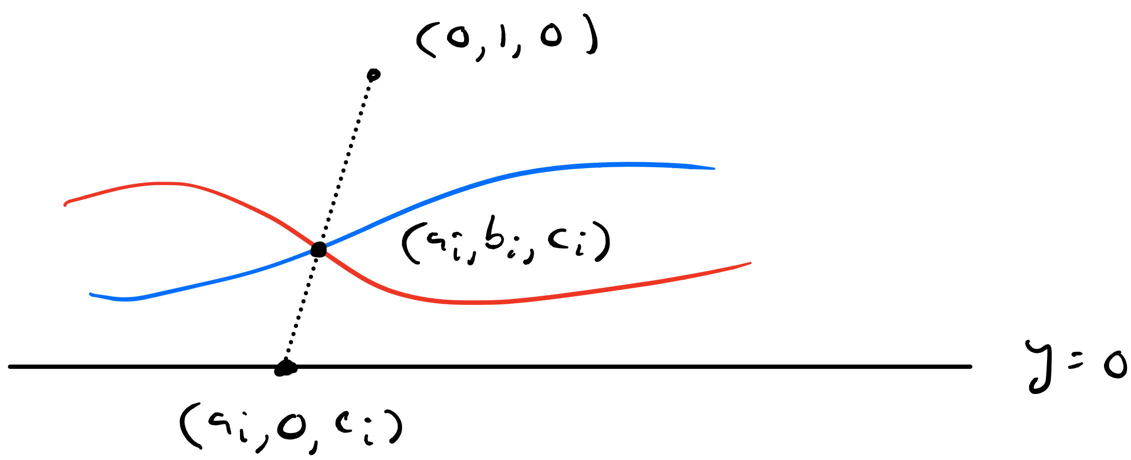
Recall: To compute $I_{\bar{p}}(C, D)$, choose $\varphi \in PGL$ so $(0,1,0) \notin C \cup D$, and no two points of $C \cap D$ on a "vertical line." Suppose $\bar{p}_i = (a_i, b_i, c_i)$, $C = V(F)$, $D = V(G)$.



Let $H(x,z) = \text{Res}_y(F, G)$

$$= \prod (c_i x - a_i z)^{m_i}$$

Then $I_{\bar{p}_i}(C, D) := m_i$.

Example: $F(x,y,z) = y^3 - x^2 z$,
$G(x,y,z) = y^3 + x^2 z$.

Compute the resultant:

$$H = \det \begin{pmatrix} 1 & 0 & 0 & -x^2z & & \\ & 1 & 0 & 0 & -x^2z & \\ & & 1 & 0 & 0 & -x^2z \\ 1 & 0 & 0 & x^2z & & \\ & 1 & 0 & 0 & x^2z & \\ & & 1 & 0 & 0 & x^2z \end{pmatrix}$$

$$= 8x^6z^3$$

Factor:

$$H(x,z) = 8\,(1x - 0z)^6\,(0x - (-1)z)^3$$

Point of multiplicity 6 at

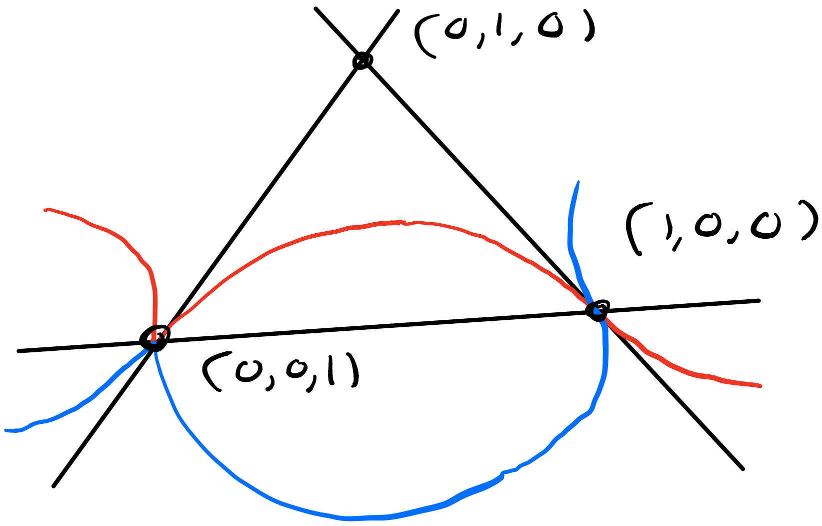
$$(x,z) = (0,1) \longrightarrow (x,y,z) = (0,0,1)$$

Point of multiplicity 3 at

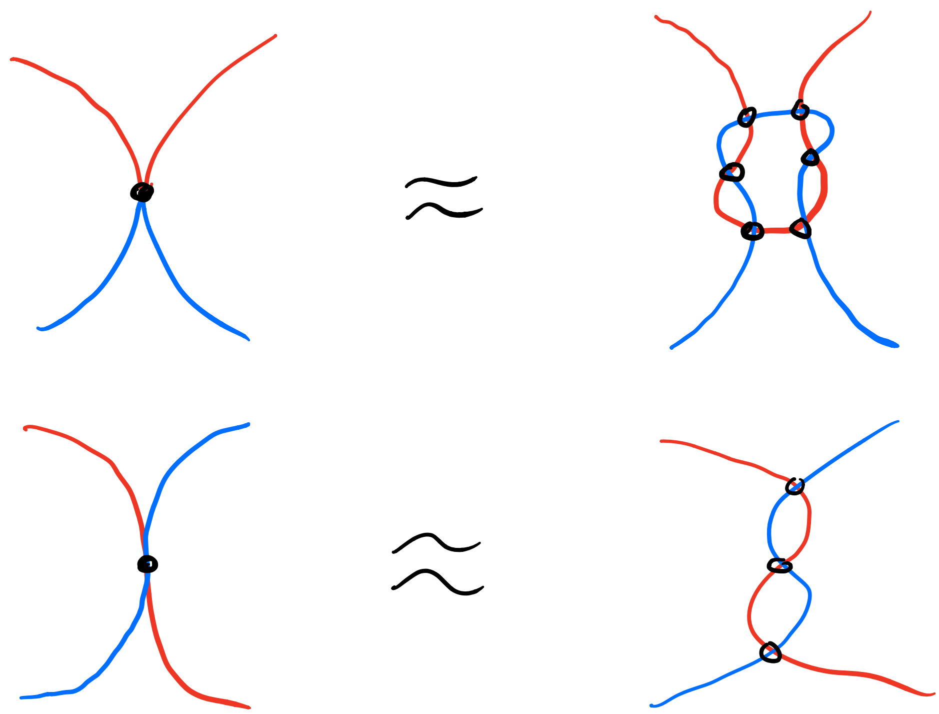
$$(x,z) = (-1,0) \longrightarrow (x,y,z) = (-1,0,0)$$
$$= (1,0,0)$$

Bézout: $6 + 3 = 3 \cdot 3$ ✓

Picture:



Intuition:



(under small perturbation of the coefficients of F & G.)

Sometimes the intersections cannot be visualized in $\mathbb{R}^2$

Applications of Bézout

Theorem: Any algebraic curve has finitely many singularities.

Proof: Let $C = V(F)$, with F square-free. I claim that

F, F_x are coprime.

To see this, assume for contradiction that F & F_x have a common prime factor π:

$$F = \pi \varphi \quad (\pi \nmid \varphi)$$

$$F_x = \pi \psi$$

But then

$$\pi \psi = F_x = \pi \varphi_x + \pi_x \varphi$$

$$\pi(\psi - \varphi_x) = \pi_x \varphi.$$

$$\Longrightarrow \pi \mid \pi_x \varphi.$$

Since π is prime and $\pi \nmid \varphi$ this implies that $\pi \mid \pi_x$, which contradicts the fact that

$$\deg(\pi_x) < \deg(\pi).$$

Since F, F_x are coprime of degrees $d, d-1$ then Bézout's theorem implies that

$$\# V(F) \cap V(F_x) \leq d(d-1) < \infty.$$

But any singular point $\bar{p} \in V(F)$ must be in this intersection.

We have shown that any curve of degree d has $\leq d(d-1)$ singular points.

But we can do much better!

Theorem: Let C have degree d.

1. $\# \mathrm{Sing}(C) \leq \frac{1}{2} d(d-1)$

2. If C is irreducible,

   $\# \mathrm{Sing}(C) \leq \frac{1}{2}(d-1)(d-2)$.

And both of these bounds are sharp.

First we will prove that 2 $\Rightarrow$ 1.

Proof by induction on degree:

If C is irreducible, done because

$$\frac{1}{2}(d-1)(d-2) \leq \frac{1}{2} d(d-1) \quad \checkmark$$

Otherwise, suppose

$$C = D \cup E$$

where $\deg(D) = d_1 < d$
$\deg(E) = d_2 < d$
$\deg(C) = d = d_1 + d_2$.

By induction:

$$\# \operatorname{Sing}(D) \leq \frac{1}{2} d_1(d_1-1) = \binom{d_1}{2}$$

$$\# \operatorname{Sing}(E) \leq \frac{1}{2} d_2(d_2-1) = \binom{d_2}{2}$$

But

$$\# \operatorname{Sing}(C) \leq \# \operatorname{Sing}(D) + \# \operatorname{Sing}(E) + \# D \cap E.$$

$$\leq \binom{d_1}{2} + \binom{d_2}{2} + d_1 d_2$$

miracle!

$$= \binom{d_1+d_2}{2}$$

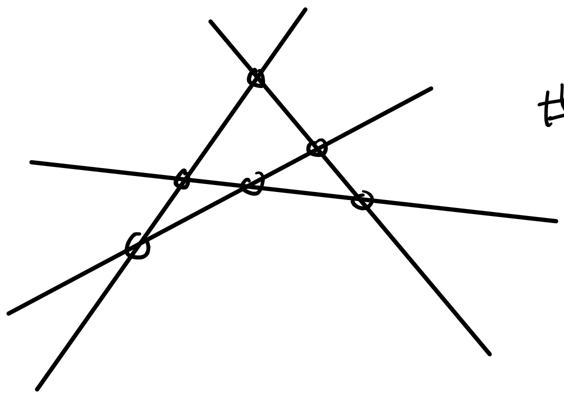
$$= \binom{d}{2} = \frac{1}{2} d(d-1) \quad \checkmark$$

To see that this is sharp, let C be union of d lines in general position.

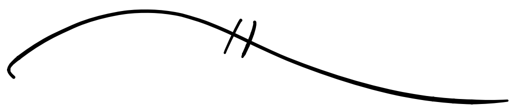
$\operatorname{Sing}(C)$ = intersection points of pairs of lines

$$\# \operatorname{Sing}(C) = \# \text{pairs} = \binom{d}{2} \quad \checkmark$$

Example : $d=4$,



$\# \text{Sing} = \binom{4}{2} = 6.$



To prove ② we need a preliminary discussion of the

"Veronese Embedding"

Given d, n let $N = \binom{d+n}{d} - 1.$

We define a function

$$\varphi_d : \mathbb{CP}^n \longrightarrow \mathbb{CP}^N$$

$$(a_0, a_1, \dots, a_n) \longmapsto (\underbrace{a_0^d, a_0^{d-1}a_1, \dots, a_n^d}_{\text{how many coordinates?}})$$

General term looks like

$$a_0^{d_0} a_1^{d_1} \cdots a_n^{d_n}, \quad d_0 + d_1 + \cdots + d_n = d.$$

Encode each possibility as a binary string of length $d+n$ with d 0's, n 1's:

$$\underbrace{0 \cdots 0}_{d_0} 1 \underbrace{0 \cdots 0}_{d_1} 1 \cdots 1 \underbrace{0 \cdots 0}_{d_n}$$

Number of choices : $\binom{d+n}{d}$ ✓

Claim : $\varphi_d : \mathbb{CP}^n \to \mathbb{CP}^N$ is injective (in fact "biregular onto its image") and the image

$$\varphi_d(\mathbb{CP}^n) \subseteq \mathbb{CP}^N$$

is a variety.

Example :

$$\varphi_3 : \mathbb{CP}^1 \to \mathbb{CP}^3$$

is the twisted cubic curve.

We are interested in the maps

$$\varphi_d : \mathbb{CP}^2 \hookrightarrow V \subseteq \mathbb{CP}^{d(d+3)/2}$$

$$\left[\binom{2+d}{2} - 1 = \frac{d(d+3)}{2} \right]$$

Claim: $C \subseteq \mathbb{CP}^2$ is a degree d curve if and only if

$$\varphi_d(C) = V \cap H$$

for some hyperplane $H \subseteq \mathbb{CP}^{d(d+3)/2}$.

Proof: let u_{ijk} be coordinates on $\mathbb{CP}^{d(d+3)/2}$ corresponding to $x^i y^j z^k$.

$$\sum_{i+j+k=d} a_{ijk} x^i y^j z^k = 0 \iff \sum_{i+j+k=d} a_{ijk} u_{ijk} = 0$$

curve deg d in $\mathbb{CP}^2$ hyperplane in $\mathbb{CP}^{d(d+3)/2}$

Corollary: For any $d(d+3)/2$ points in $\mathbb{CP}^2$, there exists a curve of degree d through these points.

Indeed, let $p_1, p_2, \dots, p_N \in \mathbb{CP}^2$ $[N = d(d+3)/2]$. Then points

$$\varphi_d(p_1), \varphi_d(p_2), \dots, \varphi_d(p_N) \in \mathbb{CP}^N$$

lie on a hyperplane H, hence

$$p_1, \dots, p_N \in \underbrace{\varphi_d^{-1}(V \cap H)}_{\text{a curve of degree } d.}$$

///

Proof of (2), i.e., an irreducible curve C of degree d has $\leq \frac{1}{2}(d-1)(d-2)$ singular points.

True for $d = 1, 2$. So let $d \geq 3$.

Suppose for contradiction that C has $\geq s = \frac{1}{2}(d-1)(d-2) + 1$ sing. points.

Choose s of these singular points:

$$p_1, p_2, \cdots, p_s.$$

Now consider further $d-3$ points on C:

$$\underbrace{p_1, p_2, \cdots, p_s, q_1, \cdots, q_{d-3}}_{} \in C.$$

$$\begin{aligned}
\# \text{ points} &= \delta + (d-3) \\
&= \frac{1}{2}(d-1)(d-2) - 1 + d-3 \\
&\ \vdots \quad \text{magic!} \\
&= \frac{1}{2}(d-2)(d-2+3).
\end{aligned}$$

By previous discussion, $\exists$ curve D of degree $d-2$ passing through these points. But C is irreducible & $\deg D = d-2 < d = \deg C$, so C, D have no common component ("are coprime"). Hence from Bézout:

$$\sum I_q(C, D) = d(d-2)$$

On the other hand,

$$I_{\bar{p}_i}(C, D) \geq \overbrace{\mathrm{mult}_{\bar{p}_i}(C)}^{\geq 2} \cdot \mathrm{mult}_{\bar{p}_i}(D) \geq 2$$

$$I_{\bar{q}_i}(C, D) \geq 1.$$

so that

$$d(d-2) = \sum I_{\bar{\eta}}(C, D)$$

$$\geq 2s + (d-3)$$

$$= 2\left(\tfrac{1}{2}(d-1)(d-2) + 1\right) + d - 3$$

$$\vdots \quad \text{miracle !}$$

$$= d(d-2) + 1.$$

Contradiction. QED.

It is more difficult to show that this bound is sharp.

Example: Chebyshev Curves
(Fisher, Alg. Curves, § 3.9)

Consider the Chebyshev polynomials:

$$\begin{array}{l|l} \cos(2\theta) = 2\cos^2\theta - 1 & T_2(x) = 2x^2 - 1 \\ \cos(3\theta) = 4\cos^3\theta - 3\cos\theta & T_3(x) = 4x^3 - 3x \end{array}$$

etc.

For coprime $m < n$ define the curve

$$C_{m,n} := V(T_m(x) = T_n(y)) \subseteq \mathbb{R}^2$$

Theorem: $C_{m,n}$ is an irreducible curve of degree n with

$$\frac{1}{2}(m-1)(n-1)$$

ordinary double points, located at

$$(x,y) = \left(\cos\left(\frac{l\pi}{n}\right), \cos\left(\frac{k\pi}{m}\right)\right)$$

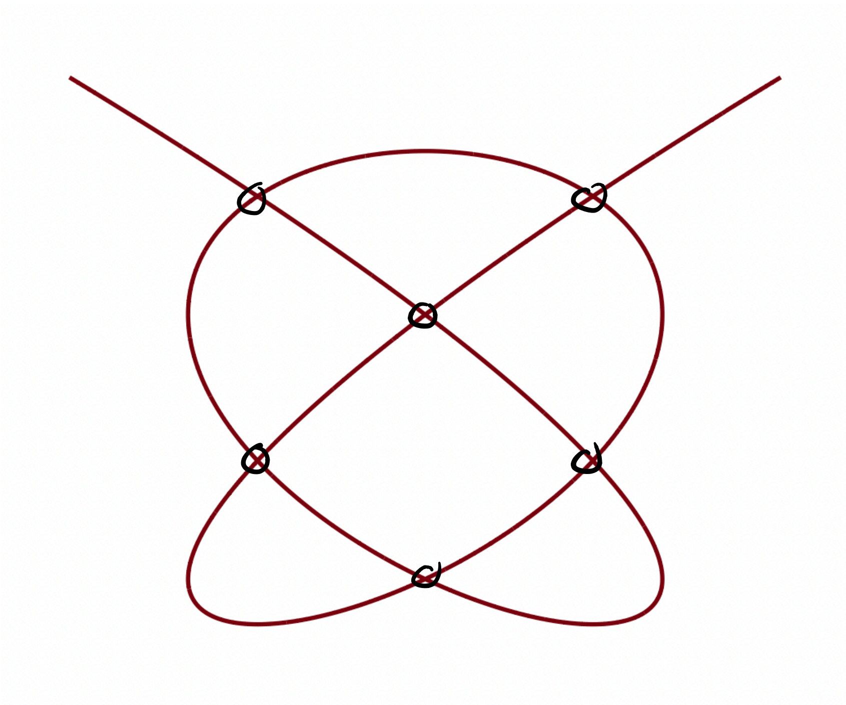
where $k = 1, 2, .., m-1$
$l = 1, 2, .., n-1$

and where $k - l$ is <u>even</u>.

Corollary: The Chebyshev curve $C_{d-1,d}$ is an irreducible curve of degree d with the maximum possible number of singularities

$$\frac{1}{2}(d-2)(d-1).$$

Picture of $C_{4,5}$:



$\deg(C_{4,5}) = 5$ with

$\frac{1}{2}(5-2)(5-1) = 6$ singular points.