From last time: Let $f(x,y) \in \mathbb{C}[x,y]$ be irreducible & non-singular of degree d. Choose coordinates so

$$f(x,y) = y^d + \varphi_1(x) y^{d-1} + \cdots + \varphi_d(x)$$

where $\varphi_k(x) \in \mathbb{C}[x]$ has degree $\leq k$.

Consider the projection

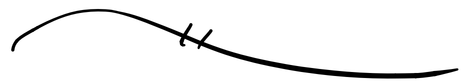
$$\begin{aligned} \mathbb{C}^2 &\longrightarrow \mathbb{C} \\ (x,y) &\longmapsto x \end{aligned}$$

For each $x \in \mathbb{C}$ there exist d preimages $\bar{p}_1, \ldots, \bar{p}_d \in C$ where
with possible repitition

$$\bar{p}_i = (x, y_i(x)), \quad f(x, y_i(x)) = 0$$

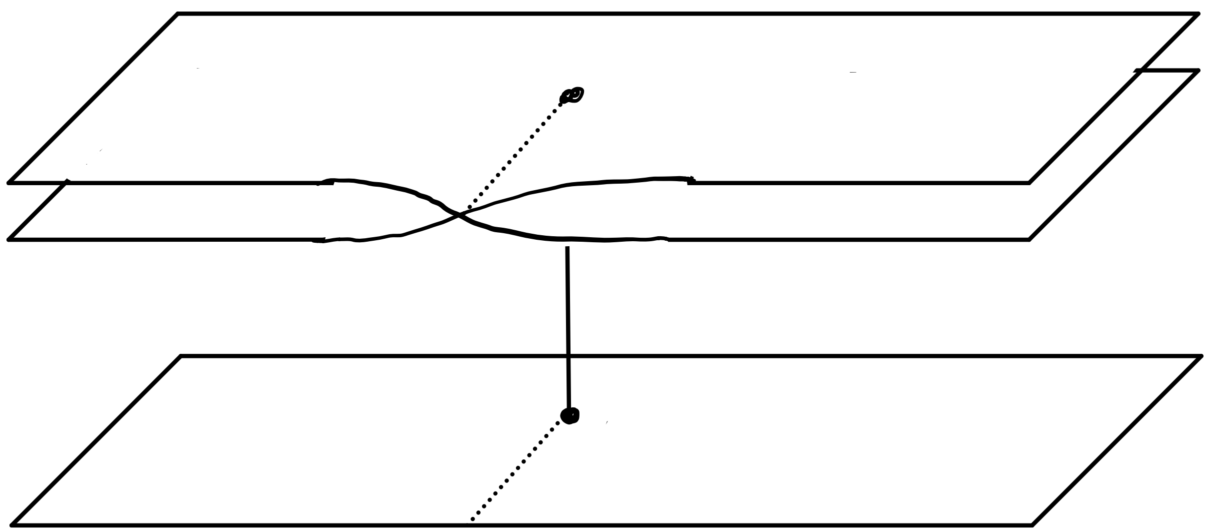
and $y_i(x)$ is locally a holomorphic function of x. Furthermore, there is a finite "branch locus" where the fiber has fewer than d points:
i.e. no repetition

$$\Delta(\pi) = \{x \in \mathbb{C} : \#\pi^{-1}(x) < d\}$$



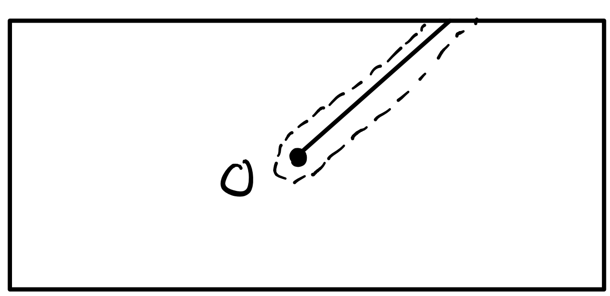
Example: $f(x,y) = y^2 - x$

The projection is branched at $x = 0$:

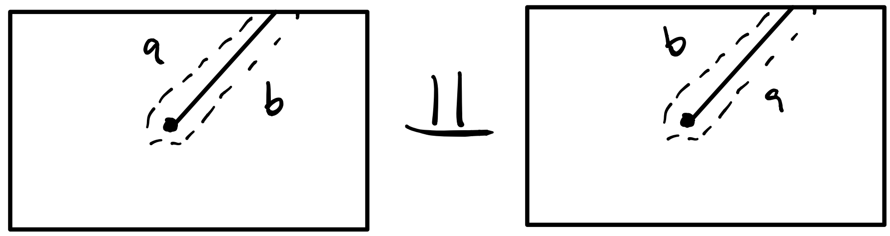


The self-intersection is an illusion so we might as well delete it. Choose an arbitrary ray from 0 to ∞ and delete it from $\mathbb{C}$:

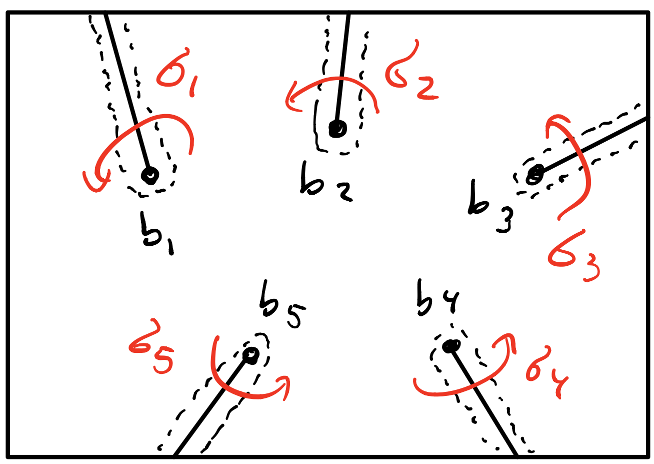
$\mathbb{C}' :=$



We can view $C = V(y^2 - x)$ as two copies of $\mathbb{C}'$ glued together:

$C =$ 

More generally, for a curve C of degree d with branch locus $\Delta(\pi)$ we pick non-intersecting rays from the branch points and delete them:

$\mathbb{C}' :=$ 

Then the curve (topologically) consists of d copies of $\mathbb{C}'$ together with gluing instructions:

$$C = \coprod_{i=1}^{d} \mathbb{C}_i'$$

called the "sheets" of the projection

If the branch points are $b_1, \dots, b_r \in \mathbb{C}$ the gluing instructions consist of r permutations $\sigma_1, \sigma_2, \dots, \sigma_r \in S_d$

$\sigma_i :=$ the permutation of the sheets obtained by circling the branch point counterclockwise

We observe that C is connected if and only if the permutations $\sigma_1, \dots, \sigma_r$ act transitively on the set of sheets $\mathbb{C}_1', \dots, \mathbb{C}_d'$, i.e., for any sheets $\mathbb{C}_i'$ & $\mathbb{C}_j'$ $\exists$ a sequence of permutations in $\{\sigma_1, \dots, \sigma_r\}$ taking $\mathbb{C}_i'$ to $\mathbb{C}_j'$.

We want to prove the following.

Connectedness Theorem:

If $f(x,y) \in \mathbb{C}[x,y]$ is <u>irreducible</u> then the curve $C = V(f) \subseteq \mathbb{C}^2$ is <u>connected</u> (in the usual topology).

The surprising thing about this theorem is that it is difficult to prove & the proof looks like

Galois Theory!

Review (or Crash Course):

Let $\mathbb{F} \supseteq \mathbb{Q}$ be a field extension of degree d, meaning

$$\dim_{\mathbb{Q}} \mathbb{F} = d.$$

Let $G = \mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ denote the group of ring automorphisms

$$\varphi : \mathbb{F} \to \mathbb{F}$$

that fix $\mathbb{Q}$ (this last condition is redundant because $\varphi(1) = 1$ implies $\varphi(a) = a$ for all $a \in \mathbb{Q}$).

Example: $$\varphi : \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{2}]$$
$$a + b\sqrt{2} \longmapsto a - b\sqrt{2}.$$

If $\mathbb{F}/\mathbb{Q}$ satisfies a certain nice condition ("normality") then we obtain an order-reversing bijection:

$$\{\text{subgroups of } G\} \longleftrightarrow \{\text{subfields of } \mathbb{F} \text{ (containing } \mathbb{Q})\}$$

$$H \longleftrightarrow \mathbb{K}$$

where $H = \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$

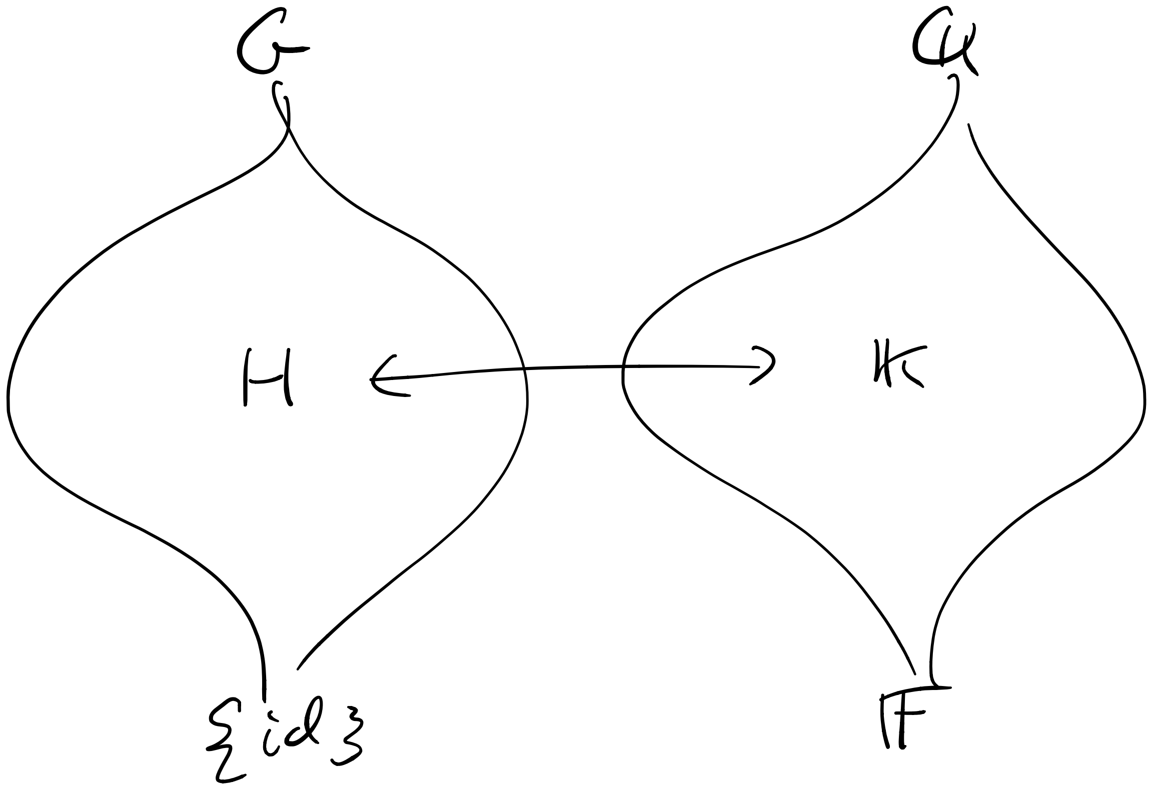
& $\mathbb{K} = \mathrm{Fix}(H)$

$$= \{\alpha \in \mathbb{F} : \varphi(\alpha) = \alpha \ \forall \varphi \in H\}$$

Furthermore, $H \subseteq G$ is a normal subgroup if and only if $\mathbb{K}/\mathbb{Q}$ is a "normal" field extension, in which case:

$$\frac{G}{H} = \frac{\mathrm{Gal}(\mathbb{F}/\mathbb{Q})}{\mathrm{Gal}(\mathbb{K}/\mathbb{Q})} \approx \mathrm{Gal}(\mathbb{F}/\mathbb{K})$$

Picture:



Presumably you know what a "normal subgroup" is. [$H \leq G$ is normal if

$h \in H,\ g \in G \Rightarrow ghg^{-1} \in H$.]

Next time I will explain what I mean by a "normal field extension".