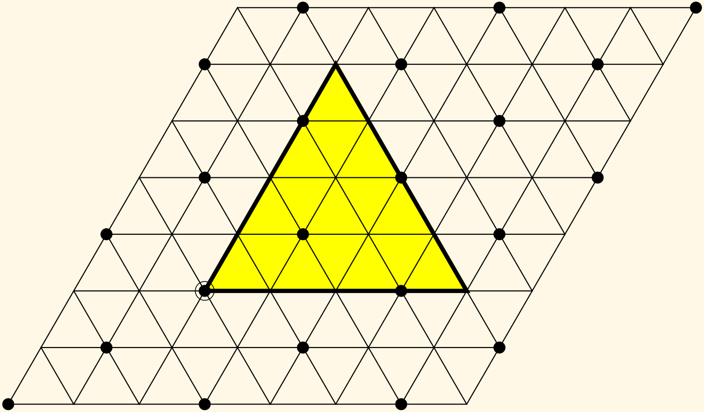
Here's The Picture

- It contains b^{a-1} elements (these are the "parking functions").



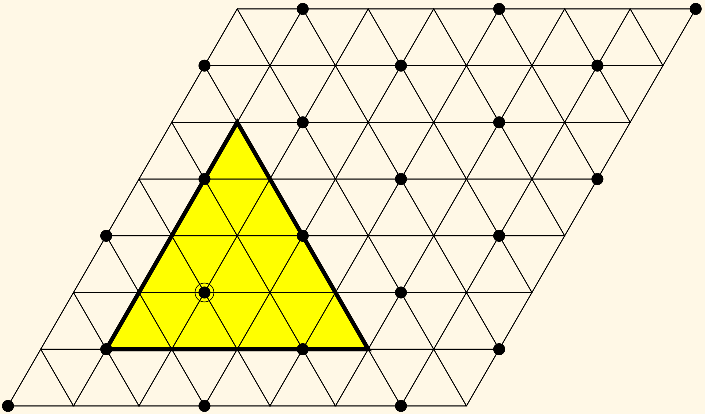
Here's The Picture

- But they look better as a simplex...



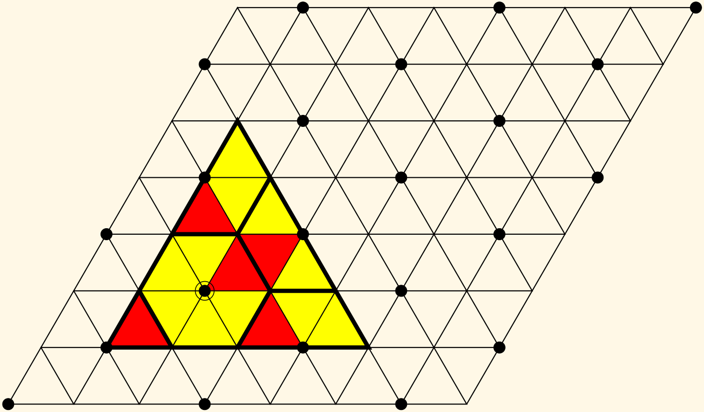
Here's The Picture

- ...which is congruent to a nicer simplex.



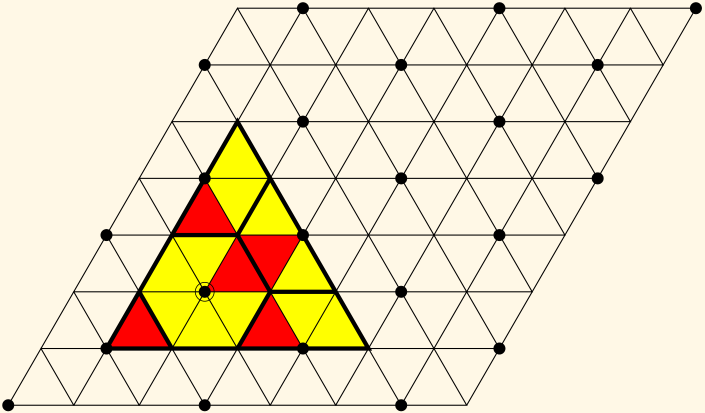
Here's The Picture

- There are $\text{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a,b}$ elements of the root lattice inside.



Here's The Picture

- These are the (*a*, *b*)-Dyck paths (via Anderson, James-Kerber).



Other Weyl Groups?

Definition

Consider a Weyl group W with Coxeter number h and let $p \in \mathbb{N}$ be **coprime** to h. We define the **Catalan number**

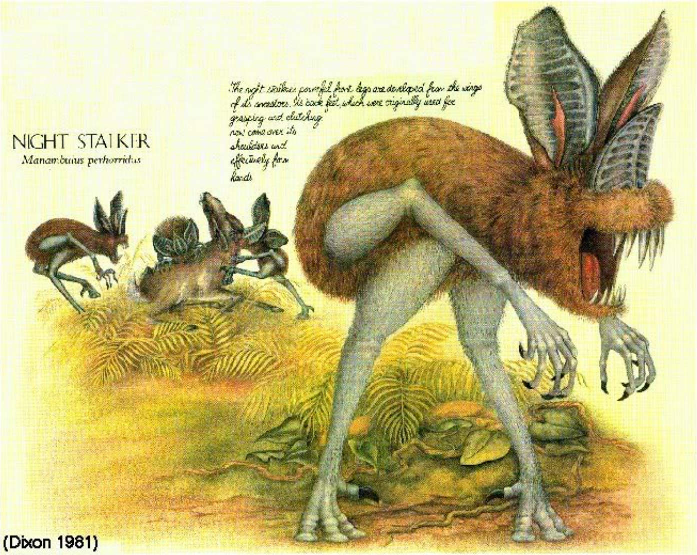
$$\mathrm{Cat}_q(W,p) := \prod_j \frac{[p+m_j]_q}{[1+m_j]_q}$$

where $e^{2\pi i m_j/h}$ are the eigenvalues of a Coxeter element.

Observation

$$\mathrm{Cat}_q(\mathfrak{S}_a, b) = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q$$

Thank You


NIGHT STALKER
Manambulus perhorridus
The night stalker's powerful front legs are developed from the wings
of its ancestors. Its back feet, which were originally used for
grasping and clutching
now come over its
shoulders and
effectively form
hands
(Dixon 1981)