Introduction Continued ...

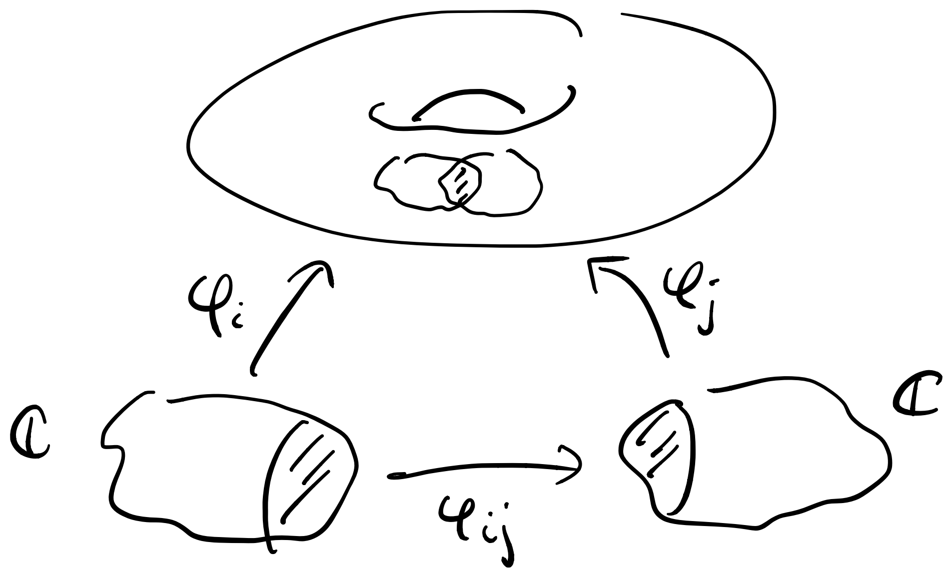
Three equivalent subjects:

① Algebraic curves: Sets of the form

$$C = \{ (a,b) : f(a,b) = 0 \} \subseteq \mathbb{C}^2$$

for some polynomial $f(x,y) \in \mathbb{C}[x,y]$.

② Riemann Surfaces: One dimensional compact complex manifolds



③ Function Fields: Field extensions $\mathbb{F} \supseteq \mathbb{C}$ with

$$\mathrm{tr.deg}(\mathbb{F}/\mathbb{C}) = 1.$$

Today I will sketch the equivalences:

$$\begin{tikzcd}
 & 2 \arrow[dl, leftrightsquigarrow] \arrow[dr, leftrightsquigarrow] & \\
1 \arrow[rr, leftrightsquigarrow] & & 3
\end{tikzcd}$$

Remarks:

- $1 \leftrightsquigarrow 3$ is modern abstract algebra (algebraic geometry). Historically it was dicovered by passing through 2.

- If $\mathbb{K}$ is general alg. closed field, then 2 does not exist, but we still have $1 \leftrightsquigarrow 3$.

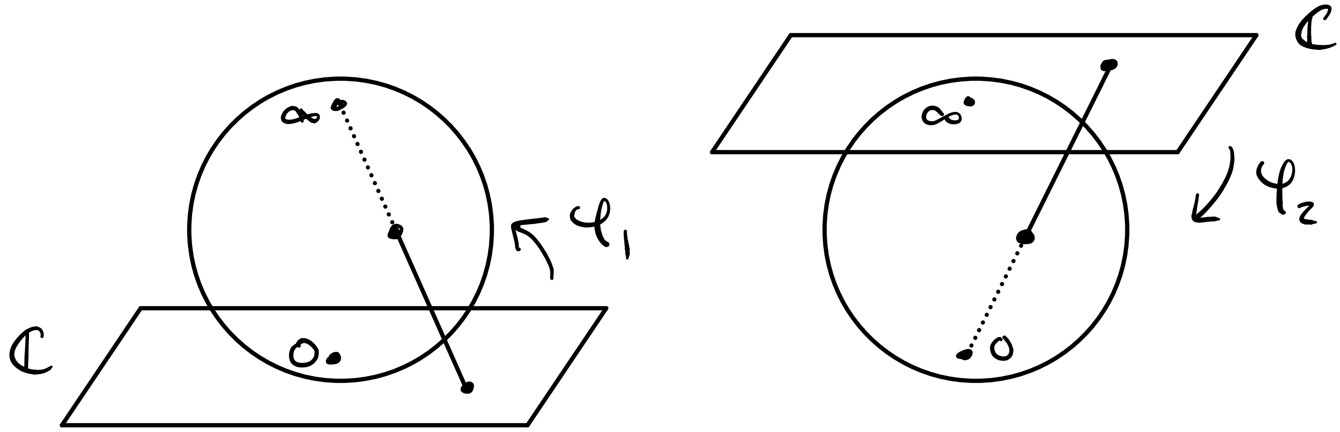
We saw $1 \rightsquigarrow 2$ last time...

Now $2 \rightsquigarrow 3$.

Warning: This is HARD.

$\mathbb{CP}^1 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \simeq S^2$

Two charts:



$$\varphi_{12}(z) = \frac{1}{z}$$

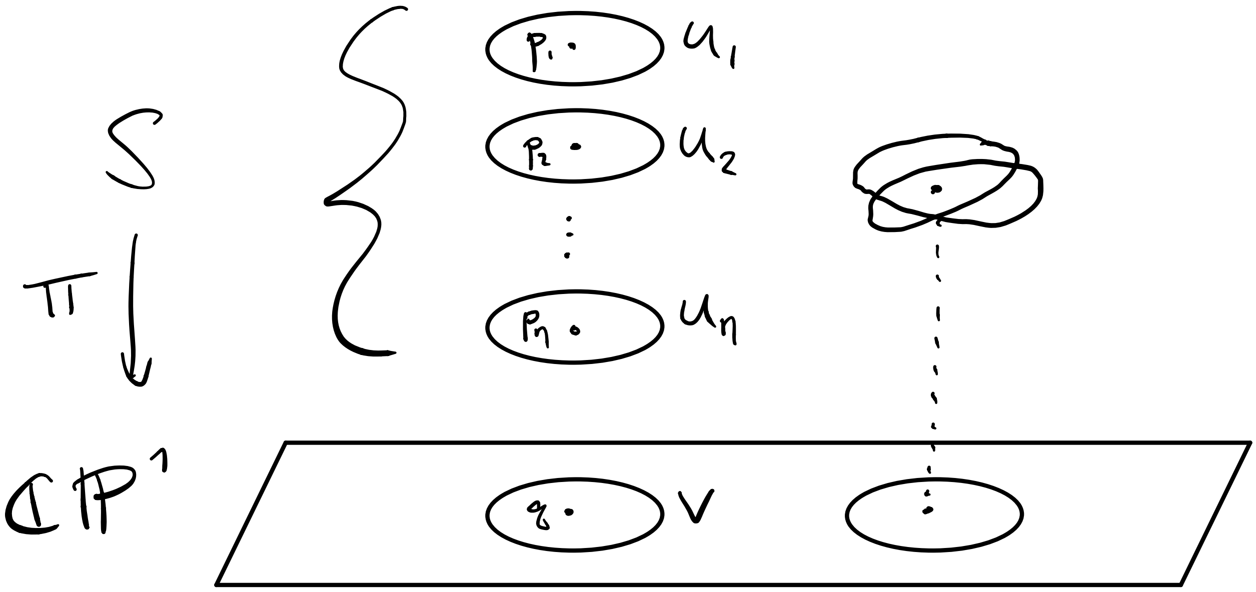
Let S be compact Riemann surface.

Liouville: $\nexists$ non-constant holomorphic functions $S \to \mathbb{C}$

Riemann Existence: $\exists$ non-constant holomorphic $\pi : S \to \mathbb{CP}^1$

[called a "meromorphic function" $S \to \mathbb{C}$, i.e., some points "go to ∞ in a holomorphic way," called the poles of π.]

Geometrically: Think of π as a "ramified covering map"



Let $\mathbb{F}$ = field of meromorphic functions $S \to \mathbb{CP}^1$

We have

$$\underset{\substack{\uparrow \\ \text{constant} \\ \text{functions}}}{\mathbb{C}} \subsetneq \underset{\substack{\uparrow \\ \text{rational} \\ \text{expressions} \\ \text{in } \pi}}{\mathbb{C}(\pi)} \subseteq \mathbb{F}$$

Since $\mathbb{C}$ is algebraically closed, this implies π is transcendental $/ \mathbb{C}$

Theorem: $\mathbb{F}/\mathbb{C}(\pi)$ is algebraic.

In fact, $\dim(\mathbb{F}/\mathbb{C}(\pi)) =$
$\#$ sheets $=: n < \infty$ (S compact)

Proof Sketch: To see $\dim(\mathbb{F}/\mathbb{C}(\pi)) \leq n$. Consider any $f_1, \ldots, f_{n+1} \in \mathbb{F}$. We will show $\exists\ a_1(\pi), \ldots, a_{n+1}(\pi) \in \mathbb{F}$ such that $a_1(\pi) f_1 + \cdots + a_{n+1}(\pi) f_{n+1} \equiv 0 \in \mathbb{F}$.

Note: Function f_i is uniquely determined by the restiction $f_{ij} = f_i|_{u_j}$ for any j.
[Reason: S is connected. Analytic continuation.] Now the $(n+1) \times n$ $\mathbb{C}(\pi)$-linear system

$$\sum f_{ij}\, a_i(\pi) \equiv 0$$

has a non-trivial solution: $a_1(\pi), \ldots, a_{n+1}(\pi)$.
The same relation must hold

$$\sum f_i\, a_i(\pi) \equiv 0 \qquad ///$$

To see $\dim(\mathbb{F}/\mathbb{C}(\pi)) \geq n$.

Riemann Existence: $\exists$ some $f \in \mathbb{F}$ "separating the sheets" i.e. where

$$f(p_1), f(p_2), \dots, f(p_n) \in \mathbb{CP}^1$$

are all distinct. Since $\dim(\mathbb{F}/\mathbb{C}(\pi))$ is $< \infty$ we know $1, f, f^2, \dots$ is $\mathbb{C}(\pi)$-linearly dependent, hence f is algebraic $/\mathbb{C}(\pi)$. Suppose for contradiction

$$a_0(\pi) + a_1(\pi) f + \cdots + a_{n-1}(\pi) f^{n-1} \equiv 0$$

for some $a_j(\pi) \in \mathbb{C}$.

Evaluate at point $p_i \in S$:

$$a_0(z) + a_1(z) f(p_i) + \cdots + a_{n-1}(z) f(p_i)^{n-1} = 0$$

Hence the poly $\sum a_j(z) x^j \in \mathbb{C}[x]$ of degree $\leq n-1$ has n distinct roots $f(p_1), f(p_2), \dots, f(p_n) \in \mathbb{C}$. ///

Conclusion: $\mathrm{tr.deg}(\mathbb{F}/\mathbb{C}) = 1$,

$\dim(\mathbb{F}/\mathbb{C}(\pi)) = n$,

hence $\exists$ single "primitive element" so that

$$\mathbb{F} = \mathbb{C}(\pi)[f]$$

③ $\rightsquigarrow$ ①: Let $\mathrm{tr.deg}(\mathbb{F}/\mathbb{C}) = 1$.

By definition (& prim elt thm.):

$$\mathbb{F} = \mathbb{C}(x)[y],$$

where y satisfies an algebraic relation

$$a_0(x) + a_1(x)y + \cdots + a_n(x)y^n = 0,$$

and where $a_i(x) = p_i(x)/q_i(x)$ for some polynomials $p_i, q_i \in \mathbb{C}[x]$.

Clear Denominators: Let $q = \prod q_i$,

Let $p_i'(x) = p_i q = \frac{p_i}{q_i} \prod q_j \in \mathbb{C}[x]$

Multiply both sides by $g(x)$:

$$\underbrace{p_0'(x) + p_1'(x)y + \cdots + p_n'(x)y}_{f(x,y)} = 0$$

$$= 0.$$

Finally ① ⇝ ③: This is completely modern. Given curve

$$C = \{(a,b) : f(a,b) = 0\} \in \mathbb{C}^2$$

define an ideal in the ring $\mathbb{C}[x,y]$:

$$I(C) = \{ g \in \mathbb{C}[x,y] : g(a,b) = 0 \text{ for all } (a,b) \in C \}$$

Define the "coordinate ring" of C:

$$\mathbb{C}[C] := \mathbb{C}[x,y] / I(C).$$

$$= \text{"polynomial functions } C \to \mathbb{C}\text{"}$$

Let $f = f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n}$ be prime factorization in the UFD $\mathbb{C}[x,y]$.

Study's Lemma (Last Semester):

$$I(C) = (f_1 f_2 \cdots f_n)$$
$$= \{ f_1 f_2 \cdots f_n g : g \in \mathbb{C}[x,y] \}$$

Assume f is prime, so that

$I(C) = (f)$, which is a <u>prime ideal</u>.

Follows that $\mathbb{C}[C] = \mathbb{C}[x,y]/(f)$

is a <u>domain</u>, so it has a field

of fractions:

$$\mathbb{F} = \mathbb{C}(C) := \operatorname{Frac}\left(\mathbb{C}[x,y]/(f)\right).$$

"function field of C"

Trancendence degree 1 ?

$$\begin{array}{ccccccc}
\mathbb{C} & \hookrightarrow & \mathbb{C}[x,y] & \longrightarrow & \mathbb{C}[x,y]/(f) & \longrightarrow & \mathbb{F} \\
\alpha & \longmapsto & \alpha & \longmapsto & \alpha + (f) & \longmapsto & \alpha + (f) \\
 & & x & \longmapsto & x + (f) & \longmapsto & x + (f) =: \bar{x} \\
 & & y & \longmapsto & y + (f) & \longmapsto & y + (f) =: \bar{y}
\end{array}$$

At least one of $\bar{x}, \bar{y} \in \mathbb{F}$ is transcendental over $\mathbb{C} = \{\alpha + (f) : \alpha \in \mathbb{C}\}$.

Since $\mathbb{C}$ is alg closed, only need to show $\bar{x}$ or $\bar{y}$ is non-constant.

Say $\qquad \bar{x} = \text{const}$

$$x + (f) = \alpha + (f)$$

$$x - \alpha \in (f)$$

$$f \mid x - \alpha$$

$$\Longrightarrow f(x,y) = \beta(x - \alpha).$$

Similarly: $\quad \bar{y} = \text{const}$

$$\Longrightarrow f(x,y) = \beta(y - \alpha).$$

Can't both be true! $\qquad$ ///

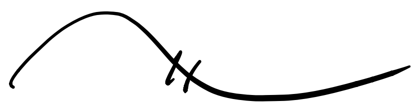
But $\bar{x}, \bar{y}$ generate $\mathbb{F}$ as a $\mathbb{C}$-algebra because $\mathbb{C}[x,y] \twoheadrightarrow \mathbb{C}[x,y]/(f)$ is surjective. And $\bar{x}, \bar{y}$ are algebraically dependent over $\mathbb{C}$

because $f(x,y) \in \mathbb{C}[x,y]$ evaluated at $\bar{x}, \bar{y} \in \mathbb{F}$ gives

$$
\begin{aligned}
f(\bar{x}, \bar{y}) &= f(x,y) + (f) \\
&= 0 + (f) \\
&= \text{zero element of } \mathbb{F}.
\end{aligned}
$$

We conclude that

$$\operatorname{tr.deg}(\mathbb{F}/\mathbb{C}) = 1.$$



End of Introduction.