# Rational Catalan Combinatorics 2

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# This talk will *further* advertise a definition.

Here is it.

**Definition**

Let $0 < a < b$ be coprime and consider $x = a/(b-a) \in \mathbb{Q}$.
Then we define the **Catalan number**

$$\mathrm{Cat}(x) := \frac{1}{a+b}\binom{a+b}{a,b} = \frac{(a+b-1)!}{a!b!}.$$

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# Special cases.

## When $b = 1 \bmod a \ldots$

- *Eugène Charles Catalan (1814-1894)*

  $(a < b) = (n < n + 1)$ gives the **good old Catalan number**

$$\mathrm{Cat}(n) = \mathrm{Cat}\left(\frac{n}{1}\right) = \frac{1}{2n+1}\binom{2n+1}{n}.$$

- *Nicolaus Fuss (1755-1826)*

  $(a < b) = (n < kn + 1)$ gives the **Fuss-Catalan number**

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# Euclidean Algorithm & Symmetry.

## Definition

Again let $x = a/(b-a)$ for $0 < a < b$ coprime.
Then we define the **derived Catalan number**

$$\mathrm{Cat}'(x) := \frac{1}{b}\binom{b}{a} = \begin{cases} \mathrm{Cat}(1/(x-1)) & \text{if } x > 1 \\ \mathrm{Cat}(x/(1-x)) & \text{if } x < 1 \end{cases}$$

This is a "categorification" of the Euclidean algorithm.

## Remark

If we define $\mathrm{Cat} : \mathbb{Q} \setminus [-1,0] \to \mathbb{N}$ by $\mathrm{Cat}(-x-1) := \mathrm{Cat}(x)$ then the formula is simpler:

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# Catalan "Number Theory"?

## Problem

Describe a recurrence for the Cat function, perhaps in terms of the *Calkin-Wilf sequence*

$$\frac{1}{1} \mapsto \frac{1}{2} \mapsto \frac{2}{1} \mapsto \frac{1}{3} \mapsto \frac{3}{2} \mapsto \frac{2}{3} \mapsto \frac{3}{1} \mapsto \frac{1}{4} \mapsto \frac{4}{3} \mapsto \cdots$$

which is defined by

$$x \mapsto \frac{1}{2\lfloor x \rfloor + 1 - x}.$$

See Aigner and Ziegler: *"Proofs from THE BOOK"*, Chapter 17.

# Motivation?

**Motivation 1:** Core Partitions

**Motivation 2:** Parking Functions

**Motivation 3:** "Lie Theory"

**Motivation 4:** Noncrossing Partitions

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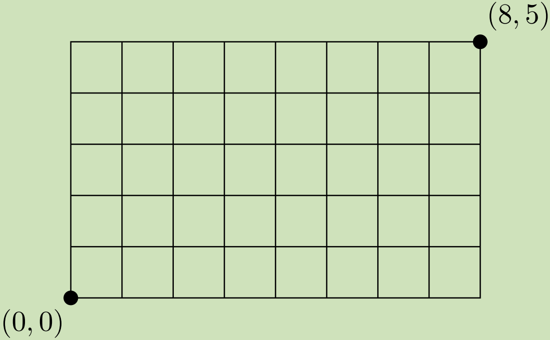
**Motivation 5:** Associahedra (with B. Rhoades and N. Williams)

# The Prototype: Actuarial Science.

- Consider the "Dyck paths" in an $a \times b$ rectangle.

**Example $(a < b) = (5 < 8)$**



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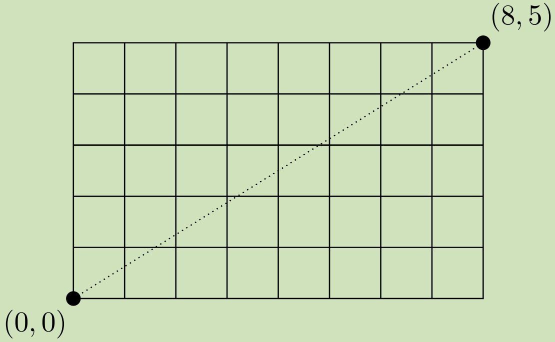

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- Let $\mathcal{D}(x)$ denote the set of Dyck paths.

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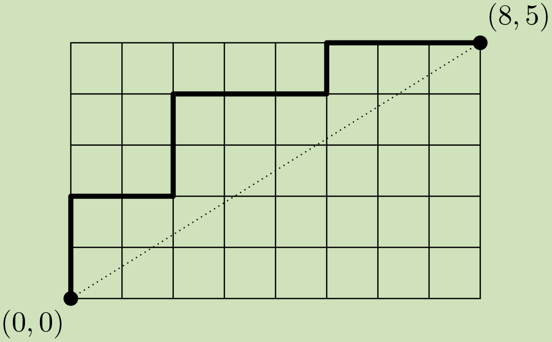

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## Theorem (Grossman, 1950, Bizley, 1954)

*For $a, b$ coprime, the number of Dyck paths is the Catalan number:*

$$|\mathcal{D}(x)| = \mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a, b}.$$

- *Claimed by Grossman (1950), "Fun with lattice points".*
  *(He wrote 8 articles with this name.)*
- *Proved by Bizley (1954), in Journal of the Institute of Actuaries.*
- *Proof: Break $\binom{a+b}{a,b}$ lattice paths into cyclic orbits of size $a+b$.*
  *Each orbit contains a unique Dyck path.*

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## Theorem (Armstrong, 2010, Loehr, 2010)

- *Consider the rectangle of height $a$ and width $b$ with $0 < a < b$ coprime . The number of Dyck paths with $i$ vertical runs equals*

$$\mathrm{Nar}(x, i) := \frac{1}{a}\binom{a}{i}\binom{b-1}{i-1}.$$

*Call these the* **Narayana numbers**.

- *And the number with $r_j$ vertical runs of length $j$ equals*

$$\mathrm{Krew}(x, \mathbf{r}) := \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a} = \frac{b!}{r_0! r_1! \cdots r_a!}.$$

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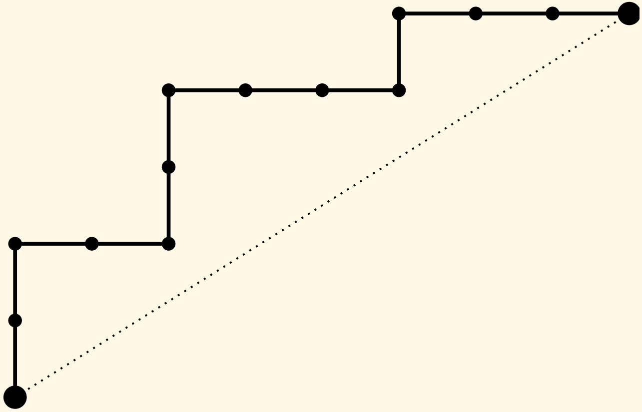
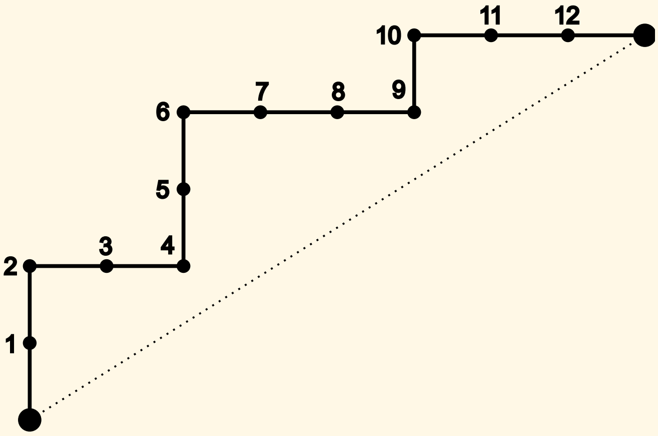
# To **create** a noncrossing partition...

- Start with a Dyck path. (E.g. $(a, b) = (5, 8)$.)



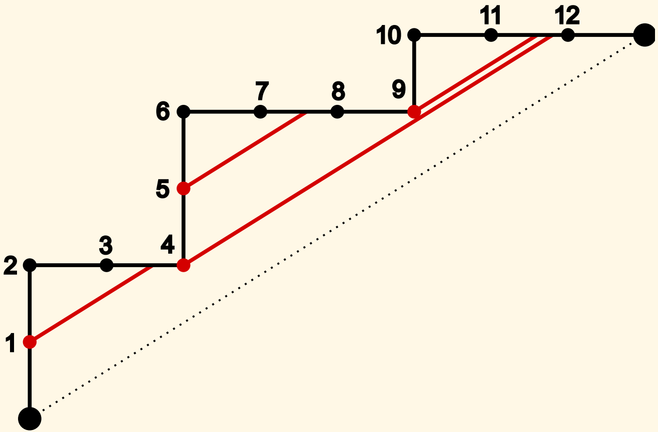
# To **create** a noncrossing partition...

- Label the internal vertices by $\{1, 2, \dots, a+b-1\}$.



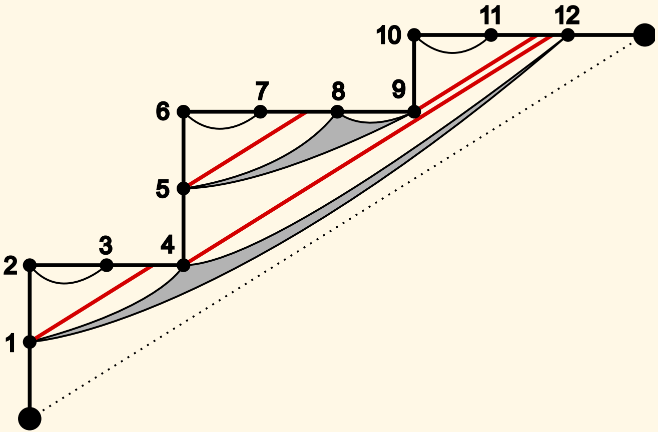
# To **create** a noncrossing partition...

- Shoot lasers from the bottom left.



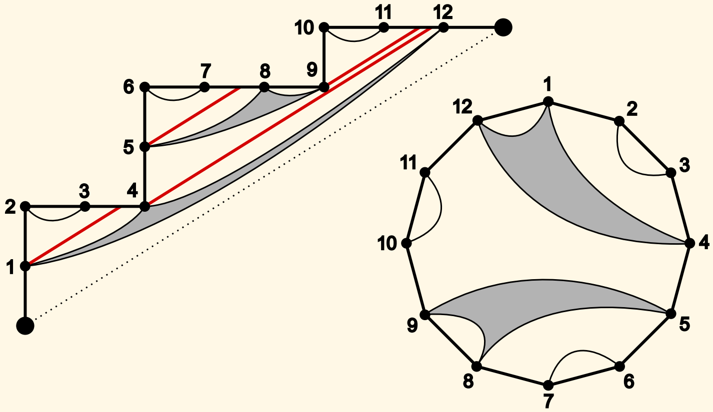
# To **create** a noncrossing partition. . .

- Who can see each other?



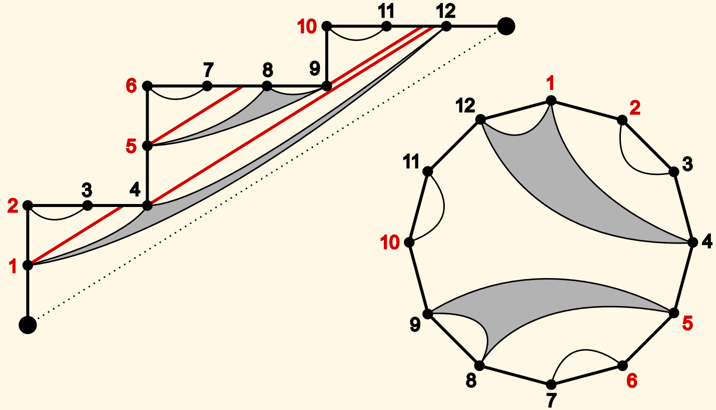
# To **create** a noncrossing partition. . .

- There you go!



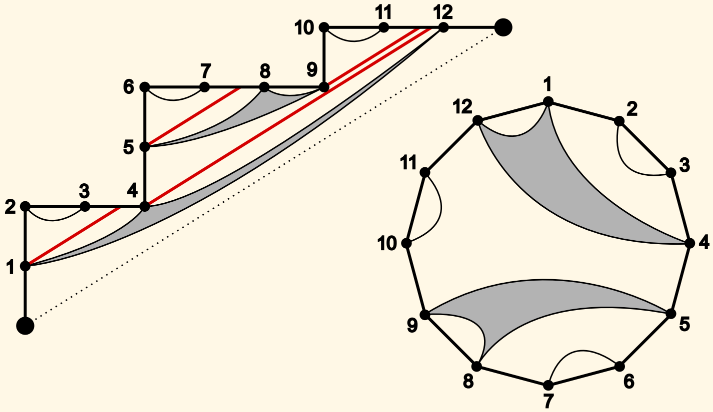
# To **create** a noncrossing partition...

- We have created $\mathrm{Cat}(x) = \frac{1}{a}\binom{a+b}{a,b}$ different noncrossing partitions of the cycle $[a+b-1]$, and each of them has $a$ blocks.



# To **rotate** a noncrossing partition...

- Q: What does "rotation" of the partition correspond to?



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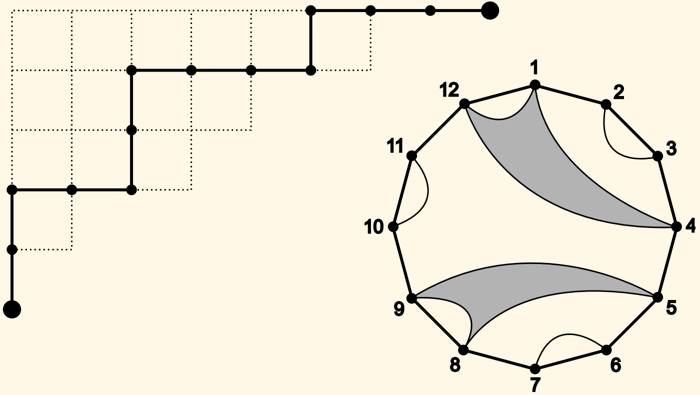
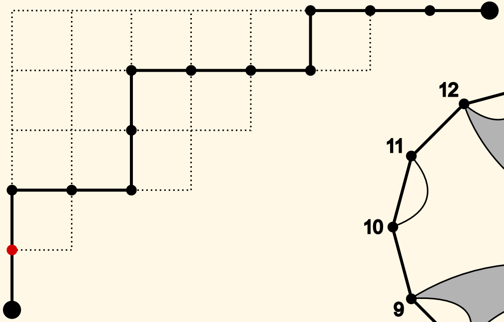
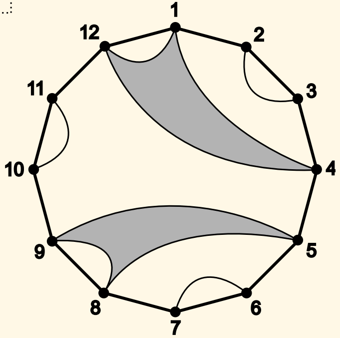
- A: Think of the path as a maximal chain in a poset.



# To **rotate** a noncrossing partition. . .

- Perform "promotion" on the chain.





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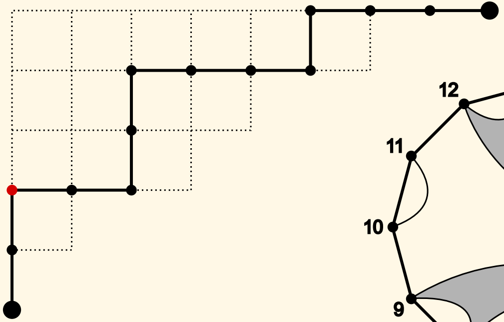

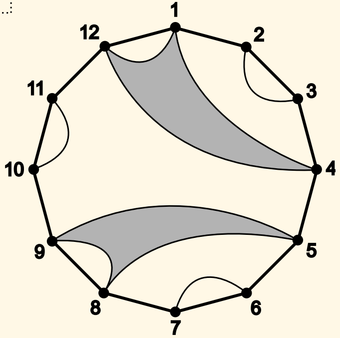

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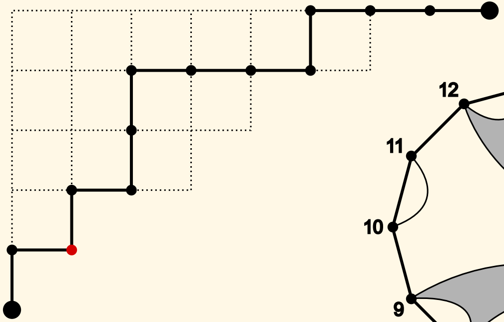

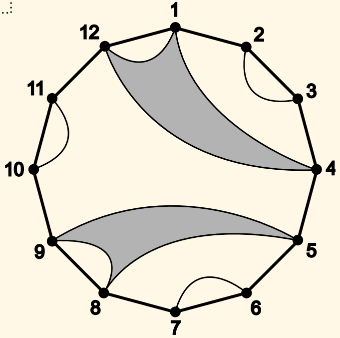

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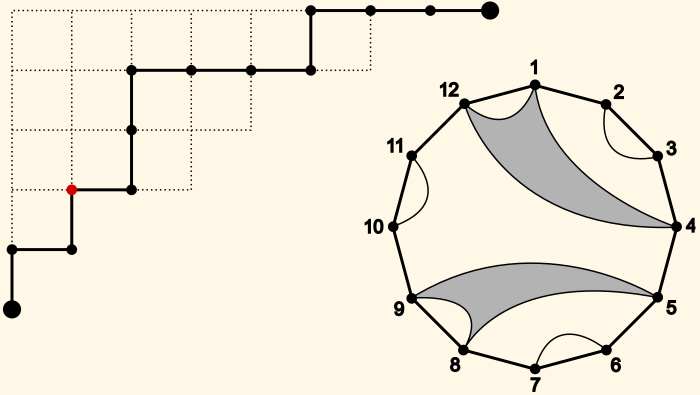

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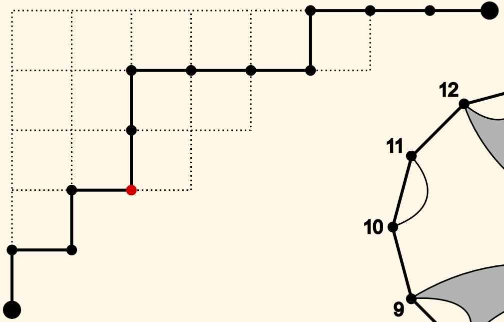

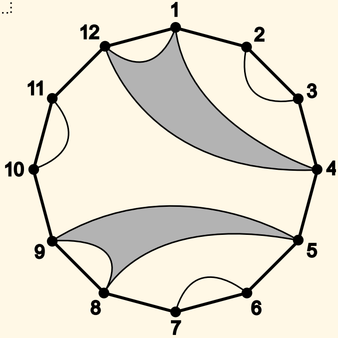

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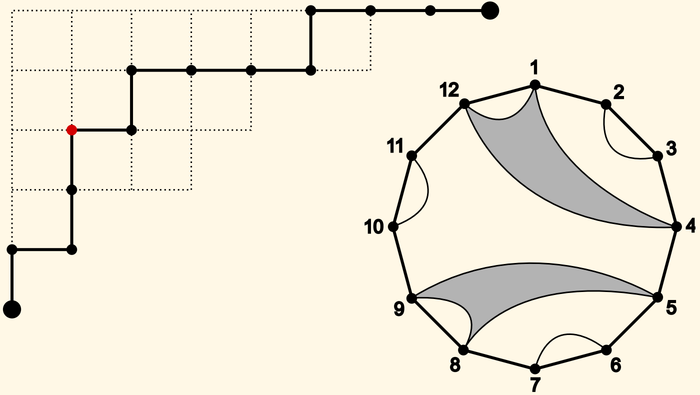

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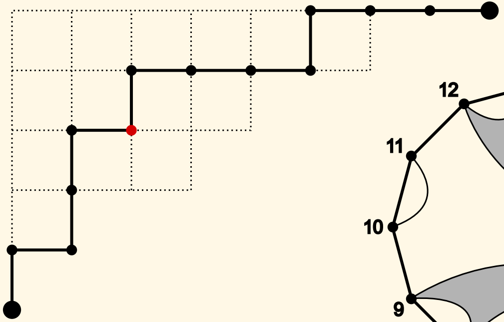

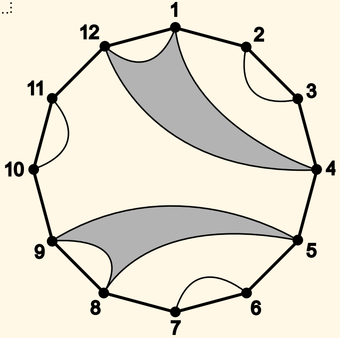

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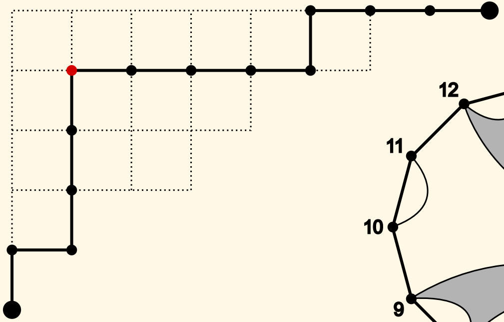

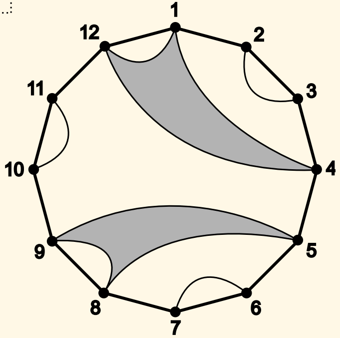

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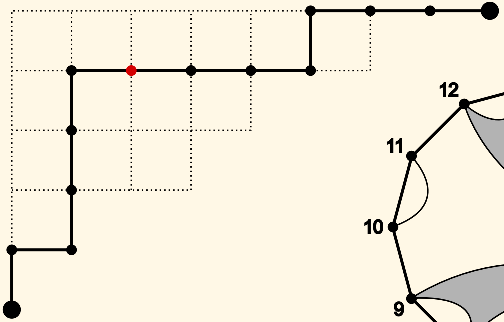

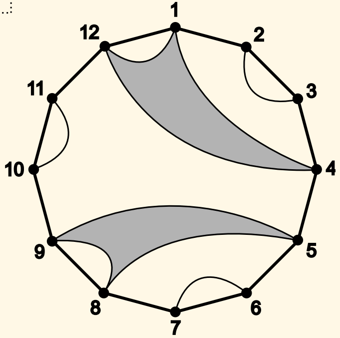

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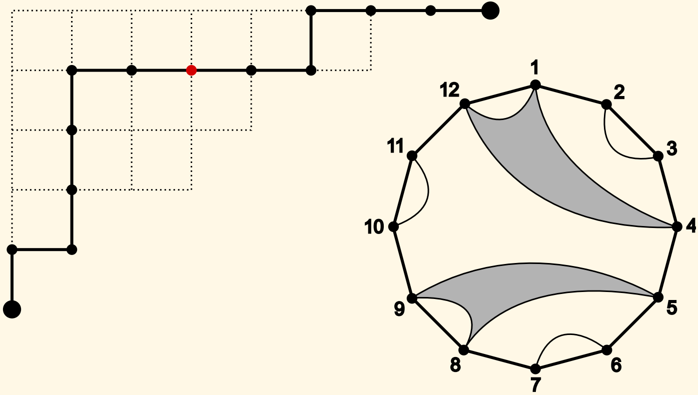

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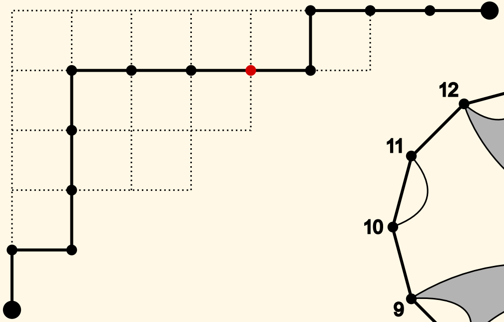

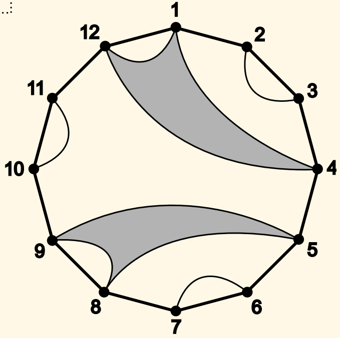

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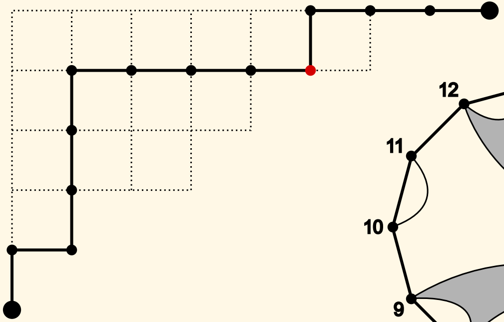

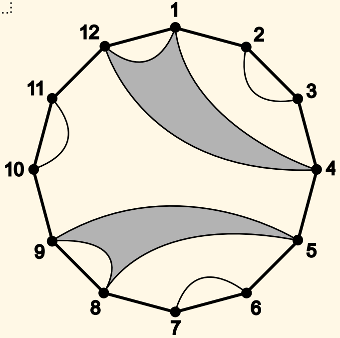

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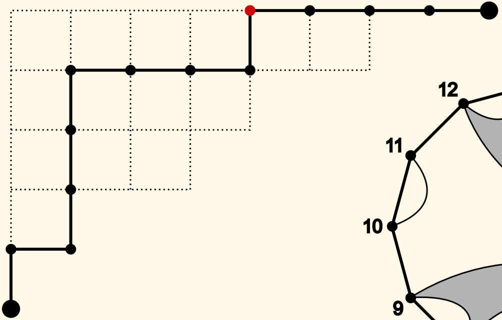

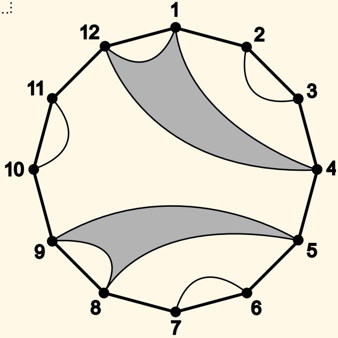

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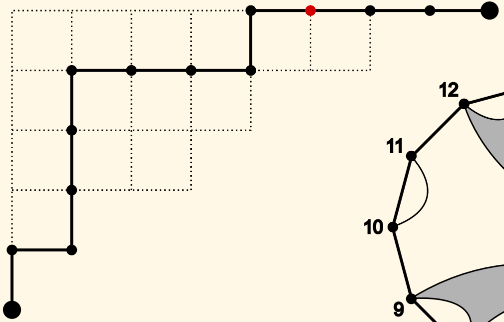

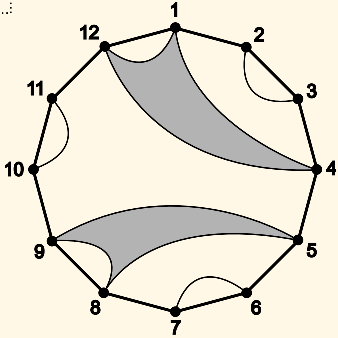

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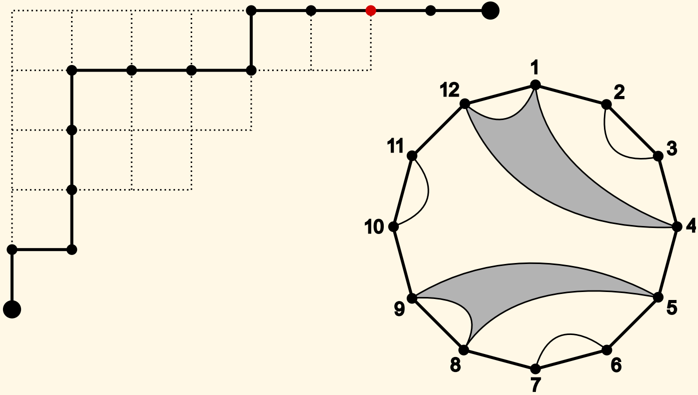

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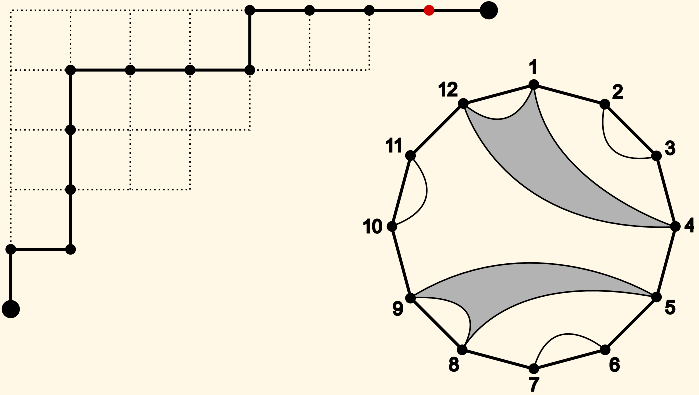

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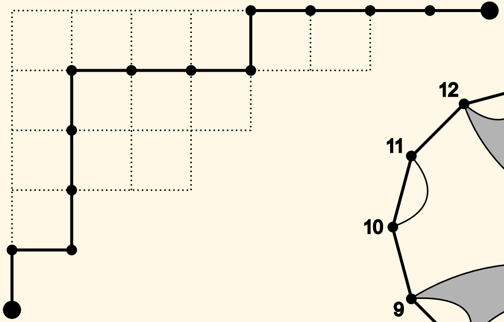
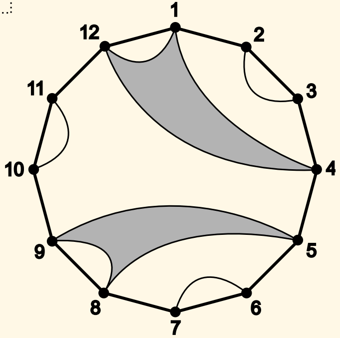
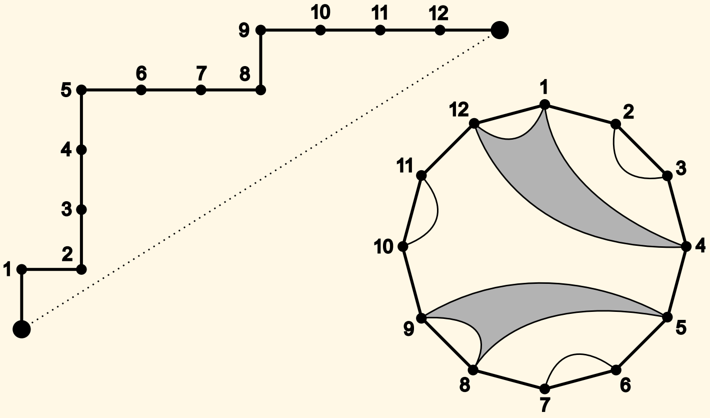




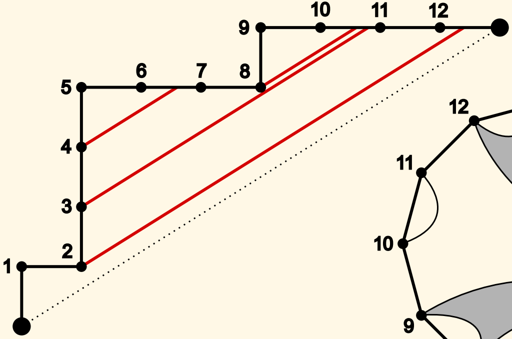
# To **rotate** a noncrossing partition. . .

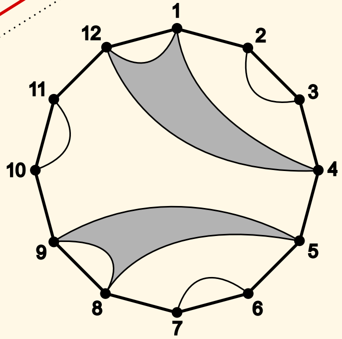
- Think of it as a path again.



# To **rotate** a noncrossing partition. . .

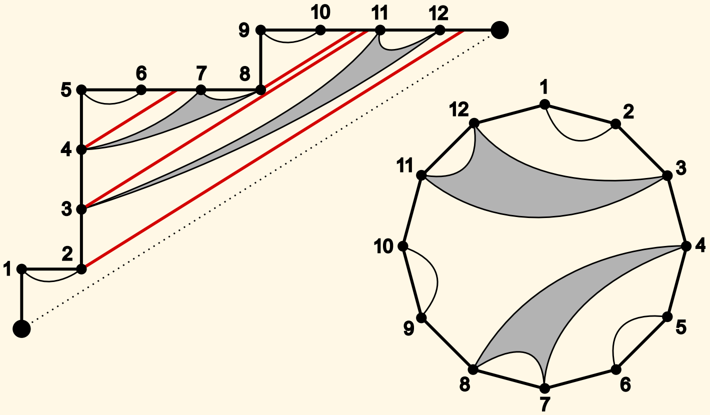
- Again with the lasers.



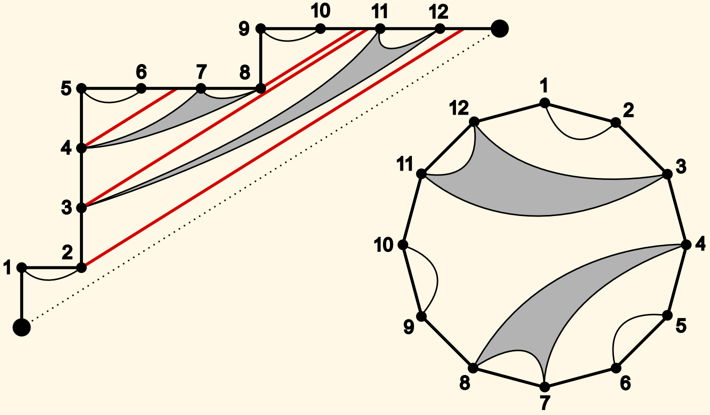
# To **rotate** a noncrossing partition. . .

- And there you go!



# To **rotate** a noncrossing partition. . .

- *Psst . . . mention the case* $(a < b) = (n < n(k-1)+1)$.



# What have we done?

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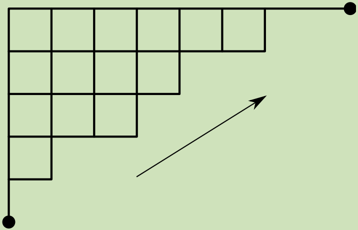
## Definition

For $0 < a < b$ coprime, consider the **triangle poset**

$$\mathcal{T}(a,b) := \{(x,y) \in \mathbb{Z}^2 : y \leq a,\ x \leq b,\ yb - xa \geq 0\}.$$

## As you see here.



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## Theorem (with Nathan Williams)

- *Promotion on $\mathcal{T}(a,b)$ has order $a+b-1$.*
- *Furthermore, the number of orbits* Orb *with $d$ dividing $\frac{a+b-1}{|\mathrm{Orb}|}$ is (most likely) the coefficient of $q^d$ in*

$$\frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q \mod (q^{a+b-1}-1).$$

!

- $\mathcal{T}(n,n+1)$ is a root poset.
- Credit: D.White, Panyushev, Striker-Williams, Me-Stump-Thomas.

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**Observation**

We have some “rational NC partitions” but they don’t form a poset. (They all have $a$ blocks!)

**Question**

Can one define a **poset** of “rational NC partitions”?

**Answer**

Yes.

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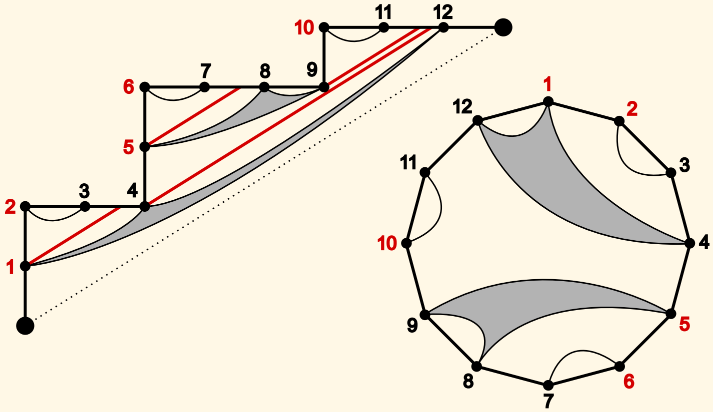
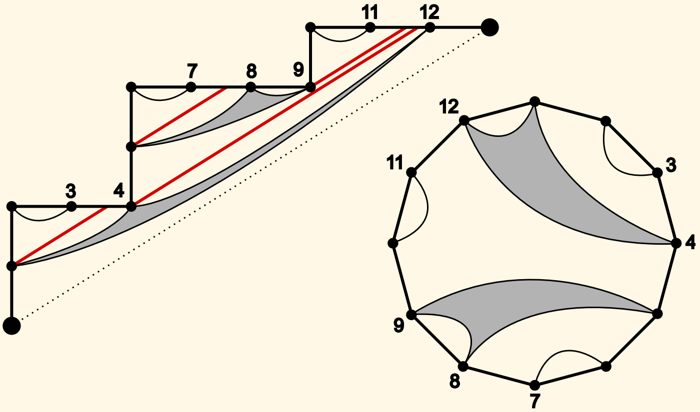
# To **de-homogenize** a noncrossing partition. . .

- Remember this thing?



# To **de-homogenize** a noncrossing partition. . .

- Now we label only the horizontal steps.



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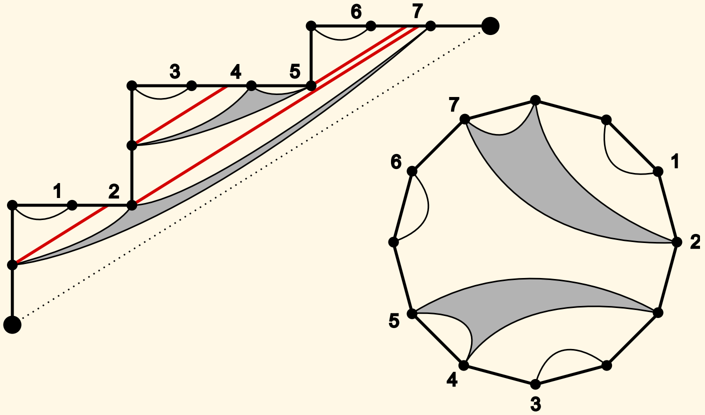
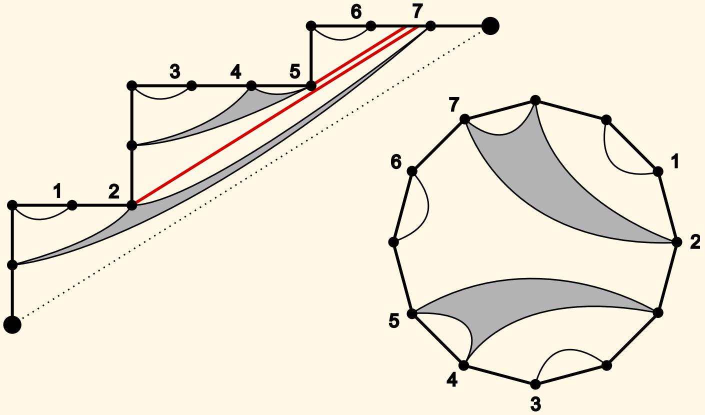


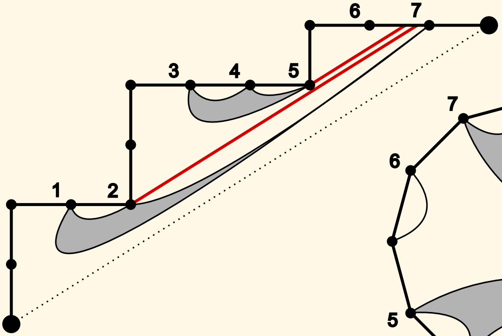
# To **de-homogenize** a noncrossing partition. . .

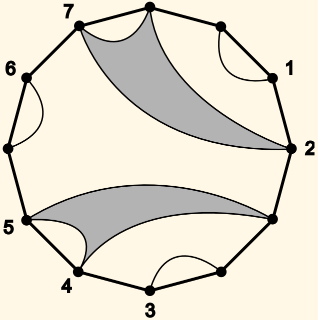
- Now we shoot lasers only from the corners.



# To **de-homogenize** a noncrossing partition...

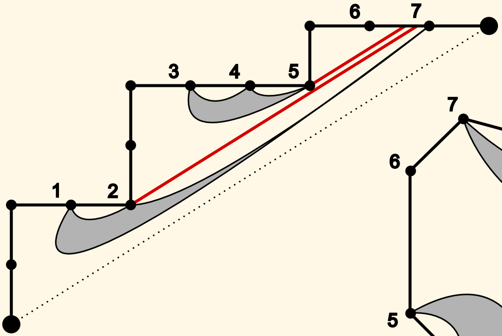
- Now who can see each other?

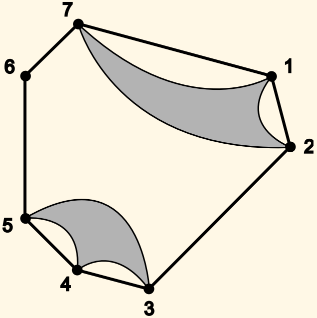


# To **de-homogenize** a noncrossing partition. . .

- There you go!



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## Definition (with Nathan Williams)

Consider $x = a/(b-a)$ with $0 < a < b$ coprime. We have constructed a poset of NC partitions called $\mathrm{NC}(x) = \mathrm{NC}(a, b)$.

## Facts (with Nathan Williams)

- $\mathrm{NC}(n, n+1) = \mathrm{NC}(n)$ is the **good old noncrossing partitions**.
- $\mathrm{NC}(n, (k-1)n+1)$ is the $k$**-divisible noncrossing partitions**.
- $\mathrm{NC}(a, b)$ is a (graded) order filter in $\mathrm{NC}(b-1)$.
- $\mathrm{NC}(a, b)$ is ranked by the Narayana numbers $\frac{1}{a}\binom{a}{i}\binom{b-1}{i-1}$.
- $\mathrm{NC}(x)$ has $\mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a,b}$ elements.
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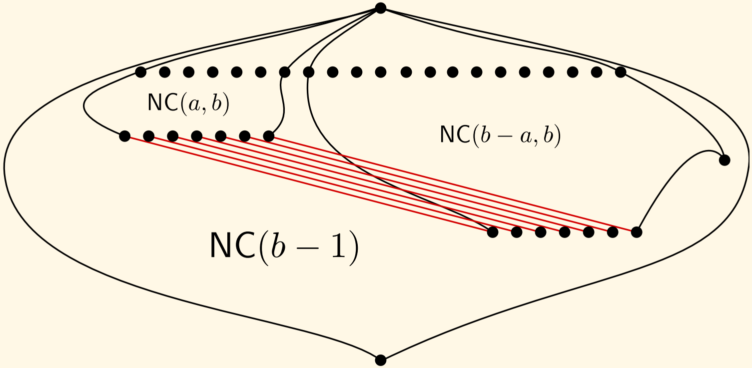
# Inversion = “Alexander Duality”?

- Note that $x \leftrightarrow 1/x$ is the same as $(a < b) \leftrightarrow (b - a < b)$.



# So now what?

## Observation

The good old associahedron is a nice polytope with *h*-vector given by the good old Narayana numbers.

## Question

Can one define a "rational associahedron" with *h*-vector given by

$$\mathrm{Nar}(x,i) = \frac{1}{a}\binom{a}{i}\binom{b-1}{i-1}?$$

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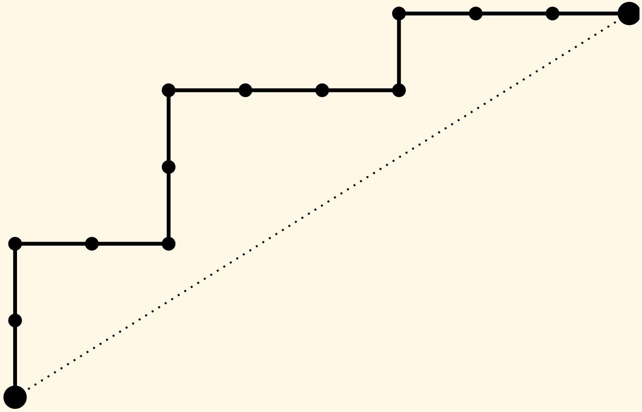
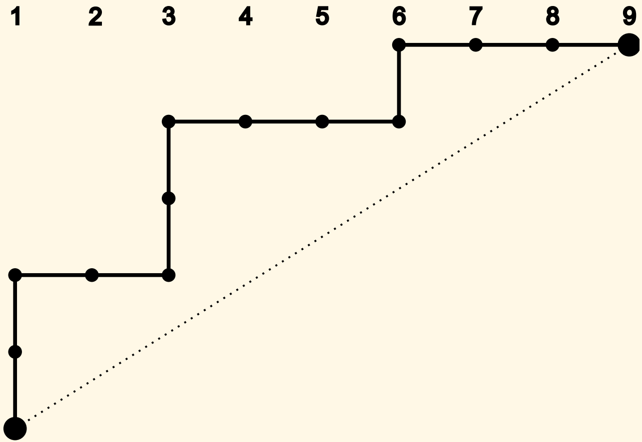
# To **create** a polygon dissection...

- Start with a Dyck path. (E.g. $(a, b) = (5, 8)$.)



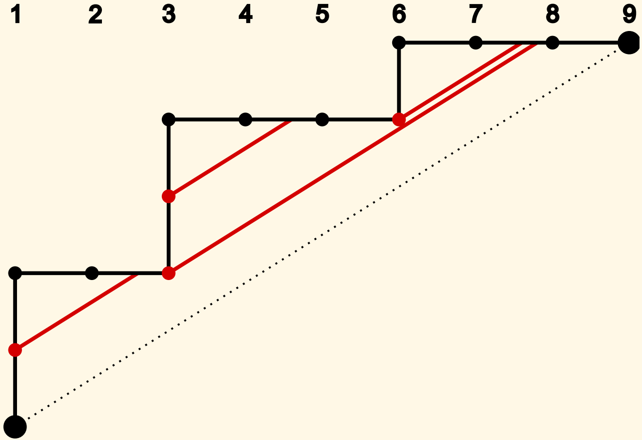
# To **create** a polygon dissection. . .

- Label the columns by $\{1, 2, \ldots, b+1\}$.



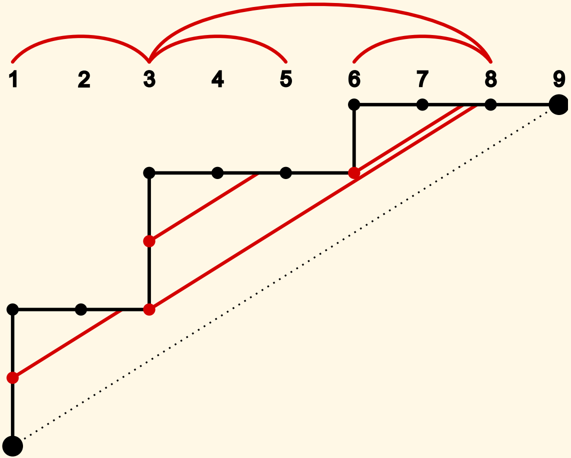
# To **create** a polygon dissection. . .

- Shoot some lasers from the bottom left.



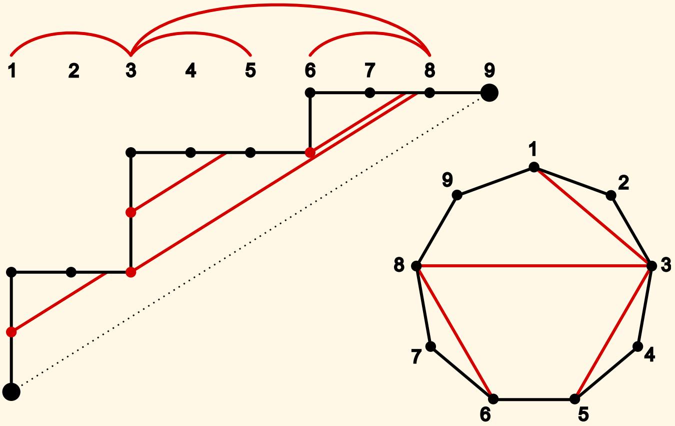
# To **create** a polygon dissection. . .

- Lift the lasers up.



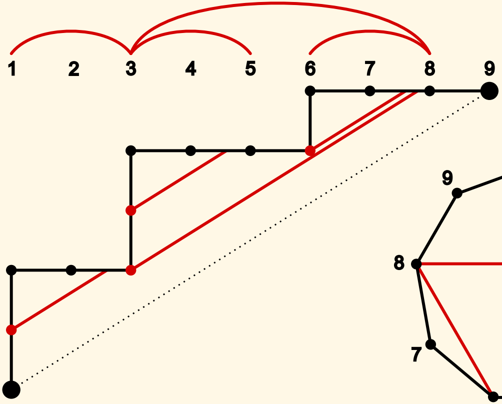
# To **create** a polygon dissection…

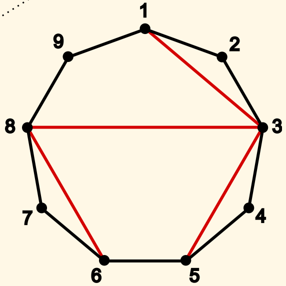
- There you go!



# To **create** a polygon dissection...

- We have created $\mathrm{Cat}(x) = \frac{1}{a}\binom{a+b}{a,b}$ different "triangulations" of the cycle $[b+1]$, and each of them has $a$ diagonals.



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## Definition (with N. Williams)

We have a simplicial complex $\mathrm{Ass}(x) = \mathrm{Ass}(a, b)$.

## Facts (with B. Rhoades and N. Williams)

- $\mathrm{Ass}(n, n+1) = \mathrm{Ass}(n)$ is the **good old associahedron**.
- $\mathrm{Ass}(n, (k-1)n+1)$ is the **generalized cluster complex** of Athanasiadis-Tzanaki and Fomin-Reading.
- $\mathrm{Ass}(x)$ has $\mathrm{Cat}(x)$ facets and Euler characteristic $\mathrm{Cat}'(x)$.
- $\mathrm{Ass}(x)$ is shellable with $h$-vector $\mathrm{Nar}(x, i) = \frac{1}{a}\binom{a}{i}\binom{b-1}{i-1}$.
- Hence its $f$-vector is given by the **Kirkman numbers**

$$\mathrm{Kirk}(x, i) := \frac{1}{a}\binom{a}{i}\binom{b+i-1}{i-1}.$$

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- $\mathrm{Ass}(n, (k-1)n+1)$ is the **generalized cluster complex** of Athanasiadis-Tzanaki and Fomin-Reading.
- $\mathrm{Ass}(x)$ has $\mathrm{Cat}(x)$ facets and Euler characteristic $\mathrm{Cat}'(x)$.
- $\mathrm{Ass}(x)$ is shellable with $h$-vector $\mathrm{Nar}(x, i) = \frac{1}{a}\binom{a}{i}\binom{b-1}{i-1}$.
- Hence its $f$-vector is given by the **Kirkman numbers**

$$\mathrm{Kirk}(x, i) := \frac{1}{a}\binom{a}{i}\binom{b+i-1}{i-1}.$$

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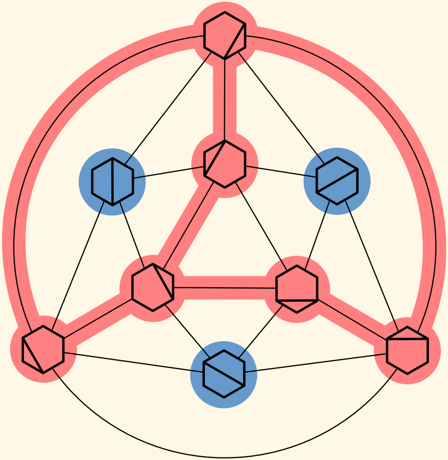
# Inversion = "Alexander Duality"?

- E.g. Ass(2/3) and Ass(3/2) are dual inside Ass(4).



# The end.