RCC

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Outline of the Talk

1. The Frobenius Coin Problem
2. Rational Dyck Paths
3. Core Partitions
4. The Double Abacus
5. q-Catalan Numbers
6. What Does It Mean?

1. The Frobenius Coin Problem

1. The Frobenius Coin Problem

Frobenius Coin Problem (late 1800s). Given two natural numbers $a, b \in \mathbb{N}$, describe the monoid

$$a\mathbb{N} + b\mathbb{N} := \{ax + by : x, y \in \mathbb{N}\}.$$

We can assume that $\gcd(a, b) = 1$ since if $a = da'$ and $b = db'$ then

$$a\mathbb{N} + b\mathbb{N} = d(a'\mathbb{N} + b'\mathbb{N}).$$

Sylvester's Theorem (1882). Let $\gcd(a, b) = 1$. The set

$$\mathbb{N} - (a\mathbb{N} + b\mathbb{N})$$

of "non-representable numbers" has size $(a-1)(b-1)/2$. The largest element of the set is $ab - a - b$, called the Frobenius number.

1. The Frobenius Coin Problem

I will present a beautiful geometric proof.

For example, suppose that $(a, b) = (3, 5)$.

1. The Frobenius Coin Problem

Label each point $(x, y) \in \mathbb{Z}^2$ by the integer $ax + by \in \mathbb{Z}$.

15					30					
10					25					
5					20					
0	3	6	9	12	15	18	21	24	27	30
					10					
					5					
					0	3	6	9	12	15

1. The Frobenius Coin Problem

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1. The Frobenius Coin Problem

We observe that every integer label occurs because

$$a\mathbb{Z} + b\mathbb{Z} = \gcd(a, b)\mathbb{Z} = \mathbb{Z}.$$

15	18	21	24	27	30	33	36	39	42	45
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0	3	6	9	12	15	18	21	24	27	30
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1. The Frobenius Coin Problem

In fact, $\mathbb{Z}$ appears without redundancy in any vertical strip of width b.

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0	3	6	9	12	15	18	21	24	27	30
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0	3	6	9	12	15	18	21	24	27	30
-5	-2	1	4	7	10	13	16	19	22	25
-10	-7	-4	-1	2	5	8	11	14	17	20
-15	-12	-9	-6	-3	0	3	6	9	12	15

1. The Frobenius Coin Problem

. . . or in any horizontal strip of height *a*, etc.

15	18	21	24	27	30	33	36	39	42	45
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5	8	11	14	17	20	23	26	29	32	35
0	3	6	9	12	15	18	21	24	27	30
-5	-2	1	4	7	10	13	16	19	22	25
-10	-7	-4	-1	2	5	8	11	14	17	20
-15	-12	-9	-6	-3	0	3	6	9	12	15

1. The Frobenius Coin Problem

...or in any horizontal strip of height a, etc.

15	18	21	24	27	30	33	36	39	42	45
10	13	16	19	22	25	28	31	34	37	40
5	8	11	14	17	20	23	26	29	32	35
0	3	6	9	12	15	18	21	24	27	30
-5	-2	1	4	7	10	13	16	19	22	25
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5	8	11	14	17	20	23	26	29	32	35
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1. The Frobenius Coin Problem

Positive labels $\mathbb{N}$ occur above a line of slope $-a/b$.

15	18	21	24	27	30	33	36	39	42	45
10	13	16	19	22	25	28	31	34	37	40
5	8	11	14	17	20	23	26	29	32	35
0	3	6	9	12	15	18	21	24	27	30
-5	-2	1	4	7	10	13	16	19	22	25
-10	-7	-4	-1	2	5	8	11	14	17	20
-15	-12	-9	-6	-3	0	3	6	9	12	15

1. The Frobenius Coin Problem

Labels from the monoid $a\mathbb{N} + b\mathbb{N}$ occur in this quadrant.

15	18	21	24	27	30	33	36	39	42	45
10	13	16	19	22	25	28	31	34	37	40
5	8	11	14	17	20	23	26	29	32	35
0	3	6	9	12	15	18	21	24	27	30
-5	-2	1	4	7	10	13	16	19	22	25
-10	-7	-4	-1	2	5	8	11	14	17	20
-15	-12	-9	-6	-3	0	3	6	9	12	15

. . . or in this quadrant, etc.

15	18	21	24	27	30	33	36	39	42	45
10	13	16	19	22	25	28	31	34	37	40
5	8	11	14	17	20	23	26	29	32	35
0	3	6	9	12	15	18	21	24	27	30
-5	-2	1	4	7	10	13	16	19	22	25
-10	-7	-4	-1	2	5	8	11	14	17	20
-15	-12	-9	-6	-3	0	3	6	9	12	15

1. The Frobenius Coin Problem

Therefore the labels $\mathbb{N} - (a\mathbb{N} + b\mathbb{N})$ occur in this triangle.

15	18	21	24	27	30	33	36	39	42	45
10	13	16	19	22	25	28	31	34	37	40
5	8	11	14	17	20	23	26	29	32	35
0	3	6	9	12	15	18	21	24	27	30
-5	-2	1	4	7	10	13	16	19	22	25
-10	-7	-4	-1	2	5	8	11	14	17	20
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1. The Frobenius Coin Problem

The largest label in the triangle is the Frobenius number

$$ab - a - b.$$

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1. The Frobenius Coin Problem

But why does the triangle have size $(a-1)(b-1)/2$?

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1. The Frobenius Coin Problem

Because it is one half of an $(a-1) \times (b-1)$ rectangle!

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1. The Frobenius Coin Problem

Indeed, for all $0 \leq n \leq ab$ with $a \nmid n$ and $b \nmid n$ we have

$$n \notin (a\mathbb{N} + b\mathbb{N}) \iff ab - n \in (a\mathbb{N} + b\mathbb{N})$$

15	18	21	24	27	30	33	36	39	42	45
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1. The Frobenius Coin Problem

This completes the proof of Sylvester's Theorem □

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0	3	6	9	12	15	18	21	24	27	30
-5	-2	1	4	7	10	13	16	19	22	25
-10	-7	-4	-1	2	5	8	11	14	17	20
-15	-12	-9	-6	-3	0	3	6	9	12	15

2. Rational Dyck Paths

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Grossman's Problem (1950). Given two natural numbers $a, b \in \mathbb{N}$ count the lattice paths from $(0,0)$ to $(b,-a)$ staying above the line $ax + by = 0$. The general problem reduces to the coprime case ($\gcd(a,b) = 1$) via inclusion-exclusion.

Bizley's Theorem (1954). Let $\gcd(a,b) = 1$. Then the number of such "rational Dyck paths" is given by the "rational Catalan number"

$$\mathrm{Cat}(a,b) := \frac{1}{a+b}\binom{a+b}{a,b} = \frac{(a+b-1)!}{a!\,b!}.$$

Why do we call it that?

2. Rational Dyck Paths

Observe that the "rational Catalan numbers"

$$\mathrm{Cat}(a, b) := \frac{1}{a+b}\binom{a+b}{a, b} = \frac{(a+b-1)!}{a!\, b!}$$

generalize the traditional Catalan numbers

$$\mathrm{Cat}(n, 1n+1) = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n}\binom{2n}{n-1}$$

and the even-more-traditional Fuss-Catalan numbers

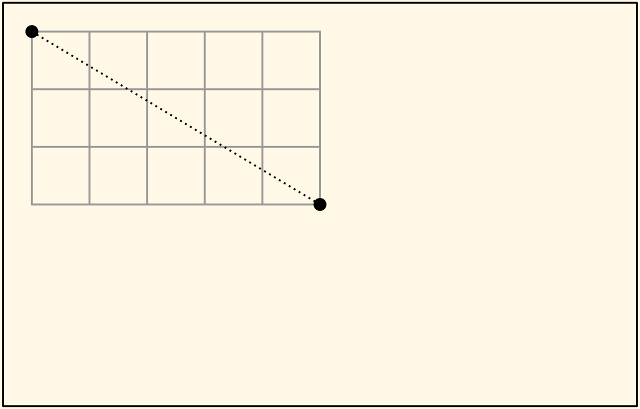
$$\mathrm{Cat}(n, kn+1) = \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n} = \frac{1}{n}\binom{(k+1)n}{n-1}.$$

[We call $b = 1 \pmod{a}$ the "Fuss level of generality."]

2. Rational Dyck Paths

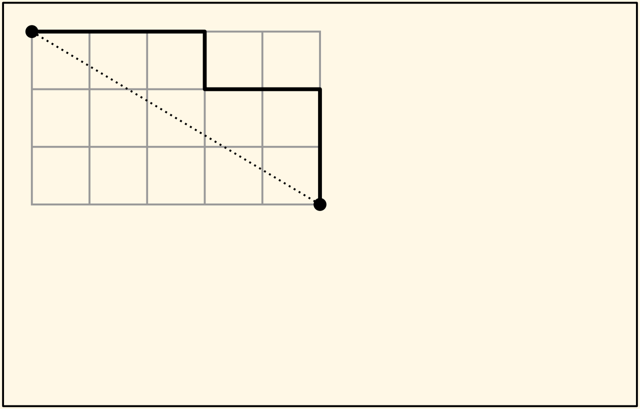
I will present Bizley's proof of the theorem.

For example, suppose that $(a, b) = (3, 5)$.



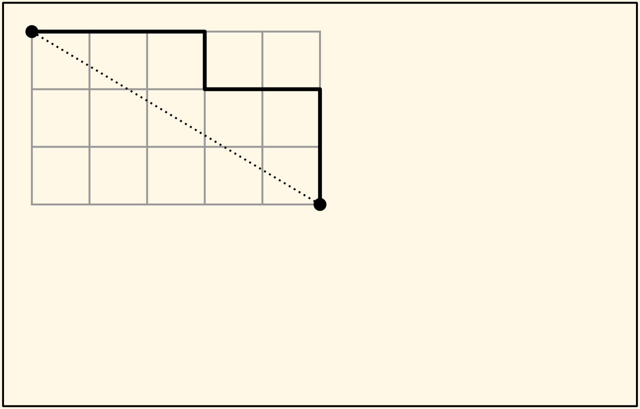
2. Rational Dyck Paths

There are a total of $\binom{a+b}{a,b}$ lattice paths from $(0,0)$ to $(b,-a)$.



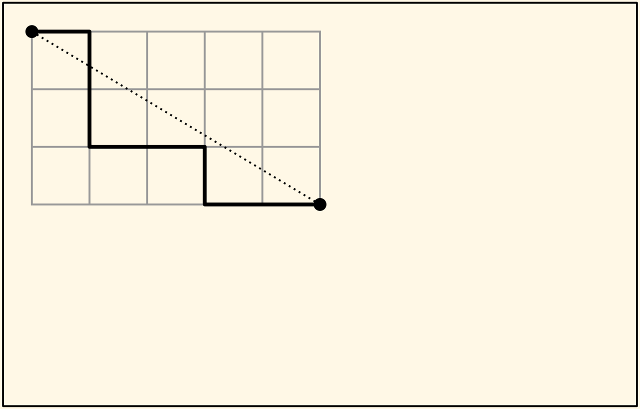
2. Rational Dyck Paths

Some of them are above the diagonal.



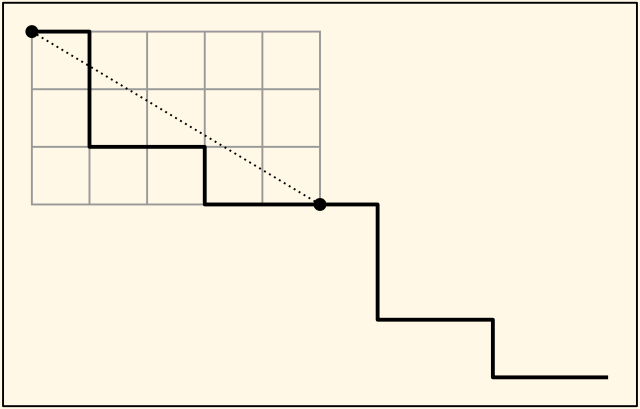
2. Rational Dyck Paths

...and some of them are not.



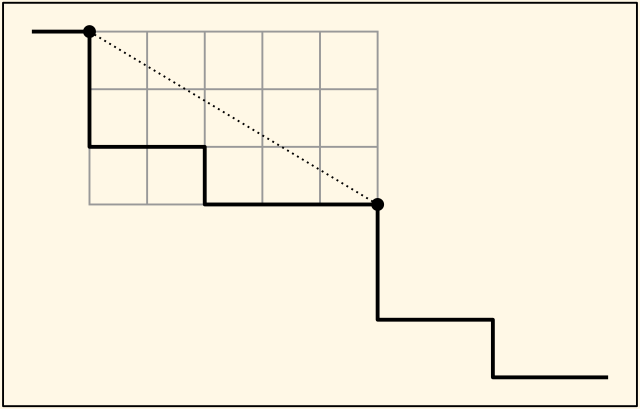
2. Rational Dyck Paths

If we double a given path . . .



2. Rational Dyck Paths

. . . then we can rotate it to create more paths.



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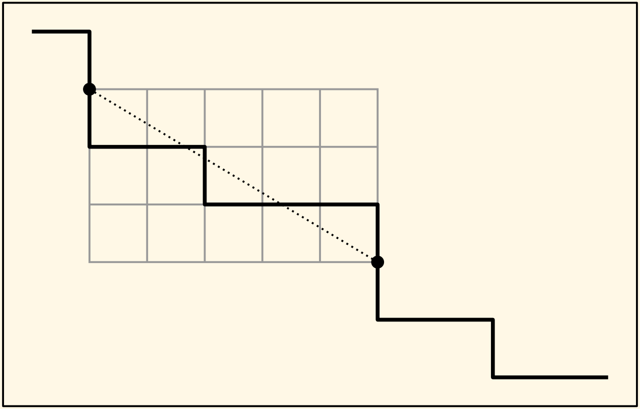

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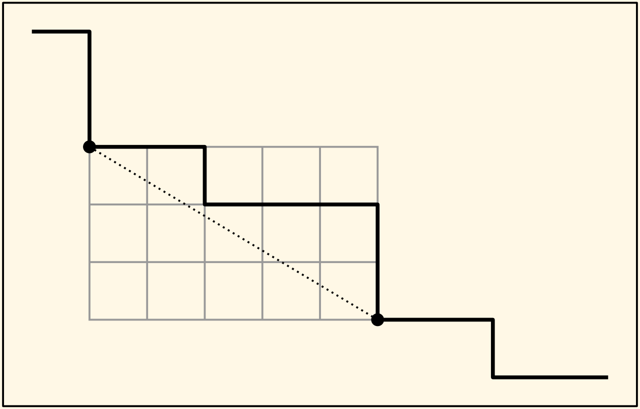

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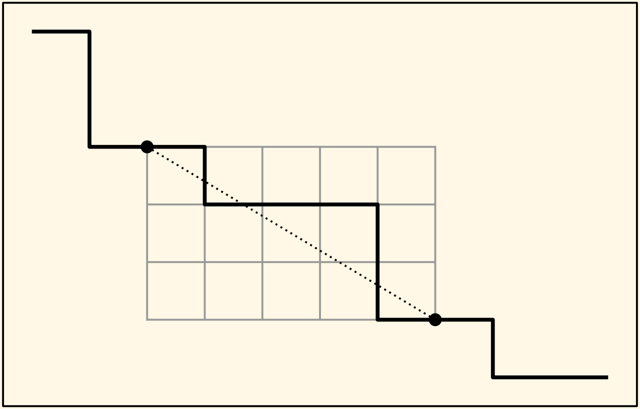

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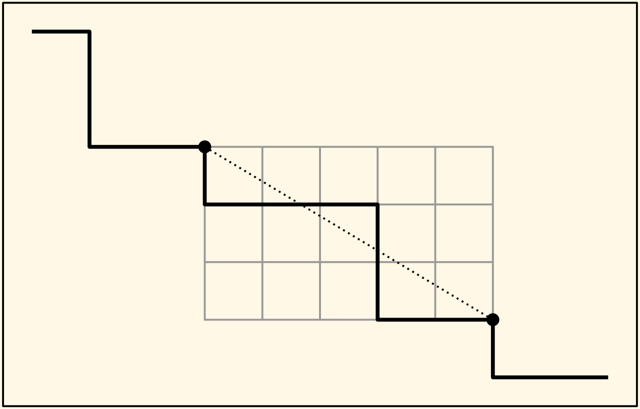

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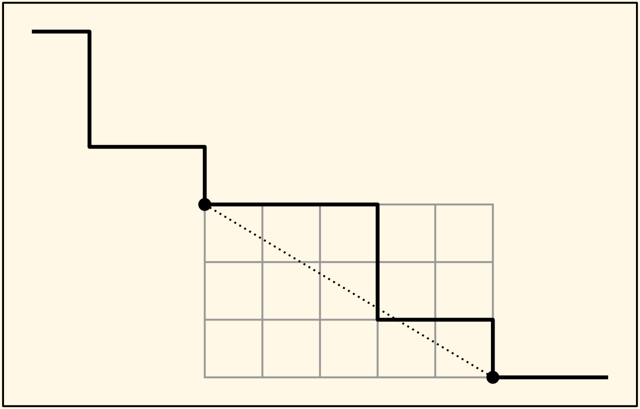

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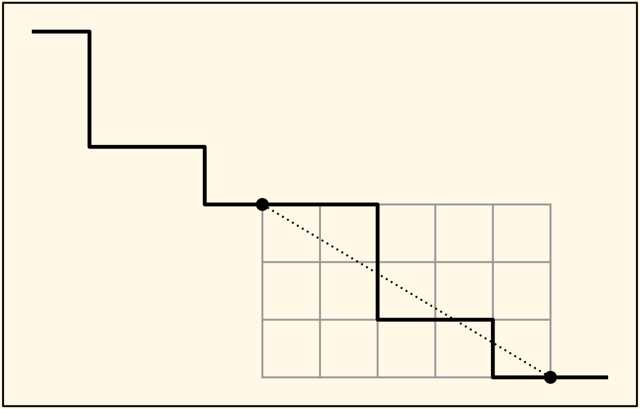

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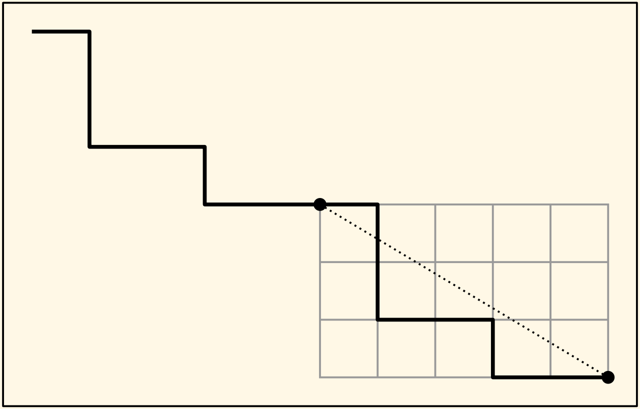
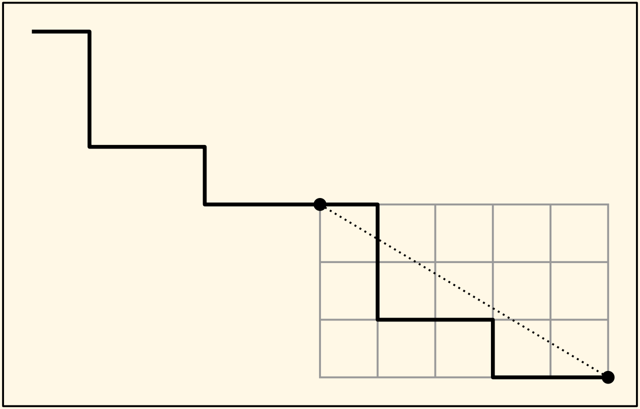


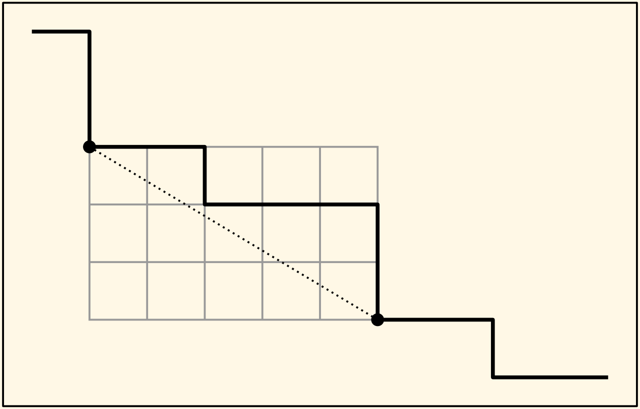
2. Rational Dyck Paths

Since $\gcd(a, b) = 1$, there are $a + b$ distinct rotations of each path.



2. Rational Dyck Paths

. . . and exactly one of them is above the diagonal.



Thus we obtain a bijection

$$\text{(Dyck paths)} \quad \longleftrightarrow \quad \text{(rotation classes of paths)}$$

and it follows that

$$\#\text{(Dyck paths)} = \binom{a+b}{a,b} / (a+b).$$

This completes the proof of Bizley's Theorem. $\square$

3. Core Partitions

3. Core Partitions

I presume everyone here knows the definition of integer partitions.

I will define them anyway.

Definition. An integer partition is an infinite binary string that begins with 0 s and ends with 1 s.

Example.

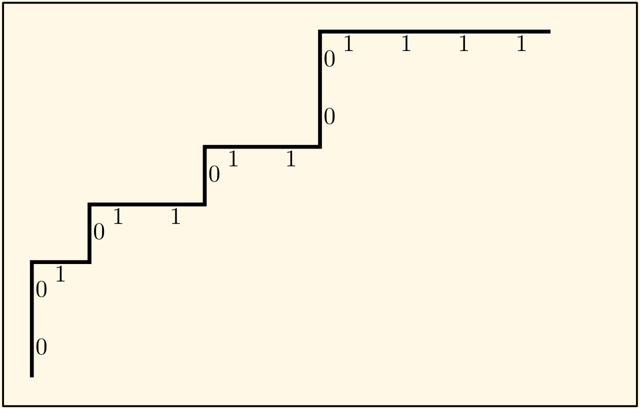
$$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots$$

Let's see a picture?

3. Core Partitions

We view 0 s as up steps and 1 s as right steps.

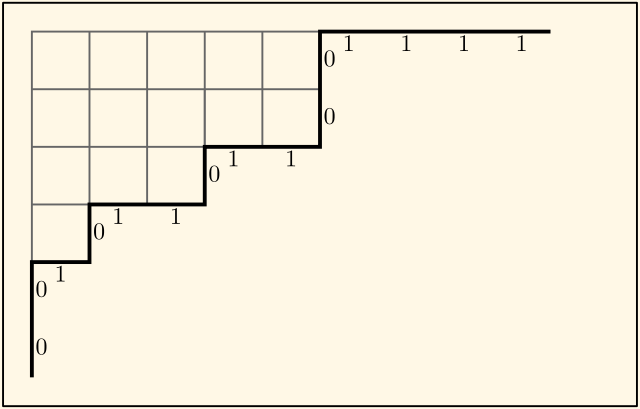
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

Observe that there is a hidden shape in the corner.

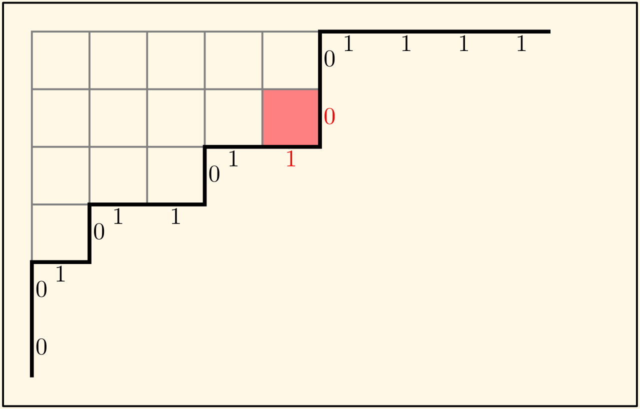
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

Each cell is an inversion of the binary string.

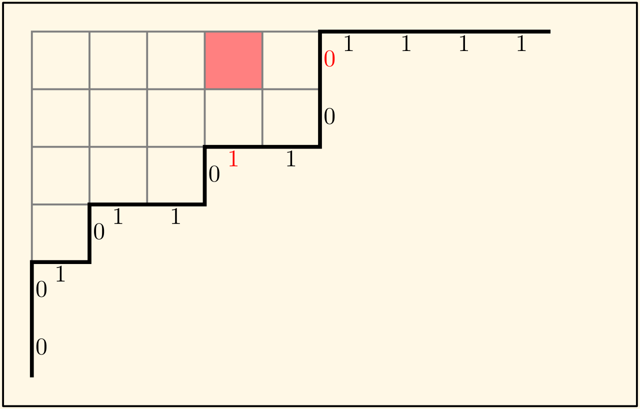
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

Each cell is an inversion of the binary string.

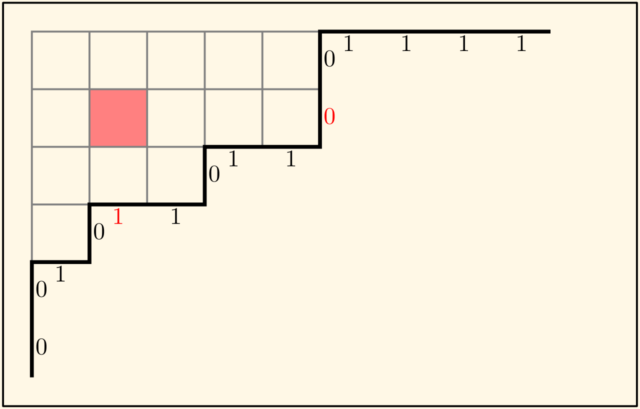
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

Each cell is an inversion of the binary string.

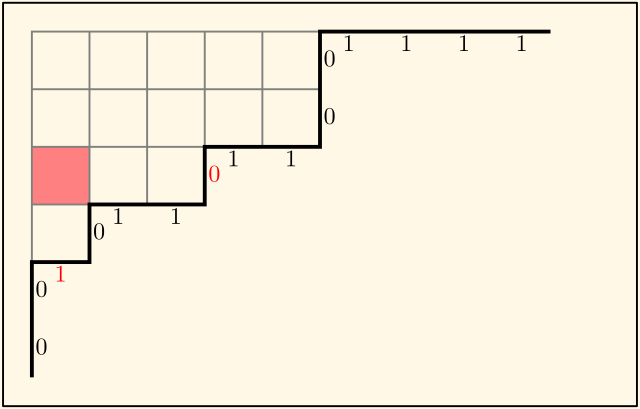
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

Each cell is an inversion of the binary string.

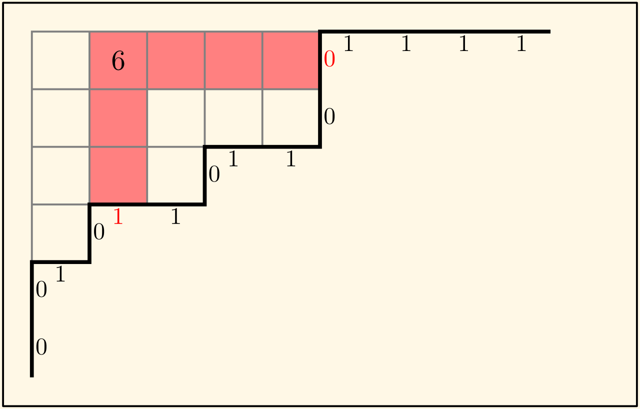
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

Observe that the length of the inversion is the hook length of the cell.

$$\cdots\ 0\ 0\ 0\ 0\ 1\ 0\ \overbrace{1\ 1\ 0\ 1\ 1\ 0\ 0}^{6}\ 1\ 1\ 1\ 1\ \cdots$$

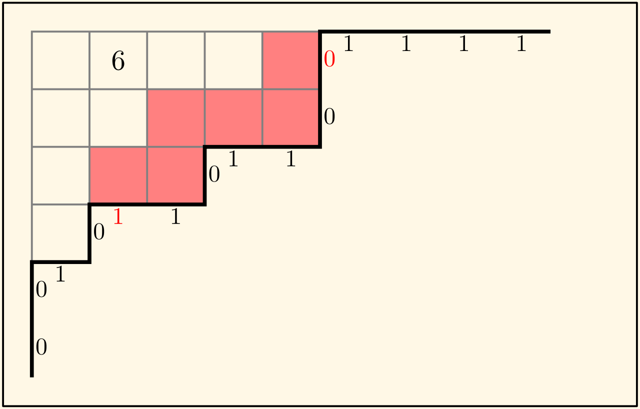


3. Core Partitions

. . . and we can think of it as a "rimhook" if we want.

6

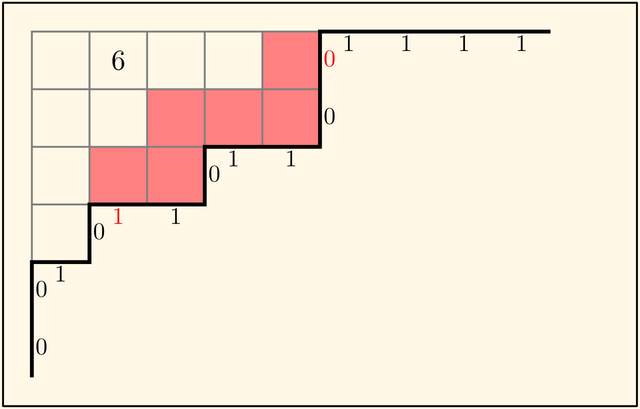
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

Question: What happens if we remove an inversion of length n?

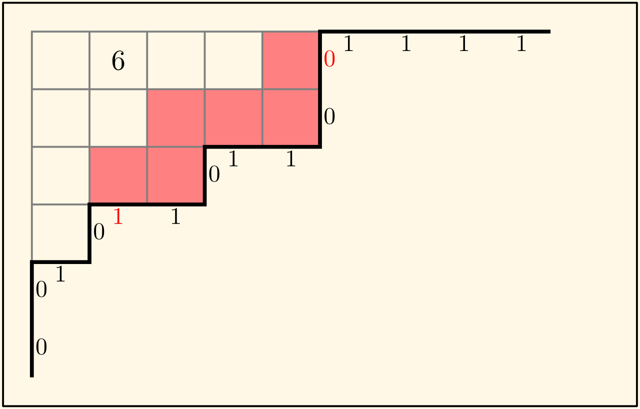
$$\overbrace{\cdots\ 0\ 0\ 0\ 0\ 1\ 0\ {\color{red}1}\ 1\ 0\ 1\ 1\ 0\ {\color{red}0}}^{6}\ 1\ 1\ 1\ 1\ \cdots$$



3. Core Partitions

Question: What happens if we remove an inversion of length n?

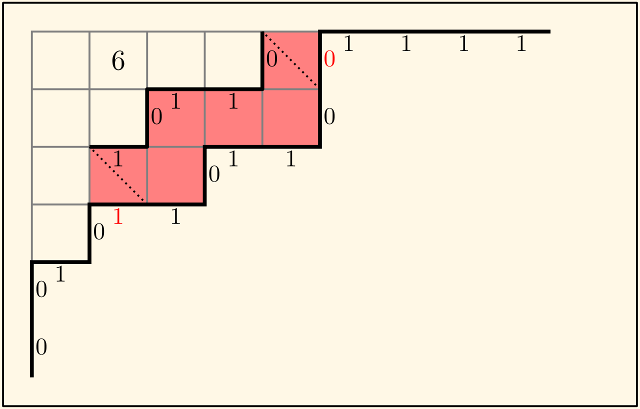
$$\cdots\ 0\ 0\ 0\ 0\ 1\ 0\ \overbrace{{\color{red}0}\ 1\ 0\ 1\ 1\ 0\ {\color{red}1}}^{6}\ 1\ 1\ 1\ 1\ \cdots$$



3. Core Partitions

Answer: The corresponding rimhook of length n gets stripped away.

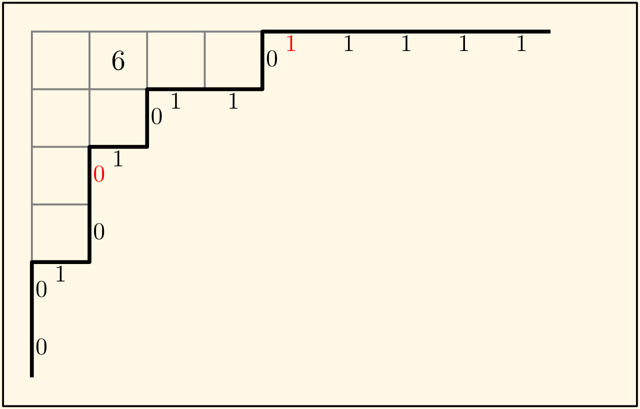
$$\cdots\ 0\ 0\ 0\ 0\ 1\ 0\ \overbrace{\color{red}{0}\ 1\ 0\ 1\ 1\ 0\ \color{red}{1}}^{6}\ 1\ 1\ 1\ 1\ \cdots$$



3. Core Partitions

Answer: The corresponding rimhook of length n gets stripped away.

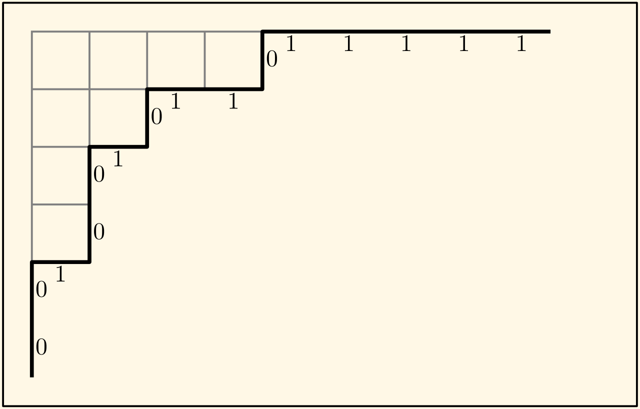
$$\cdots\ 0\ 0\ 0\ 0\ 1\ 0\ \overbrace{{\color{red}0}\ 1\ 0\ 1\ 1\ 0\ {\color{red}1}}^{6}\ 1\ 1\ 1\ 1\ \cdots$$



3. Core Partitions

Answer: The corresponding rimhook of length n gets stripped away.

$$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots$$



3. Core Partitions

We make the following definition.

Definition. Fix a positive integer $n \in \mathbb{N}$ and let λ be any integer partition. By successively removing inversions of length n, we obtain an integer partition $\tilde{\lambda}$ with no inversions of length n.

We call this $\tilde{\lambda}$ an n-core partition.

Question. Is the resulting partition $\tilde{\lambda}$ well-defined?

Theorem (Nakayama, 1941). Yes.

We call this $\tilde{\lambda}$ the n-core of λ.

3. Core Partitions

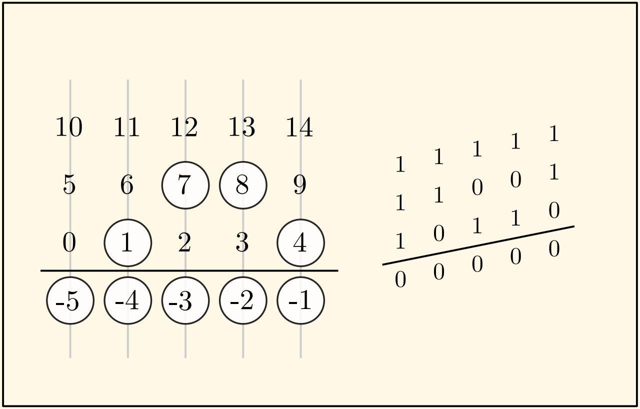
I will present a proof by James and Kerber (1981).

For example, suppose that $n = 5$.

3. Core Partitions

The Idea: Wrap the infinite binary string around an *n*-cylinder.

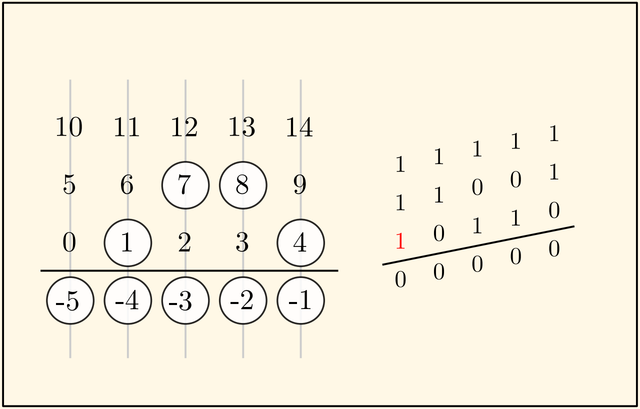
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

We place the first 1 in the zeroth position.

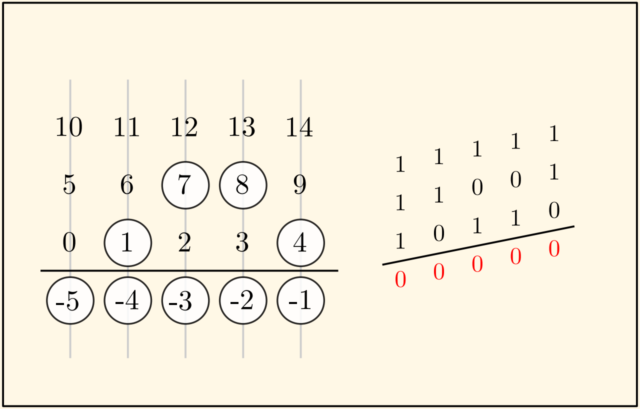
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

Everything below ground level is 0.

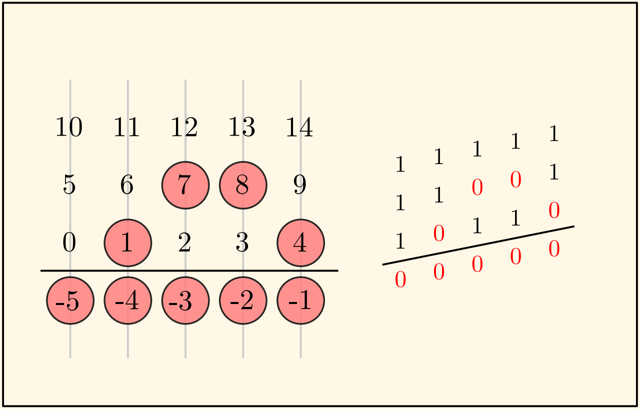
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

We think of the 0s as "beads on an abacus."

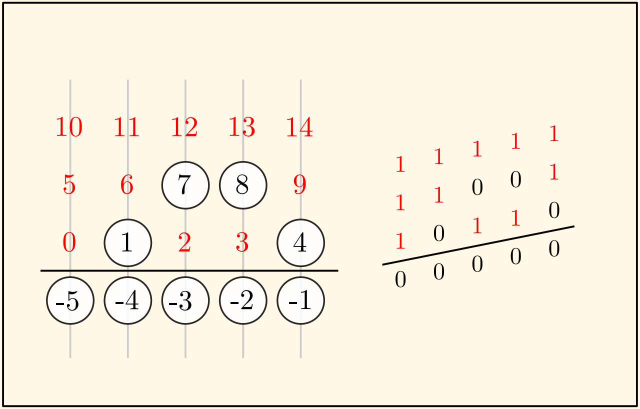
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

The 1s are "empty spaces."

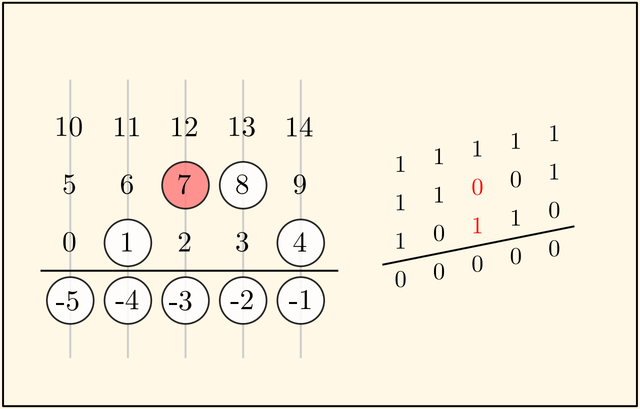
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

Removing a length n inversion means “sliding a bead down.”

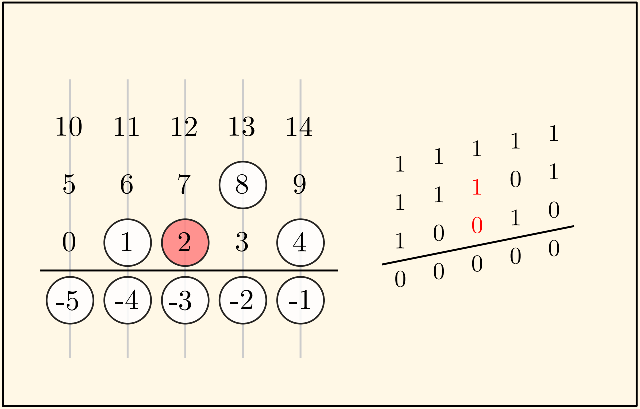
$$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad {\color{red}1} \quad 1 \quad 0 \quad 1 \quad 1 \quad {\color{red}0} \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots$$



3. Core Partitions

Removing a length n inversion means "sliding a bead down."

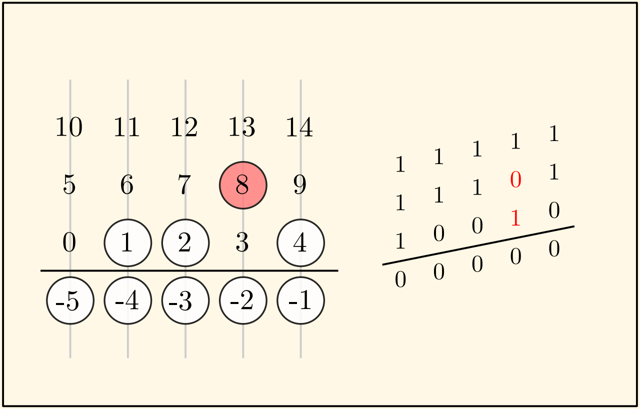
$\cdots$ 0 0 0 0 1 0 0 1 0 1 1 1 0 1 1 1 1 $\cdots$



3. Core Partitions

Continue sliding beads until there are no more length n inversions.

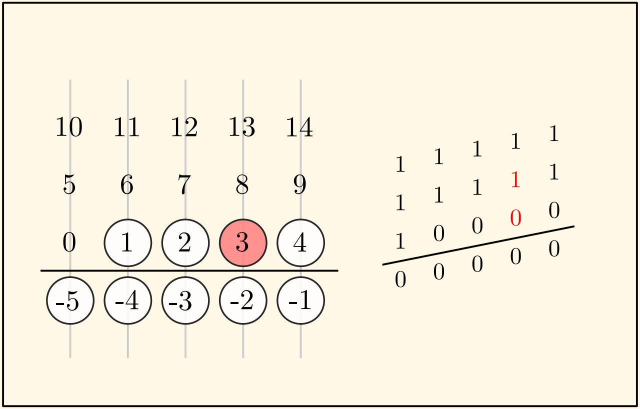
$\cdots$ 0 0 0 0 1 0 0 1 0 1 1 1 0 1 1 1 1 $\cdots$



3. Core Partitions

Continue sliding beads until there are no more length n inversions.

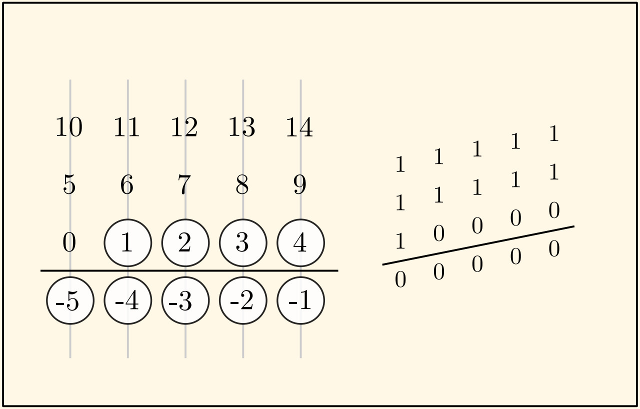
$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad \color{red}{0} \quad 0 \quad 1 \quad 1 \quad 1 \quad \color{red}{1} \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots$



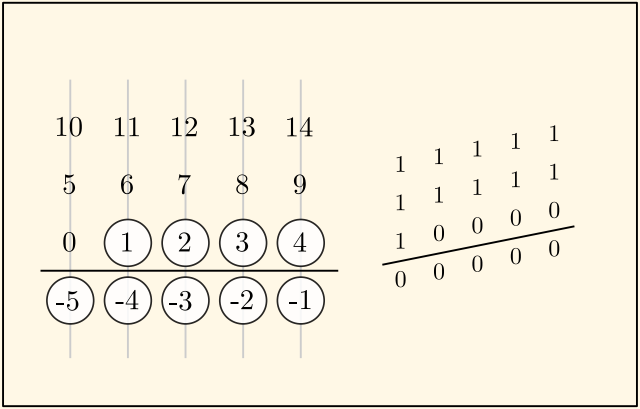
3. Core Partitions

Gravity tells us that the n-core is unique.

$\cdots$ 0 0 0 0 1 0 0 0 0 1 1 1 1 1 1 1 1 $\cdots$



This completes James and Kerber's proof. □



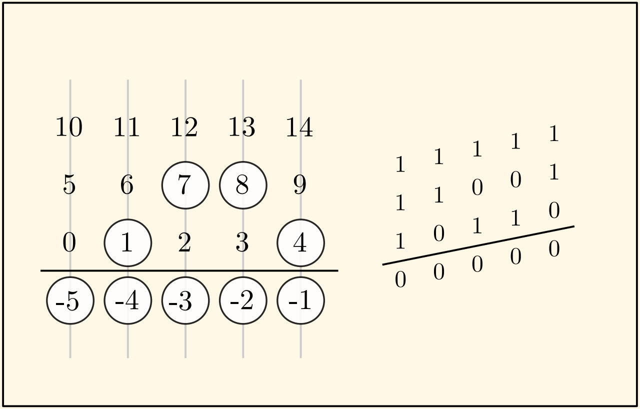
3. Core Partitions

Now let's see how it looks in terms of rimhooks.

3. Core Partitions

Go back to the original partition.

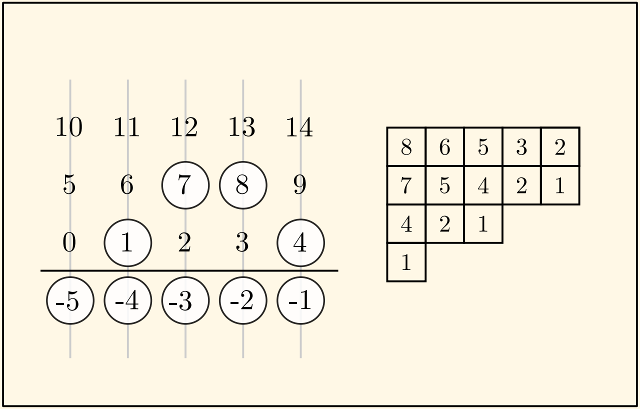
$$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots$$



3. Core Partitions

Here is the corresponding diagram with hook lengths shown.

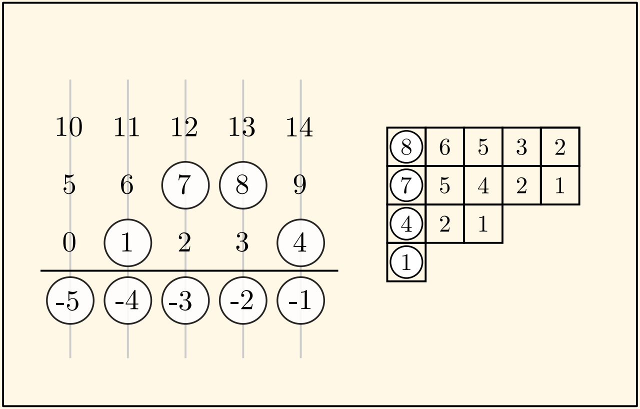
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

Observe that positive beads = hook lengths in the first column.

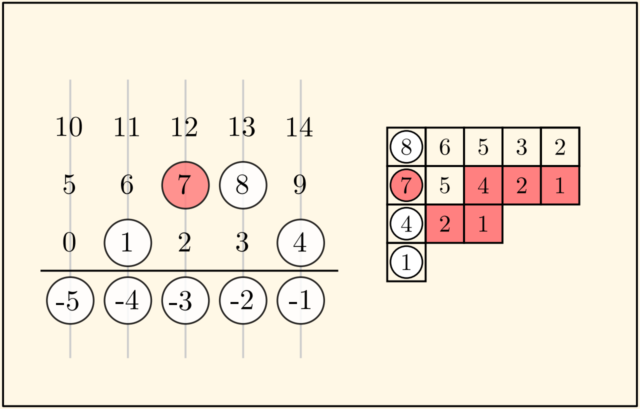
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

We can remove the n-rimhooks in this order.

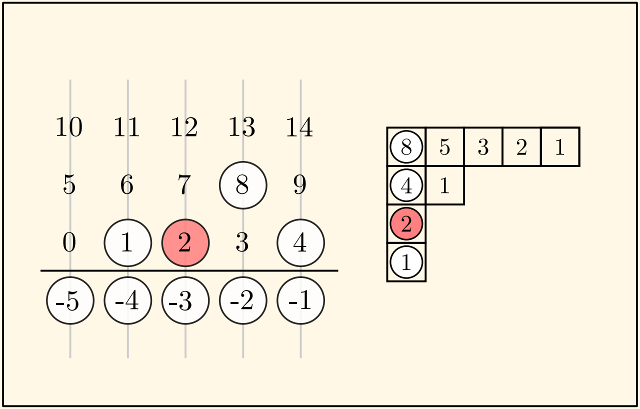
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

We can remove the n-rimhooks in this order.

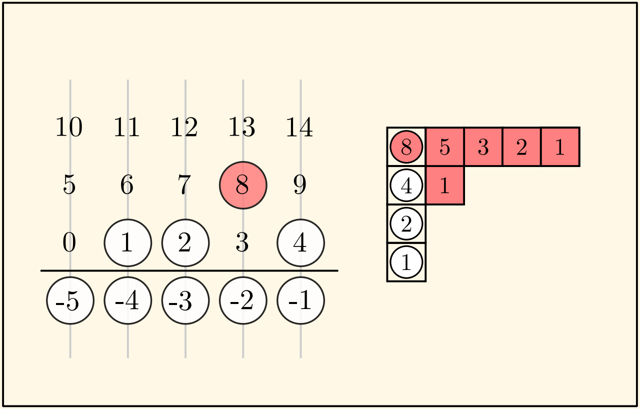
$$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad {\color{red}0} \quad 1 \quad 0 \quad 1 \quad 1 \quad {\color{red}1} \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots$$



3. Core Partitions

We can remove the n-rimhooks in this order.

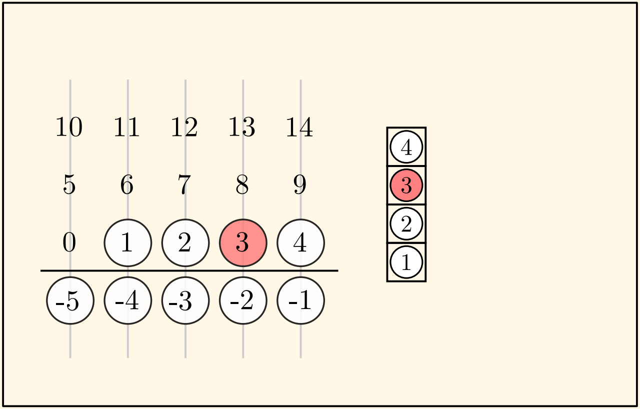
$\cdots$ 0 0 0 0 1 0 0 1 0 1 1 1 0 1 1 1 1 $\cdots$



3. Core Partitions

We can remove the *n*-rimhooks in this order.

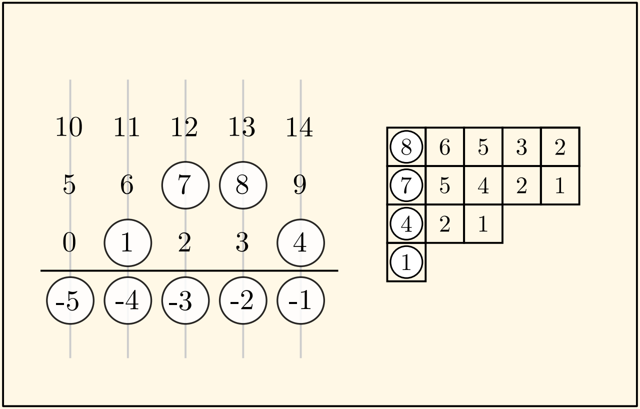
$\cdots$ 0 0 0 0 1 0 0 0 0 1 1 1 1 1 1 1 1 $\cdots$



3. Core Partitions

. . . or we can remove them in this order.

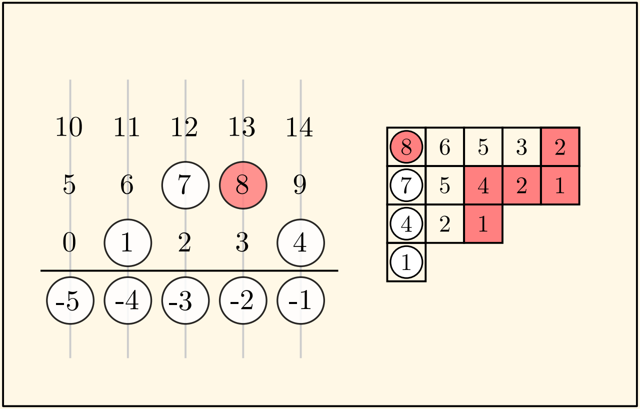
$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots$



3. Core Partitions

...or we can remove them in this order.

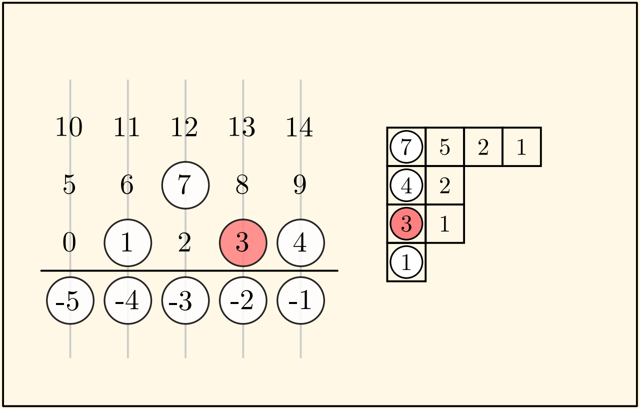
$\cdots$ 0 0 0 0 1 0 1 1 0 1 1 0 0 1 1 1 1 $\cdots$



3. Core Partitions

...or we can remove them in this order.

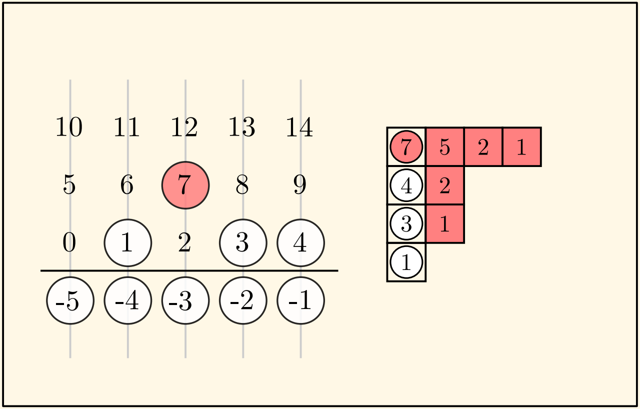
$\cdots$ 0 0 0 0 1 0 1 0 0 1 1 0 1 1 1 1 1 $\cdots$



3. Core Partitions

. . . or we can remove them in this order.

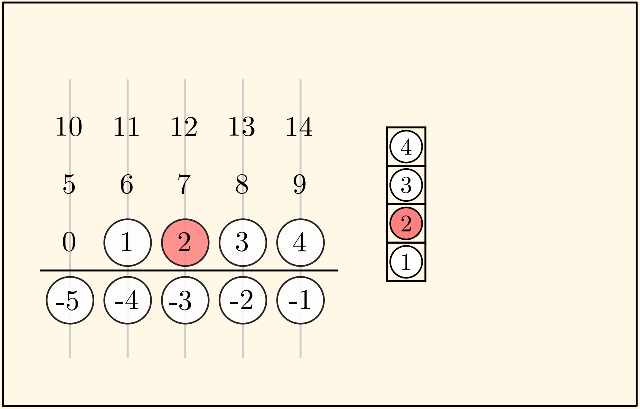
$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad \color{red}{1} \quad 0 \quad 0 \quad 1 \quad 1 \quad \color{red}{0} \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots$



3. Core Partitions

. . . or we can remove them in this order.

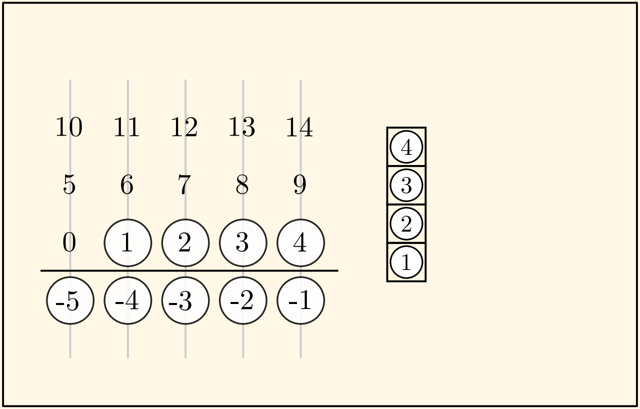
$\cdots\ 0\ 0\ 0\ 0\ 1\ 0\ \color{red}{0}\ 0\ 0\ 1\ 1\ \color{red}{1}\ 1\ 1\ 1\ 1\ 1\ \cdots$



3. Core Partitions

They look different but the resulting n core is the same.

$$\cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \cdots$$



4. The Double Abacus

4. The Double Abacus

Now it's time to put everything together.

Problem/Definition. Fix two positive integers $a, b \in \mathbb{N}$ and let λ be an integer partition. We say that λ is an (a, b)-core if it is simultaneously a-core and b-core. What can be said about such partitions?

Theorem (Anderson, 2002). If $\gcd(a, b) = 1$ then the number of (a, b)-cores is finite. Furthermore, they are counted by the rational Catalan number:

$$\#(a, b)\text{-cores} = \frac{1}{a+b}\binom{a+b}{a, b}.$$

4. The Double Abacus

I will present Anderson's proof.

For example, suppose that $(a, b) = (3, 5)$.

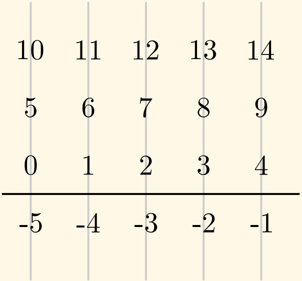
4. The Double Abacus

Consider again the standard vertical *b*-abacus.

10	11	12	13	14
5	6	7	8	9
0	1	2	3	4
-5	-4	-3	-2	-1

4. The Double Abacus

The criterion for detecting *b*-cores (i.e., gravity) is unaffected by permuting the runners.



4. The Double Abacus

The criterion for detecting *b*-cores (i.e., gravity) is unaffected by permuting the runners.

10	13	11	14	12
5	8	6	9	7
0	3	1	4	2
-5	-2	-4	-1	-3

4. The Double Abacus

. . . or by shifting them up and down.

5	13	11	14	12
0	8	6	9	7
-5	3	1	4	2
-10	-2	-4	-1	-3

4. The Double Abacus

. . . or by shifting them up and down.

5	8	11	14	12
0	3	6	9	7
-5	-2	1	4	2
-10	-7	-4	-1	-3

4. The Double Abacus

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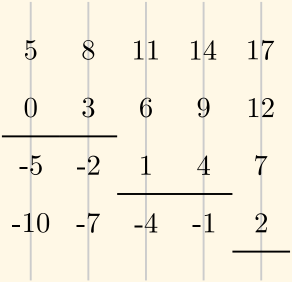
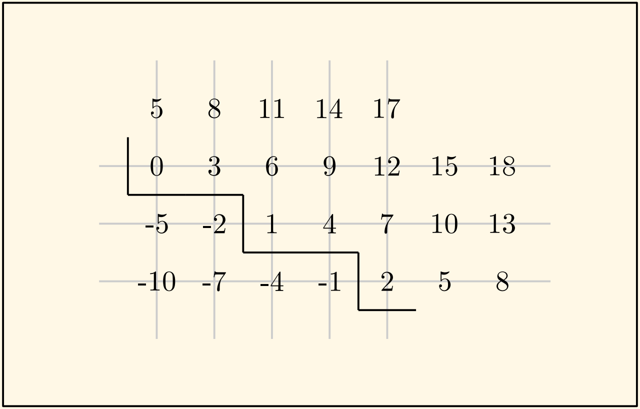


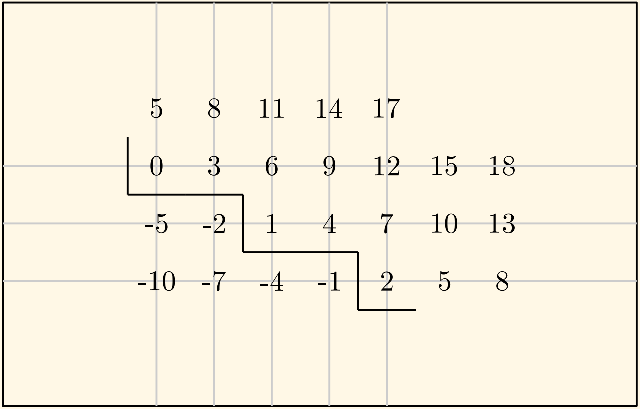
4. The Double Abacus

If we do it correctly then we obtain a horizontal *a*-abacus.



4. The Double Abacus

The correct labeling comes from the Frobenius Coin Problem.



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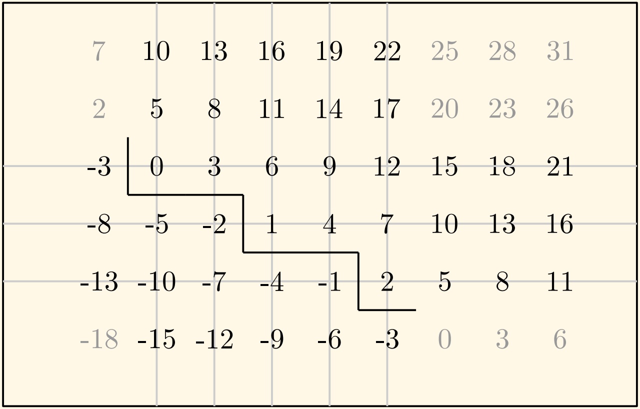

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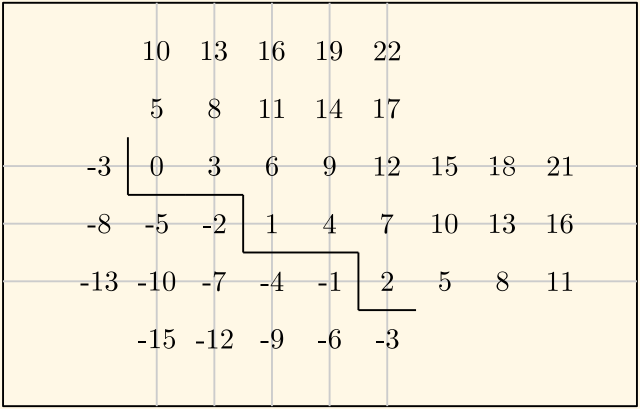
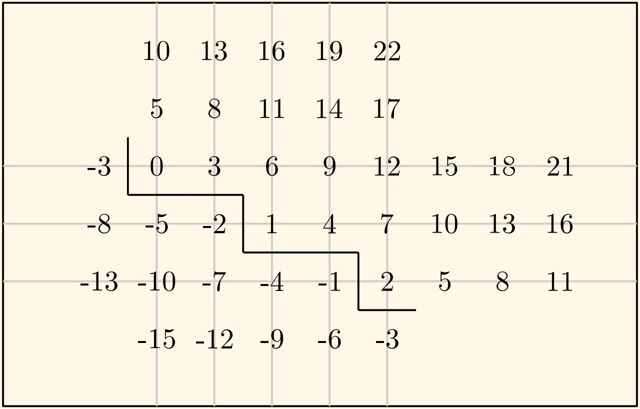


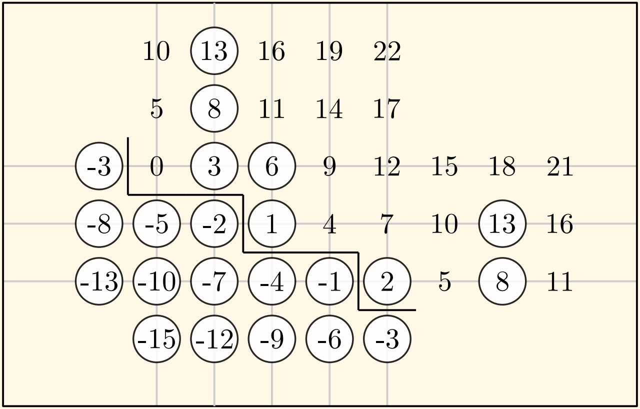
4. The Double Abacus

Finite subsets of $\mathbb{N} - \{0\}$ correspond to integer partitions.
[Recall: These are the hook lengths in the first column of a shape.]



4. The Double Abacus

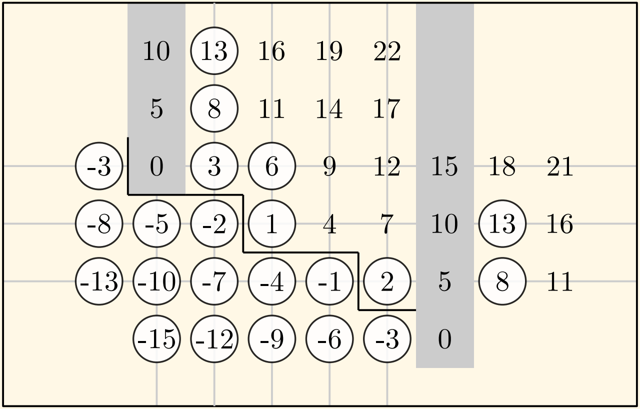
Example: The set $\{1, 2, 3, 6, 8, 13\}$ is 5-core but not 3-core.



4. The Double Abacus

In General:

- *b*-cores are down-aligned and avoid the vertical 0-runner.
- *a*-cores are left-aligned and avoid the horizontal 0-runner.



4. The Double Abacus

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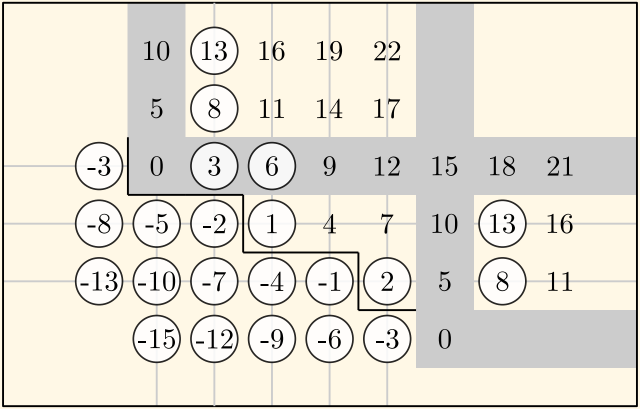
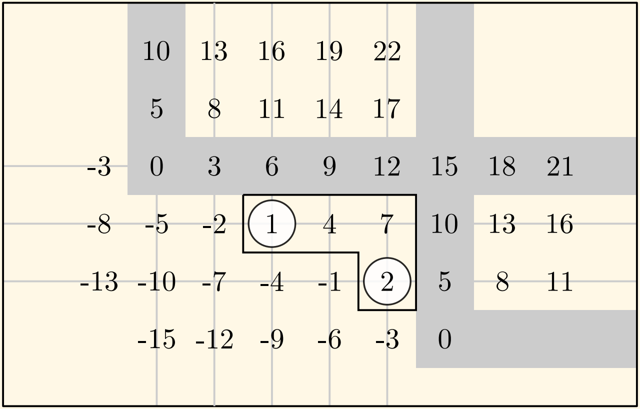


4. The Double Abacus

Hence (*a*, *b*)-cores correspond to down/left-aligned subsets of the triangle

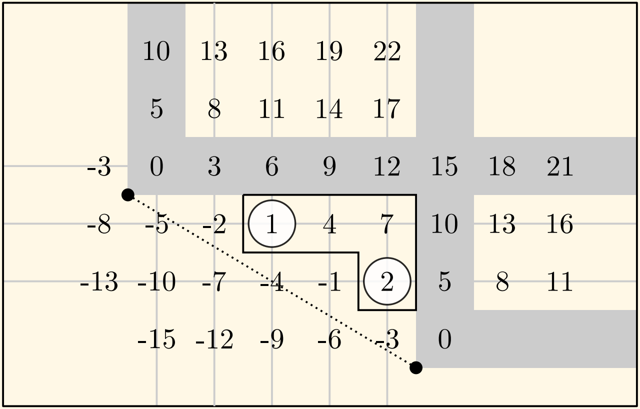
$$\mathbb{N} - (a\mathbb{N} + b\mathbb{N}).$$



4. The Double Abacus

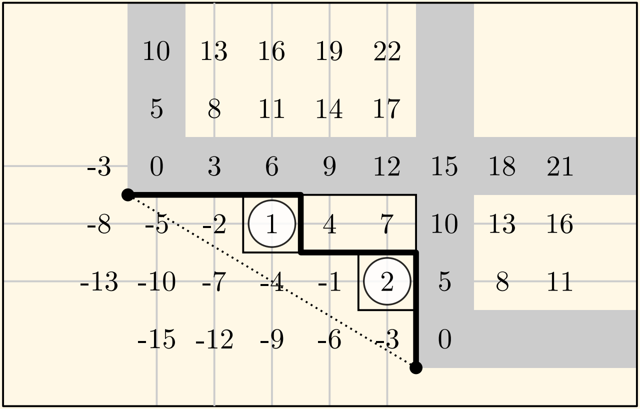
Hence (a, b)-cores correspond to down/left-aligned subsets of the triangle

$$\mathbb{N} - (a\mathbb{N} + b\mathbb{N}).$$

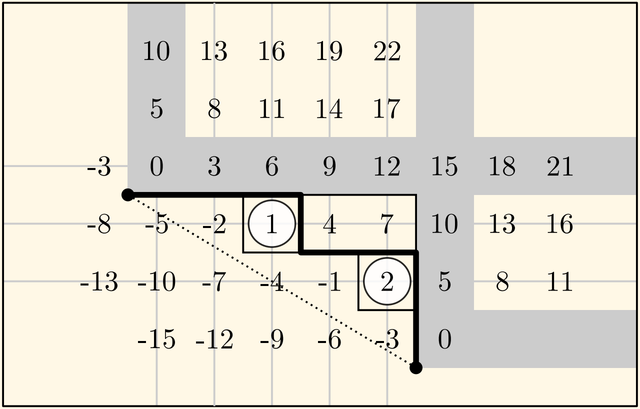


4. The Double Abacus

Finally, we observe that down/left-aligned subsets of the triangle correspond to Dyck paths in an $a \times b$ rectangle.



This completes the proof of Anderson's theorem. □



4. The Double Abacus

The length of a partition is the number of cells in its first column. As corollaries of Sylvester's Theorem we obtain the following:

- The maximum length of an (a, b)-core is $(a-1)(b-1)/2$.
- The largest hook that can occur in an (a, b)-core is $ab - a - b$.

The size of a partition is the number of cells in the full diagram. By summing over the elements of the set $\mathbb{N} - (a\mathbb{N} + b\mathbb{N})$, Olsson and Stanton proved the following.

Theorem (Olsson and Stanton, 2005). Let $\gcd(a, b) = 1$. The largest size of an (a, b)-core is

$$\frac{(a^2-1)(b^2-1)}{24}.$$

4. The Double Abacus

Going further, I conjectured and then Paul Johnson proved the following.

Theorem (Johnson, 2015). Let $\gcd(a,b) = 1$. The average size of an (a,b)-core is

$$\frac{(a+b+1)(a-1)(b-1)}{24}.$$

Proof. Use Ehrhart theory to show that the average size is a degree 2 polynomial in a and b. Then use interpolation. $\square$

But why **this** degree 2 polynomial and not another? Thiel and Williams (2015) showed that the number 24 comes from the "strange formula" of Freudenthal and de Vries:

$$\frac{1}{24} \cdot \text{dim(Lie group)} = \|\text{half the sum of positive roots}\|^2 .$$

5. q-Catalan Numbers

5. q-Catalan Numbers

So far, so good. Now comes the hard part.
Recall that the classical q-Catalan numbers are defined as follows.

Definition. Let q be a formal parameter. For all $n \in \mathbb{N}$ we define

$$\mathrm{Cat}_q(n) := \frac{1}{[2n+1]_q} \begin{bmatrix} 2n+1 \\ n, n+1 \end{bmatrix}_q = \frac{[2n]_q!}{[n]_q!\,[n+1]_q!}.$$

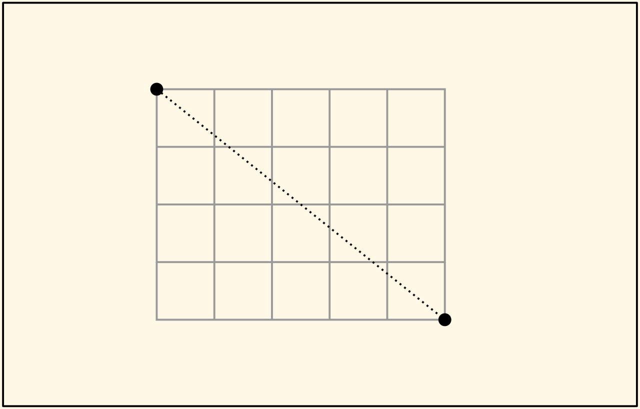
A priori, we only have $\mathrm{Cat}_q(n) \in \mathbb{Z}[[q]]$. However, it follows from a more general result of Major Percy MacMahon that $\mathrm{Cat}_q(n) \in \mathbb{N}[q]$.

Theorem (MacMahon, 1915). Let $D_{n,n+1}$ be the set of classical Dyck paths. There is a statistic $\mathrm{maj} : D_{n,n+1} \to \mathbb{N}$ (called major index) with

$$\mathrm{Cat}_q(n) = \sum_{\pi \in D_{n,n+1}} q^{\mathrm{maj}(\pi)} \in \mathbb{N}[q].$$

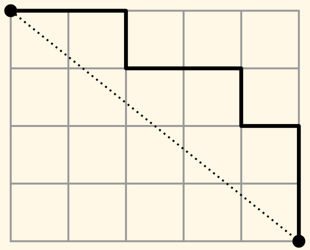
5. q-Catalan Numbers

To see what this means, let $(a, b) = (n, n+1)$ for some $n \in \mathbb{N}$.



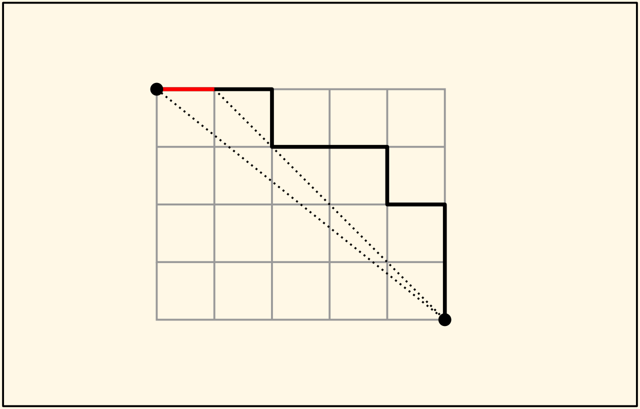
5. q-Catalan Numbers

Observe that every Dyck path begins with a right step.



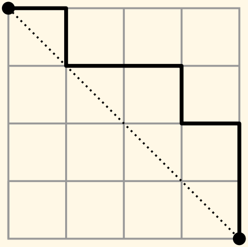
5. q-Catalan Numbers

Observe that every Dyck path begins with a right step.



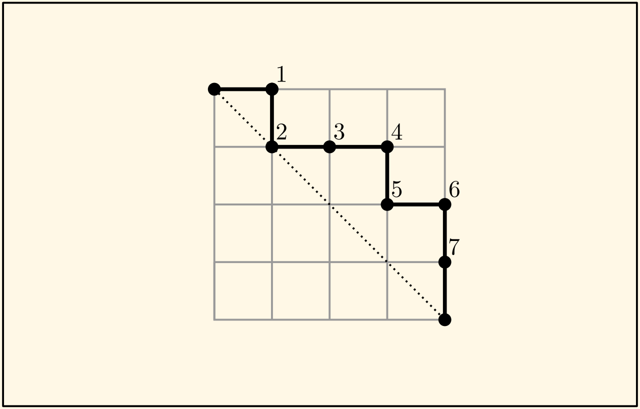
5. q-Catalan Numbers

...so we might as well consider paths in the $n \times n$ square.



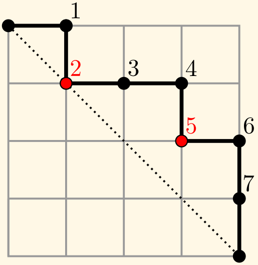
5. q-Catalan Numbers

To compute maj: Number the steps of the path,



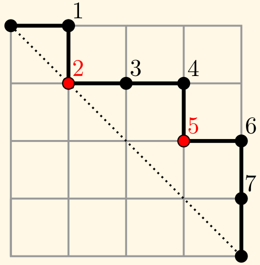
5. q-Catalan Numbers

...highlight the valleys,



5. q-Catalan Numbers

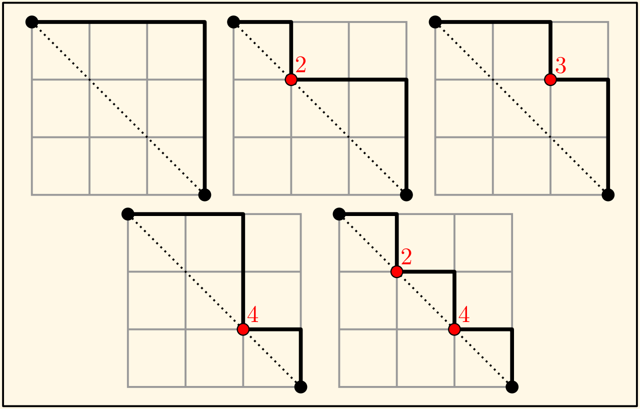
... and add the numbers of the valleys. Here: $\mathrm{maj} = 2 + 5 = 7$.



5. q-Catalan Numbers

For example, when $n = 3$ we observe that

$$\mathrm{Cat}_q(3) = \frac{[6]_q!}{[3]_q!\,[4]_q!} = q^0 + q^2 + q^3 + q^4 + q^6 = \sum q^{\mathrm{maj}}.$$



5. q-Catalan Numbers

By analogy with the classical case we define the

rational q-Catalan numbers.

Definition. Let q be a formal parameter. For any $\gcd(a, b) = 1$ we define

$$\mathrm{Cat}_q(a, b) := \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q = \frac{[a+b-1]_q!}{[a]_q! \, [b]_q!}.$$

Stanton's Problem. Let $D_{a,b}$ be the set of rational Dyck paths. Find a combinatorial statistic $\mathsf{stat} : D_{a,b} \to \mathbb{N}$ such that

$$\mathrm{Cat}_q(a, b) = \sum_{\pi \in D_{a,b}} q^{\mathsf{stat}(\pi)}.$$

This problem is **surprisingly difficult**!

5. q-Catalan Numbers

Recall that we have a bijection $D_{a,b} \leftrightarrow C_{a,b}$ between (a, b)-Dyck paths and (a, b)-core partitions. I will present a statistic

$$\text{stat} : C_{a,b} \to \mathbb{N}$$

that conjecturally satisfies

$$\text{Cat}_q(a, b) = \sum_{\pi \in C_{a,b}} q^{\text{stat}(\pi)}.$$

Let $\ell(\pi)$ denote the length of the partition π (i.e., the number of cells in the first column) and recall from Sylvester's Theorem that

$$\max\{\ell(\pi) : \pi \in C_{a,b}\} = \frac{(a-1)(b-1)}{2}.$$

5. q-Catalan Numbers

Next I will define a mysterious statistic called skew length:

$$s\ell : C_{a,b} \to \mathbb{N}$$

5. q-Catalan Numbers

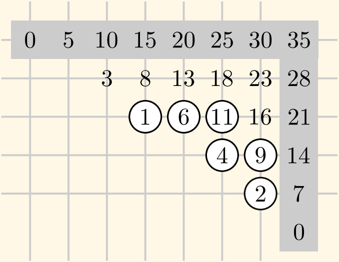
For example, let $(a, b) = (5, 7)$ and consider the Double Abacus.

0	5	10	15	20	25	30	35
		3	8	13	18	23	28
			1	6	11	16	21
					4	9	14
						2	7
							0

5. q-Catalan Numbers

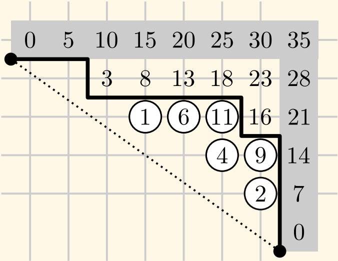
Recall that (a, b)-cores correspond to down/left-aligned sets of beads inside the triangle

$$\mathbb{N} - (a\mathbb{N} + b\mathbb{N}).$$



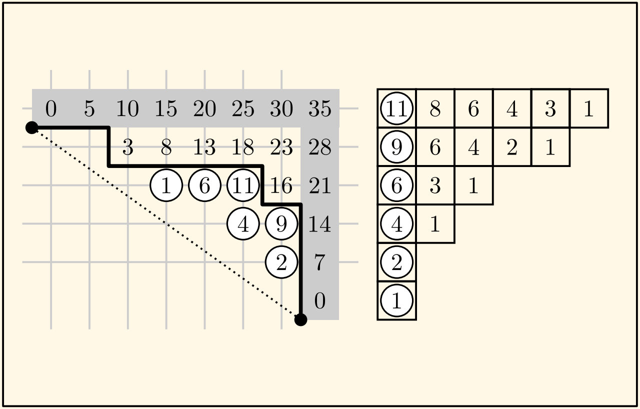
5. q-Catalan Numbers

. . . which correspond to (a, b)-Dyck paths.



5. q-Catalan Numbers

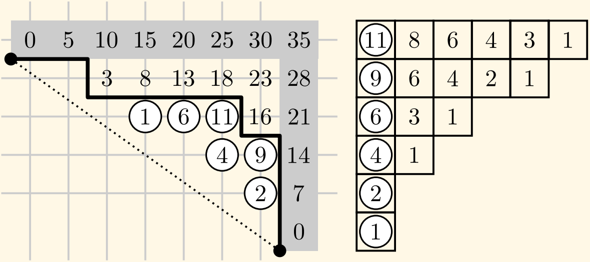
Recall that beads = hook lengths in the first column of a partition.



5. q-Catalan Numbers

Observe that this partition has no 5-hooks or 7-hooks.

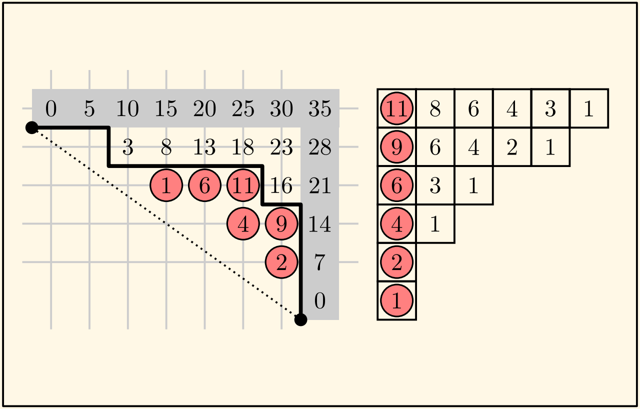
In fact, all hooks lengths come from the triangle $\mathbb{N} - (a\mathbb{N} + b\mathbb{N})$.



5. q-Catalan Numbers

Observe that the area of the Dyck path is the length of the partition:

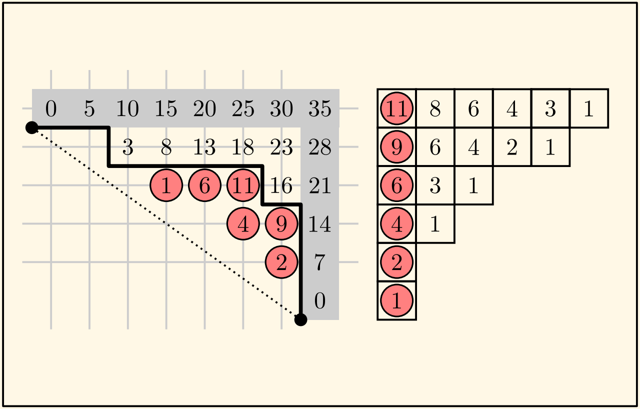
$$\text{area}(\pi) = \ell(\pi) = 6.$$



5. q-Catalan Numbers

Now for the skew length. The official definition:

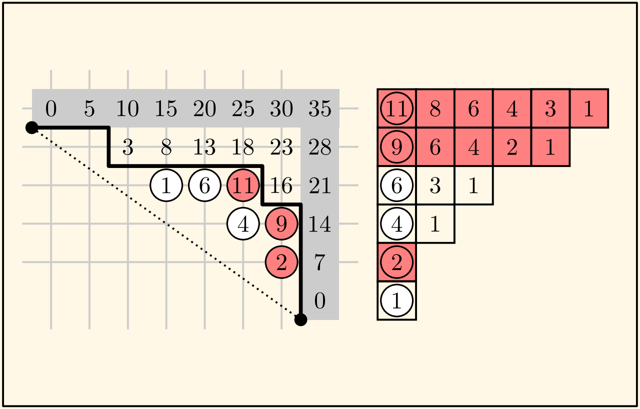
$$s\ell(\pi) := \#(a\text{-rows}) \cap (b\text{-boundary}).$$



5. q-Catalan Numbers

Let me explain.

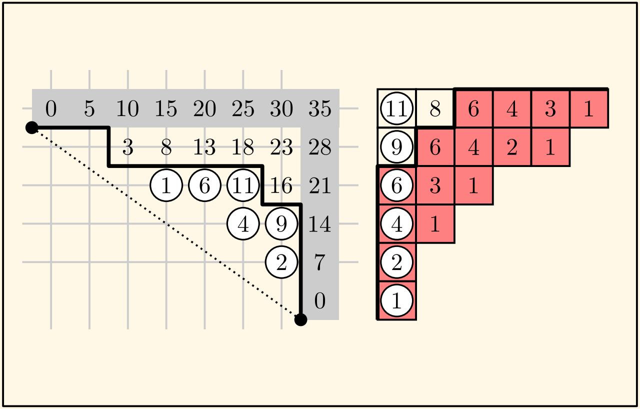
- The *a*-rows correspond to rightmost beads under the path.
- The *b*-boundary is the cells with hook length $< b$.



5. q-Catalan Numbers

Let me explain.

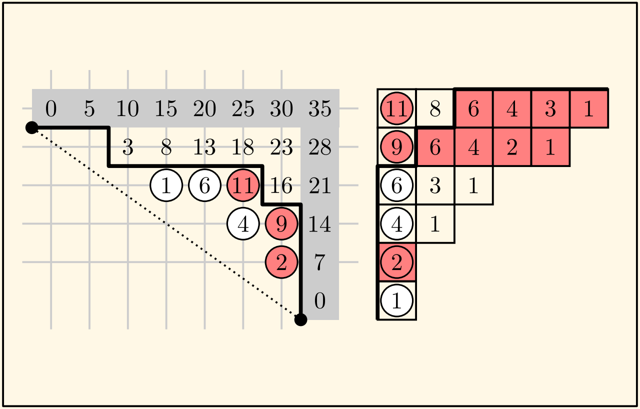
- The a-rows correspond to rightmost beads under the path.
- The b-boundary is the cells with hook length $< b$.



5. q-Catalan Numbers

The skew length is the number of cells in the intersection of the *a*-rows and the *b*-boundary. In this case,

$$s\ell(\pi) = 9.$$

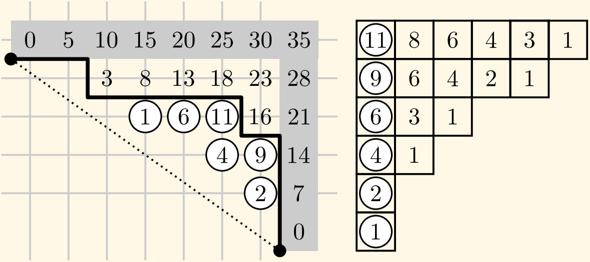


5. q-Catalan Numbers

You might wonder if the definition of $s\ell$ is symmetric in a and b:

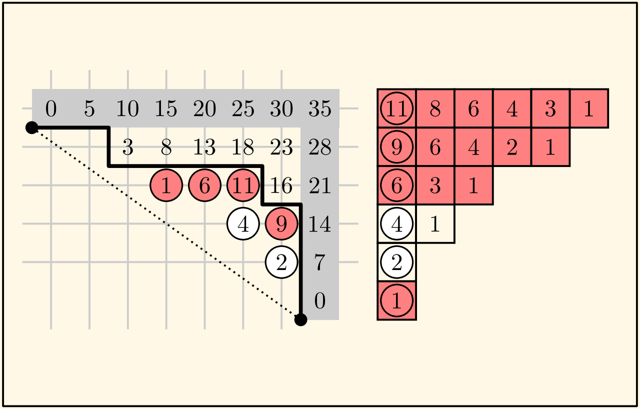
$$\#(a\text{-rows}) \cap (b\text{-boundary}) = \#(b\text{-rows}) \cap (a\text{-boundary})\,?$$

Xin (2015) and Ceballos-Denton-Hanusa (2015) proved that this is true.



Let's check:

- The b-rows correspond to uppermost beads under the path.
- The a-boundary is the cells with hook length $< a$.



5. q-Catalan Numbers

Let's check:

- The b-rows correspond to uppermost beads under the path.
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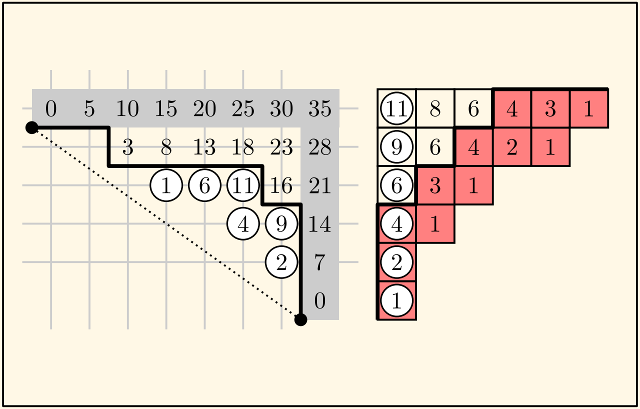
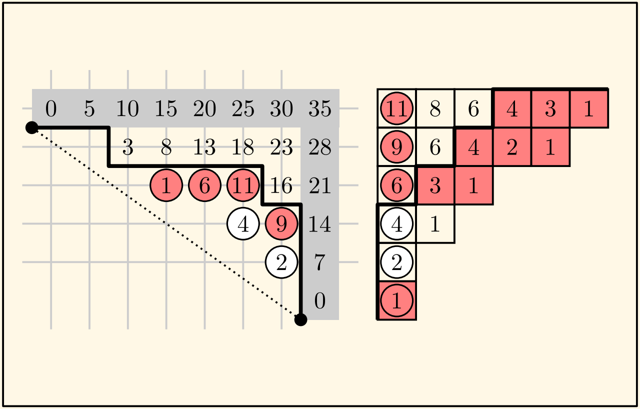


Intersecting the b-rows and a-boundary gives $s\ell(\pi) = 9$ as before. □



5. q-Catalan Numbers

The following conjecture is the reason for defining skew length.

Conjecture 1. The sum of length and skew length is a "q-Catalan statistic." That is, we have

$$\sum_{\pi \in C_{a,b}} q^{\ell(\pi)+s\ell(\pi)} = \mathrm{Cat}_q(a,b) = \frac{[a+b-1]_q!}{[a]_q!\,[b]_q!}.$$

And the following conjecture is the reason for calling it "skew length." [Maybe you prefer the name "co-skew length."]

Conjecture 2. For all $\pi \in C_{a,b}$ let $s\ell'(\pi) := (a-1)(b-1)/2 - s\ell(\pi)$. We conjecture that ℓ and $s\ell'$ have a symmetric joint distribution:

$$\sum_{\pi \in C_{a,b}} q^{\ell(\pi)} t^{s\ell'(\pi)} = \sum_{\pi \in C_{a,b}} t^{\ell(\pi)} q^{s\ell'(\pi)}$$

5. q-Catalan Numbers

The second conjecture above suggests that we should make the following definition.

Definition. For all $\gcd(a, b) = 1$ we define the "rational q, t-Catalan number"

$$\mathrm{Cat}_{q,t}(a, b) := \sum_{\pi \in C_{a,b}} q^{\ell(\pi)} t^{s\ell'(\pi)} \in \mathbb{N}[q, t].$$

Equivalent versions of this definition were given independently by Loehr-Warrington (2014) and Gorsky-Mazin (2013). We remark that the case $(a, b) = (n, n + 1)$ coincides with the "classical q, t-Catalan numbers" of Garsia and Haiman (1996):

$$\mathrm{Cat}_{q,t}(n, n + 1) = \text{classical } q, t\text{-Catalan numbers}$$

5. q-Catalan Numbers

Example. Our previous example $\pi \in C_{5,7}$ with $\ell(\pi) = 6$ and $s\ell'(\pi) = 12 - s\ell(\pi) = 12 - 9 = 3$ contributes to the red entry.

$$\mathrm{Cat}_{q,t}(5,7) = \left(\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & 1 \\
 & & & & & & & & 1 & 1 & 1 & 1 & \\
 & & & & & & 1 & 2 & 2 & 1 & 1 & & \\
 & & & & & 2 & 3 & 2 & 1 & 1 & & & \\
 & & & & 3 & 3 & 2 & 1 & 1 & & & & \\
 & & & 2 & 3 & 2 & 1 & 1 & & & & & \\
 & & 1 & \color{red}{3} & 2 & 1 & 1 & & & & & & \\
 & & 2 & 2 & 1 & 1 & & & & & & & \\
 & 1 & 2 & 1 & 1 & & & & & & & & \\
 & 1 & 1 & 1 & & & & & & & & & \\
 & 1 & 1 & & & & & & & & & & \\
 & 1 & & & & & & & & & & & \\
1 & & & & & & & & & & & &
\end{array}\right)$$

6. What Does It Mean?

May The Force Be With You