Maximal Chains of Parabolic Subgroups

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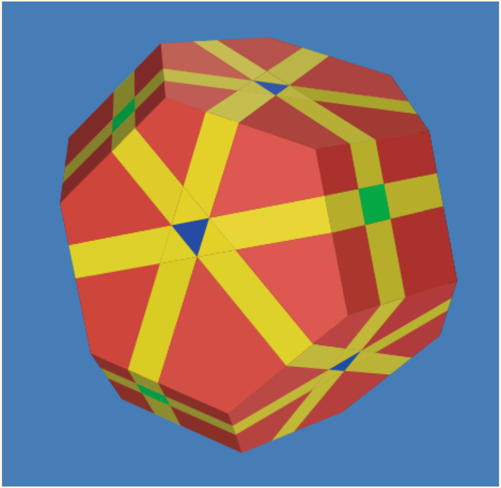
This talk is based on a picture I saw.



Gunnells, Paul E. *Cells in Coxeter Groups*. NAMS, **53** (2006)

Galois Connections

Definition (G. Birkhoff and Ø. Ore, ~1940)

Let S and S' be sets and let $\mathrm{R} \subseteq S \times S'$ be a relation (write "$a\mathrm{R}b$" to mean $(a, b) \in \mathrm{R}$). For all subsets $A \subseteq S$ and $B \subseteq S'$, define

$$A^* := \{b \in S' : a\mathrm{R}b \text{ for all } a \in A\} \subseteq S',$$
$$B^* := \{a \in S : a\mathrm{R}b \text{ for all } b \in B\} \subseteq S.$$

Thus we have a pair of maps:

$$* : 2^S \to 2^{S'},$$
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Properties

1. The maps $** : 2^S \to 2^S$ and $** : 2^{S'} \to 2^{S'}$ are closure operators, in the sense that
   - $X \subseteq X^{**}$ for all X,
   - $X \subseteq Y \Rightarrow X^{**} \subseteq Y^{**}$,
   - $(X^{**})^{**} = X^{**}$ for all X.

2. The maps $* : 2^S \to 2^{S'}$ and $* : 2^{S'} \to 2^S$ restrict to reciprocal order-reversing lattice isomorphisms between

$$\{**\text{-}\mathbf{closed} \text{ subsets of } S\} \quad \overset{*}{\longleftrightarrow} \quad \{**\text{-}\mathbf{closed} \text{ subsets of } S'\}$$

3. Of course, this is general nonsense. The real math consists in characterizing the $**$-closure for specific examples.

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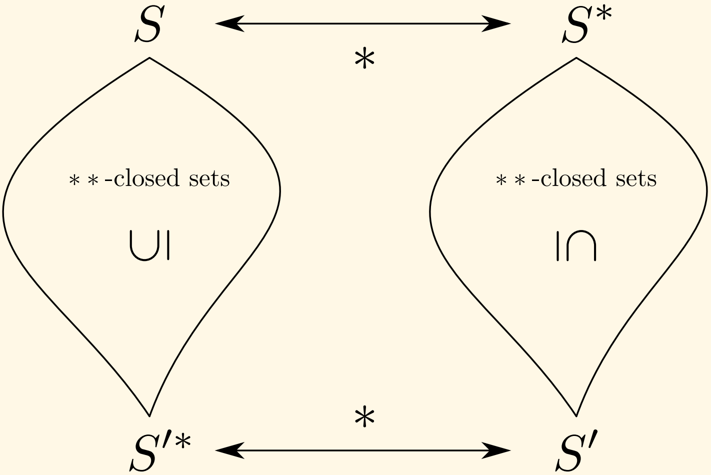

Galois Connections

Example 1 (Galois Theory)

$S = K$ is a field

$S' = G \leq \mathrm{Aut(K)}$ is a finite group of automorphisms

"aRg" means "g fixes a"

Theorem (Galois, Dedekind):

1. $k := G^* = \mathrm{Fix(G)}$ is a **subfield** of K (easy), and the $**$-closed subsets of K are **all** the intermediate fields $k \subseteq \mathbb{F} \subseteq K$.

2. Given any $X \subseteq K$, the set $X^* \subseteq G$ is a **subgroup** (easy), and the $**$-closed subsets of G are **all** the subgroups $H \leq G$.

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Example 2 (Nullstellensatz)

$S = K^n$, where K is an algebraically closed field

$S' = K[x_1, x_2, \ldots, x_n]$ is the ring of polynomials

"$\mathbf{x} \mathrm{R} f$" means "$f(\mathbf{x}) = 0$"

Theorem (Hilbert, Zariski):

Given any $X \subseteq K^n$, the set $X^* \subseteq K[x_1, \ldots, x_n]$ is an **ideal** (easy), and the $**$-closure of an ideal $I \subseteq K[x_1, \ldots, x_n]$ is its radical:

$$I^{**} = \sqrt{I} = \left\{ g \in K[x_1, \ldots, x_n] : g^\ell \in I \text{ for some } \ell \right\}.$$

Definition:

The $**$-closure on K^n is called Zariski closure.

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Example 3 (Steinberg's Theorem)

$S = V$ is a Euclidean space

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"vRg" means "g fixes v"

Theorem (Steinberg, folklore, Barcelo-Ihrig):

Given any $X \subseteq V$, the set $X^* \subseteq G$ is a **subgroup** (easy), and it is generated by the reflections that fix X pointwise.

Corollary:

The $**$-closed subsets of V are the intersections of reflecting hyperplanes.

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The $**$-closed subgroups of G are called parabolic.

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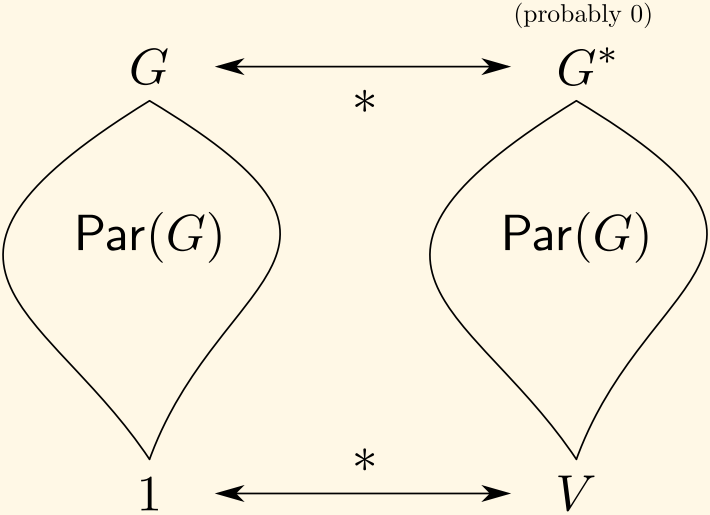

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Remarks on Steinberg

1. Par(G) is the lattice of **Par**abolic subgroups of G.

2. Parabolic subgroups are conjugate to **simple** parabolic subgroups (generated by subsets of simple reflections). The simple parabolics form a boolean sublattice inside Par(G).

3. Par(G) is a geometric lattice (Birkhoff, 1935); i.e., it is the lattice of flats of a matroid (Whitney, 1935).

4. The lattice Par(G) is graded of rank r, where

$$r := \dim(V/G^*) \quad (\text{ probably } G^* = 0 \text{ })$$

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- The symmetric group S_n acts on $\mathbb{R}^n$ by permuting a basis

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \in \mathbb{R}^n$$

- The transposition $(i, j) \in S_n$ is the reflection in $(\mathbf{e}_i - \mathbf{e}_j)^\perp \subseteq \mathbb{R}^n$.
- The rank of $S_n \curvearrowright \mathbb{R}^n$ is $r = n - 1$ because

$$S_n^* = \mathbb{R}(\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n).$$

- $\mathrm{Par}(S_n) \approx \mathrm{Par}(n)$, the lattice of **Par**titions of the set $\{1, 2, \ldots, n\}$:

  The isomorphism $\mathrm{Par}(n) \to \mathrm{Par}(S_n)$ is given by

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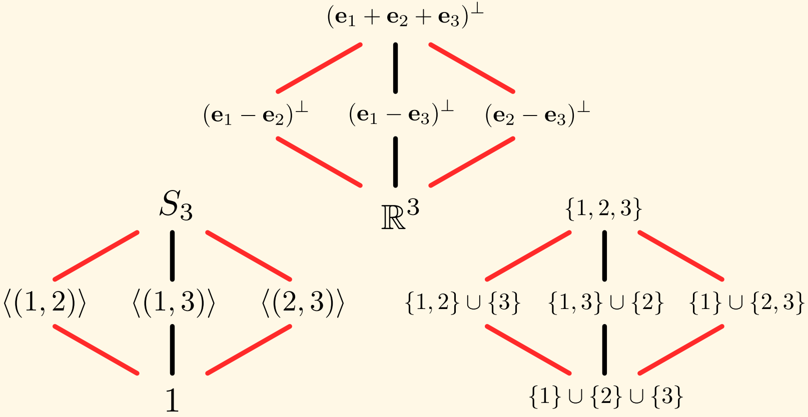
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S_3
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Theorem (Erdős-Guy-Moon)

The number of maximal chains in Par(n) is $\dfrac{(n-1)!\,n!}{2^{n-1}}$.

- Erdős, Guy and Moon, *On refining partitions*. JLMS, (1975)

Proof.

Start with $\{1\} \cup \{2\} \cup \cdots \cup \{n\}$. Choose two blocks and join them, in $\binom{n}{2}$ ways. Now you have $n-1$ blocks. Choose two blocks and join them, in $\binom{n-1}{2}$ ways. Continue until you reach $\{1, 2, \ldots, n\}$. The total number of choices was

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Other Types

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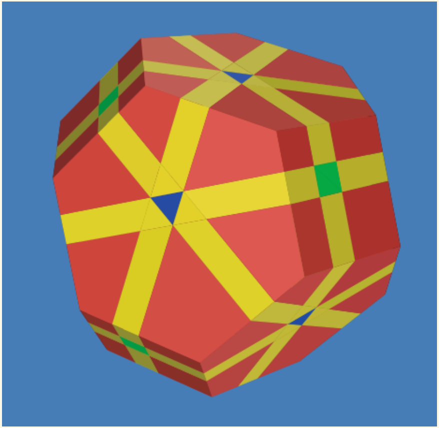
Now let G be a finite reflection group of rank r and consider the permutohedron $\mathrm{Perm}(G)$ (the dual zonotope):



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Remarks on Permutohedra

- The vertices of $\mathrm{Perm}(G)$ are the elements of the group G.
- For each corank 1 parabolic $G' \prec G$ there is a pair of facets, each isomorphic to $\mathrm{Perm}(G')$.
- Each vertex is contained in r facets (the zonotope is simple).
- We conclude that

$$r\,|G| = 2 \sum_{G' \prec G} |G'| \qquad (\star)$$

where the sum is over corank 1 parabolic subgroups $G' \prec G$.

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- The vertices of Perm(G) are the elements of the group G.
- For each corank 1 parabolic $G' \prec G$ there is a pair of facets, each isomorphic to Perm(G').
- Each vertex is contained in r facets (the zonotope is simple).
- We conclude that

$$r\,|G| = 2 \sum_{G' \prec G} |G'| \qquad (\star)$$

  where the sum is over corank 1 parabolic subgroups $G' \prec G$.

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Other Types

Theorem (Could this possibly be new?):

The number of maximal chains in $\mathrm{Par}(G)$ is $\frac{r!\,|G|}{2^r}$.

Proof.

We know from $(\star)$ that

$$|G| = \frac{2}{r} \sum_{G' \prec G} |G'|,$$

with the sum over corank 1 parabolic subgroups $G' \prec G$. Recurse:

$$|G| = \frac{2^r}{r!}\,(\#\text{ maximal flags of parabolics}).$$

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More Generally

Let C_d equal be the number of chains of parabolic subgroups

$$G \succ G_1 \succ G_2 \succ \cdots \succ G_d$$

where G_i has corank i. Let F_d be the number of codimension-d faces in the permutohedron. Then

$$d!\, F_d = 2^d\, C_d$$

Example

If G has rank r, then

- $F_r =$ number of vertices $= |G|$,
- $C_r =$ number of maximal chains in $\mathrm{Par}(G)$.

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Other Types

Humble Suggestion

Investigate the action of G on chains in $\mathrm{Par}(G)$.

Type A has been **thoroughly studied** since

- Stanley, Richard P. *Some aspects of groups acting on finite posets*. JCTA, (1982)

But maybe the formula $r!|G|/2^r$ gives new insight?

Extension of the Idea

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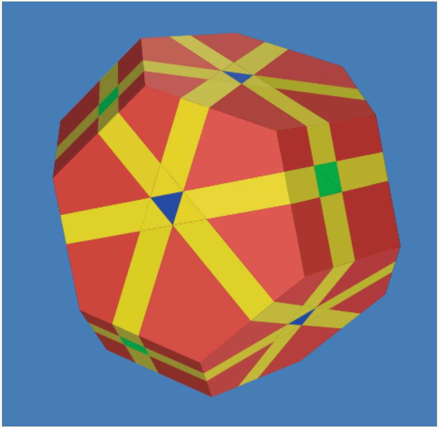
This is also a picture of the Shi hyperplane arrangement of type A_3:



Extension of the Idea

Definition

If G has crystallographic root system Φ, then the Shi arrangement is

$$\mathrm{Shi}(G) := \bigcup_{\alpha \in \Phi^+} \{H_{\alpha,0}, H_{\alpha,1}\}$$

Theorem (Yoshinaga, 2004)

The characteristic polynomial of $\mathrm{Shi}(G)$ is

$$\chi_{\mathrm{Shi}(G)}(p) = (p-h)^r,$$

where h, r are the Coxeter number and rank of G. Hence (Zaslavsky), $\mathrm{Shi}(G)$ has $(h+1)^r$ regions and $(h-1)^r$ bounded regions.

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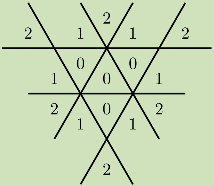
Extension of the Idea

Definition

Given a region R of Shi(G), let dof(R) be the maximal number of linearly-independent rays in R. Call this the "degrees of freedom" of R.

Note: R bounded $\iff$ dof$(R) = 0$

Example



Extension of the Idea

Definition

Define the "degrees of freedom" polynomial of the Shi arrangement

$$\mathrm{DF}_G(q) := \sum_{R \in \mathrm{Shi}(G)} q^{\mathrm{dof}(R)}.$$

Theorem

The DF *polynomial satisfies the recurrence*

$$\frac{d}{dq}\mathrm{DF}_G(q) = 2 \sum_{G' \prec G} \mathrm{DF}_{G'}(q),$$

where the sum is over corank 1 *parabolic subgroups* $G' \prec G$.

Extension of the Idea

Theorem

The recurrence can be explicitly solved:

$$\mathrm{DF}_G(q) = (-1)^r \sum_{G' \leq G} (1 - h')^{r'} \cdot \chi_{G|G'}(-1) \cdot q^{r-r'},$$

where

- *The sum is over* **all** *parabolic subgroups $G' \leq G$,*
- *h', r' are the Coxeter number and rank of G',*
- *$\chi_{G|G'}(p)$ is the char. poly. of the reflection arrangement of G* **restricted** *to the fixed space of G'.*

Extension of the Idea

Questions

1. Use the recurrence to **define** $\mathrm{DF}_G(q)$ for **non-crystallographic** groups G. For example:

$$\mathrm{DF}_{H_3}(q) = 729 + 302q + 180q^2 + 120q^3.$$

   Does this mean anything?

2. Replace $\chi_{G|G'}(-1)$ by the **unevaluated** $\chi_{G|G'}(-p)$ to define

$$\mathrm{DF}_G(p,q) = (-1)^r \sum_{G' \leq G} (1-h')^{r'} \cdot \chi_{G|G'}(-p) \cdot q^{r-r'}.$$

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3. Replace $\chi_{\mathsf{Shi}(G')}(1) = (1 - h')^{r'}$ by $\chi_{\mathsf{Shi}(G')}(-t) = (-t - h')^{r'}$:

$$\mathrm{DF}_G(t, p, q) = (-1)^r \sum_{G' \leq G} (-t - h')^{r'} \cdot \chi_{G|G'}(-p) \cdot q^{r-r'}.$$

Does this mean anything?

4. Replace $\mathsf{Shi}(G)$ by **any deformation** $\mathcal{A}(G)$ of the Coxeter arrangement in the sense of (Postnikov-Stanley, 2000):

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Thank You