Noncrossing Parking Functions

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“Non-crossing partitions in representation theory”
Bielefeld, June 2014
Plan

1. Parking Functions
2. Noncrossing Partitions
3. Noncrossing Parking Functions
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Plan

1. Parking Functions
2. Noncrossing Partitions
3. Noncrossing Parking Functions
What is a Parking Function?
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Definition

A **parking function** is a vector \( \vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n \) whose increasing rearrangement \( b_1 \leq b_2 \leq \cdots \leq b_n \) satisfies:

\[
\forall i, \ b_i \leq i
\]

Imagine a one-way street with \( n \) parking spaces.

- There are \( n \) cars.
- Car \( i \) wants to park in space \( a_i \).
- If space \( a_i \) is full, she parks in first available space.
- Car 1 parks first, then car 2, etc.
- “\( \vec{a} \) is a parking function” \( \equiv \) “everyone is able to park”.
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A parking function is a vector $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ whose increasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies:

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What is a Parking Function?

Example \((n = 3)\)

Note that \(#PF_3 = 16\) and \(\mathbb{S}_3\) acts on \(PF_3\) with 5 orbits.

In General We Have

\[
#PF_n = (n + 1)^{n-1} \quad \# \text{ orbits } = \frac{1}{n+1} \binom{2n}{n}
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“Cayley” \quad “Catalan”
What is a Parking Function?

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Structure of Parking Functions

Idea (Pollack, ∼ 1974)

Now imagine a circular street with $n + 1$ parking spaces.

- Choice functions $= (\mathbb{Z}/(n + 1)\mathbb{Z})^n$.
- Everyone can park. One empty spot remains.
- Choice is a parking function $\iff$ space $n + 1$ remains empty.
- One parking function per rotation class.

Conclusion:

- $\text{PF}_n = \text{choice functions } / \text{ rotation}$
- $\text{PF}_n \approx \mathcal{C}_n (\mathbb{Z}/(n + 1)\mathbb{Z})^n/(1, 1, \ldots, 1)$
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Why do We Care?

Culture

The symmetric group $\mathfrak{S}_n$ acts diagonally on the algebra of polynomials in two commuting sets of variables:

$$\mathfrak{S}_n \curvearrowright \mathbb{Q}[x, y] := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

After many years of work, Mark Haiman (2001) proved that the algebra of diagonal coinvariants carries the same $\mathfrak{S}_n$-action as parking functions:

$$\omega \cdot \text{PF}_n \approx_{\mathfrak{S}_n} \mathbb{Q}[x, y]/\mathbb{Q}[x, y]^{\mathfrak{S}_n}$$

The proof was hard. It comes down to this theorem:

*The isospectral Hilbert scheme of $n$ points in $\mathbb{C}^2$ is Cohen-Macaulay and Gorenstein.*
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Haiman, *Conjectures on the quotient ring. . .*, Section 7

Let $W$ be a Weyl group with rank $r$ and Coxeter number $h$. That is, $W \acts \mathbb{R}^r$ by reflections and stabilizes a “root lattice” $Q \leq \mathbb{R}^r$. We define the $W$-parking functions as

$$\text{PF}_W := Q/(h+1)Q.$$ 

This generalizes Pollack because we have

$$(\mathbb{Z}/(n+1)\mathbb{Z})^n/(1,1,\ldots,1) = Q/(n+1)Q.$$ 

Recall that $W = S_n$ has Coxeter number $h = n$, and root lattice

$$Q = \mathbb{Z}^n/(1,1,\ldots,1) = \{(r_1,\ldots,r_n) \in \mathbb{Z}^n : \sum_i r_i = 0\}.$$
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The $W$-parking space has dimension generalizing the Cayley numbers

$$\dim \text{PF}_W = (h + 1)^r \left( = (n + 1)^{n-1} \right)$$

More generally: Given $w \in W$, the character of $\text{PF}_W$ is

$$\chi(w) = \# \{ \vec{a} \in \text{PF}_W : w(\vec{a}) = w \}$$

$$= (h + 1)^{r - \text{rank}(1 - w)} \left( = (n + 1)^{\# \text{cycles}(w) - 1} \right)$$

and the number of $W$-orbits generalizes the Catalan numbers

$$\# \text{orbits} = \frac{1}{|W|} \prod_{i=1}^{r} (h + d_i) \left( = \frac{1}{n + 1} \binom{2n}{n} \right)$$
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Parking Functions $\Leftrightarrow$ Shi Arrangement
Another Language

The $W$-parking space is the same as the Shi arrangement of hyperplanes. Given positive root $\alpha \in \Phi^+ \subseteq Q$ and integer $k \in \mathbb{Z}$ consider the hyperplane $H_{\alpha,k} := \{x : (\alpha, x) = k\}$. Then we define

$$\text{Shi}_W := \{H_{\alpha,\pm 1} : \alpha \in \Phi^+\}.$$ 

Cellini-Papi and Shi give an explicit bijection:

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There are $16 = (3 + 1)^{3-1}$ chambers and $5 = \frac{1}{4} \binom{6}{3}$ orbits.
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“Ceiling Diagrams”

I like to think of Shi chambers as elements of the set

\[ \{(w, A) : w \in W, \text{ antichain } A \subseteq \Phi^+, A \cap \text{inv}(w) = \emptyset \} \].

The Shi chamber with “ceiling diagram” \((w, A)\)

- is in the cone determined by \(w\)
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How to describe the \(W\)-action on ceiling diagrams?
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- is in the cone determined by \(w\)
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How to describe the \(W\)-action on ceiling diagrams?
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What is a Noncrossing Partition?
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Definition by Example

We encode this partition by the permutation $(1367)(45)(89) \in S_9$. 
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What is a Noncrossing Partition?

Theorem (Biane, and probably others)

Let \( T \subseteq \mathcal{S}_n \) be the generating set of all transpositions and consider the Cayley metric \( d_T : \mathcal{S}_n \times \mathcal{S}_n \to \mathbb{N} \) defined by

\[
d_T(\pi, \mu) := \min\{ k : \pi^{-1}\mu \text{ is a product of } k \text{ transpositions} \}.\]

Let \( c = (123 \cdots n) \) be the standard \( n \)-cycle. Then the permutation \( \pi \in \mathcal{S}_n \) corresponds to a noncrossing partition if and only if

\[
d_T(1, \pi) + d_T(\pi, c) = d_T(1, c).
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“\( \pi \) is on a geodesic between 1 and \( c \)”
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What is Noncrossing Partition?

Definition (Brady-Watt, Bessis)

Let $W$ be any finite Coxeter group with reflections $T \subseteq W$. Let $c \in W$ be any Coxeter element. We say $w \in W$ is a "noncrossing partition" if

$$d_T(1, w) + d_T(w, c) = d_T(1, c)$$

"$w$ is on a geodesic between 1 and $c$"
The Mystery of NC and NN

Let $W$ be a Weyl group (crystallographic finite Coxeter group). Let $NC(W)$ be the set of noncrossing partitions and let $NN(W)$ be the set of antichains in $\Phi^+$ (called “nonnesting partitions”). Then we have

$$\#NC(W) = \frac{1}{|W|} \prod_{i=1}^{r} (h + d_i) = \#NN(W)$$

▶ The right equality has at least two uniform proofs.
▶ The left equality is only known case-by-case.
▶ What is going on here?
Mystery

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$$\#NC(\mathcal{W}) = \frac{1}{|\mathcal{W}|} \prod_{i=1}^{r} (h + d_i) = \#NN(\mathcal{W})$$

- The right equality has at least two uniform proofs.
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Idea and an Anecdote

**Idea:** Since the parking functions can be thought of as

\[ \{(w, A) : w \in W, A \in NN(W), A \cap \text{inv}(w) = \emptyset\} \]

maybe we should also consider the set

\[ \{(w, \sigma) : w \in W, \sigma \in NC(W), \sigma \cap \text{inv}(w) = \emptyset\} \]

where “\(\sigma \cap \text{inv}(w)\)” means something sensible.

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Pause
Now we define the $W$-action on Shi chambers

Definition of $F$-parking functions

Recall the definition of the lattice of flats for $W$

$$\mathcal{L}(W) := \{ \cap_{\alpha \in J} H_{\alpha,0} : J \subseteq \Phi^+ \},$$

and for any flat $X \in \mathcal{L}(W)$ recall the definition of the parabolic subgroup

$$W_X := \{ w \in W : w(x) = x \text{ for all } x \in X \}.$$
Now we define the $W$-action on Shi chambers

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**Definition of $\mathcal{F}$-parking functions**

For any set of flats $\mathcal{F} \subseteq \mathcal{L}(W)$ we define the $\mathcal{F}$-parking functions

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\text{PF}_{\mathcal{F}} := \{ [w, X] : w \in W, X \in \mathcal{F}, w(X) \in \mathcal{F} \} / \sim
$$

where

$$[w, X] \sim [w', X'] \iff X = X' \text{ and } wW_X = w'W_{X'}$$

This set carries a natural $W$-action. For all $u \in W$ we define

$$u \cdot [w, X] := [wu^{-1}, u(X)]$$
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Now we define the $W$-action on Shi chambers

The Prototypical Example of $\mathcal{F}$-Parking Functions

If we consider the set of nonnesting flats

$$\mathcal{F} = \mathcal{NN} := \{ \cap_{\alpha \in A} H_{\alpha,0} : \text{antichain } A \subseteq \Phi^+ \}$$

then $\text{PF}_{\mathcal{NN}}$ is just the set of ceiling diagrams of Shi chambers with the natural action corresponding to $W \curvearrowright Q/(h + 1)Q$. 
Now we define the $W$-action on Shi chambers

But There is Another Example
Noncrossing Parking Functions

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Given any \( w \in W \) there is a corresponding flat

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\ker(1 - w) = \{ x : w(x) = x \} \in \mathcal{L}(W).
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If we consider the set of noncrossing flats

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\mathcal{F} = \mathcal{N}^\mathcal{C} := \{ \ker(1 - w) : w \in \text{NC}(W) \}
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then \( \text{PF}_{\mathcal{N}^\mathcal{C}} \) is something new and possibly interesting. We call \( \text{PF}_{\mathcal{N}^\mathcal{C}} \) the set of noncrossing parking functions.
But There is Another Example

Given any \( w \in W \) there is a corresponding flat

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If we consider the set of noncrossing flats

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then \( \text{PF}_{\mathcal{NC}} \) is something new and possibly interesting. We call \( \text{PF}_{\mathcal{NC}} \) the set of noncrossing parking functions.
Type A NC parking functions are just NC partitions with labeled blocks.
Theorem

If $W$ is a Weyl group then we have an isomorphism of $W$-actions:

$$\text{PF}_{NC} \cong_W \text{PF}_{NN}$$

This is just a fancy restatement of a theorem of Athanasiadis, Chapoton, and Reiner. Unfortunately the proof is case-by-case using a computer.
However

The noncrossing parking functions have two advantages over the nonnesting parking functions.

1. $\text{PF}_{NN}$ is defined only for Weyl groups but $\text{PF}_{NC}$ is defined also for noncrystallographic Coxeter groups.

2. $\text{PF}_{NC}$ carries an extra cyclic action. Let $C = \langle c \rangle \leq W$ where $c \in W$ is a Coxeter element. Then the group $W \times C$ acts on $\text{PF}_{NC}$ by

$$(u, c^d) \cdot [w, X] := [c^d w u^{-1}, u(X)].$$
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The noncrossing parking functions have two advantages over the nonnesting parking functions.

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$$(u, c^d) \cdot [w, X] := [c^d w u^{-1}, u(X)].$$
Cyclic Sieving “Theorem”

Let $h := |\langle c \rangle|$ be the Coxeter number and let $\zeta := e^{2\pi i / h}$. Then for all $u \in W$ and $c^d \in C$ we have

$$\chi(u, c^d) = \# \{[w, X] \in \text{PF}_{NC} : (u, c^d) \cdot [w, X] = [w, X]\}$$

$$= \lim_{q \to \zeta^d} \frac{\det(1 - q^{h+1}u)}{\det(1 - qu)}$$

$$= (h + 1)^{\text{mult}_u(\zeta^d)},$$

where $\text{mult}_u(\zeta^d)$ is the multiplicity of the eigenvalue $\zeta^d$ in $u \in W$.

Unfortunately the proof is case-by-case. (And it is not yet checked for all exceptional types.)
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For more on noncrossing parking functions see my paper with Brendon Rhoades and Vic Reiner:

Vielen Dank!

Bielefeld wirklich gibt es!

picture by +Drew Armstrong and +David Roberts