787: Algebraic Curves

Introduction continued.

3 equivalent subjects:

1. Algebraic curves: $f(x,y) = 0$.
2. 1D compact complex manifolds.
3. Field extensions $\mathbb{F} \supseteq \mathbb{C}$ of transcendence degree 1.

The equivalences were worked out mostly in the 19th century, and strongly influence today's mathematics.

Sketch: (1) $\rightsquigarrow$ (2). $\mathbb{Z}$

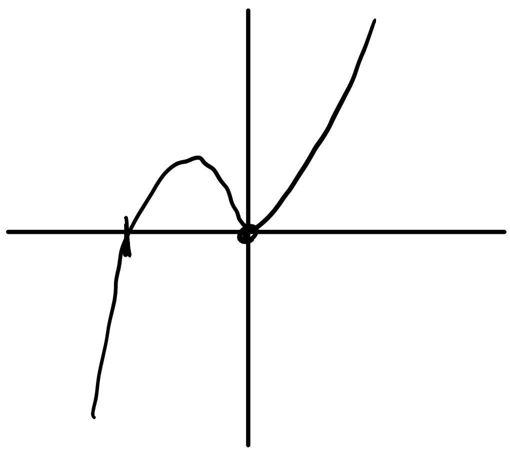
Given polynomial $f(x,y) \in \mathbb{R}[x,y]$ we have a "real affine curve"

$$C_f(\mathbb{R}) = \{(a,b) \in \mathbb{R}^2 : f(a,b) = 0\}$$

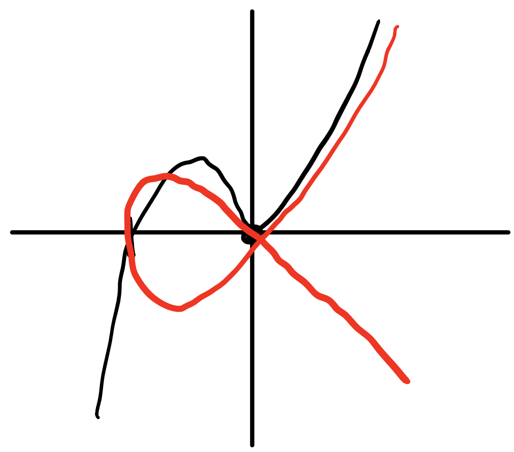
Example: $f(x,y) = y^2 - x^3 - x^2$.

What does $C_f(\mathbb{R}) \subseteq \mathbb{R}^2$ look like?

$$y^2 = x^3 + x^2$$
$$y^2 = x^2(x+1)$$



$y = x^2(x+1)$



$y = \pm\sqrt{x^2(x+1)}$

Called the "nodal cubic curve"

The origin $(0,0)$ is a singular point called a "node" or "double point".

Singular point (a,b) means:

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0.$$

$$\nabla f(a,b) = (0,0).$$

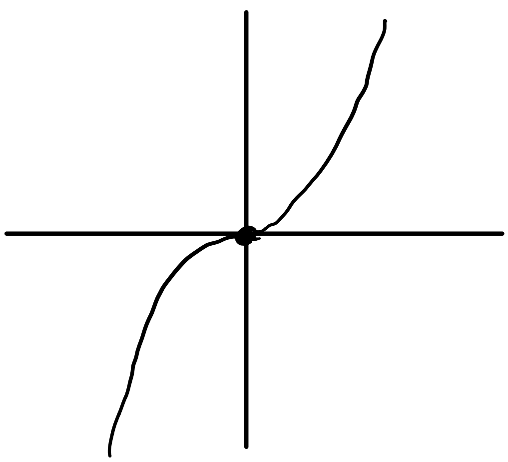
When $f(x,y) = y^2 - x^3 - x^2$,

$$\nabla f(x,y) = (-3x^2-2x,\ 2y)$$

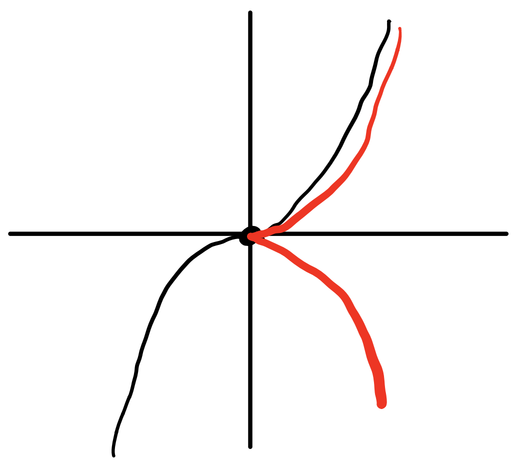
$$\nabla f(0,0) = (0,0) \quad \checkmark$$

Other main example:

$$f(x,y) = y^2 - x^3$$



$y = x^3$



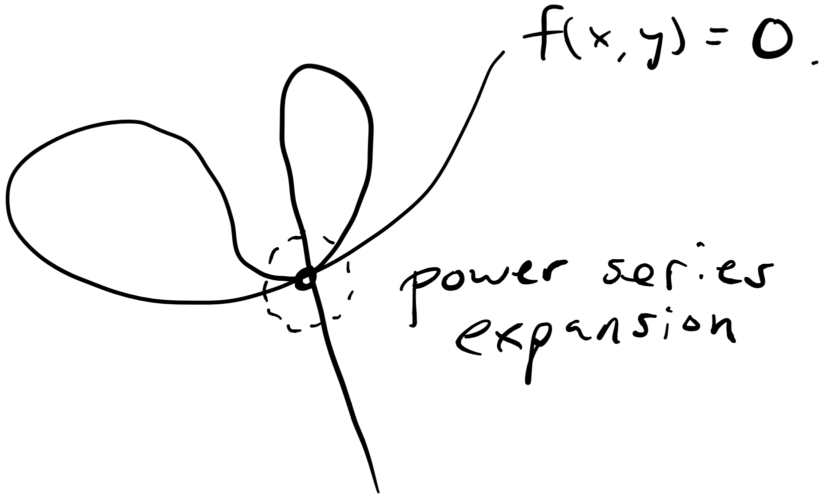
$y = \pm\sqrt{x^3}$

Called the "cuspidal cubic."

Fact: Nodes & Cusps are the two simplest kinds of singularities.

["Plücker curve": curve with only nodes & cusps.]

Worse Singularities:



Use Newton / Puiseux expansion, Weierstrass' Theorem: Formal power series rings are UFD.

[See the book by Gerd Fischer, e.g.]

(2) Riemann Surfaces.

Given $f(x,y) \in \mathbb{C}[x,y]$, we have "complex affine curve"

$$C_f(\mathbb{C}) = \{(a,b) \in \mathbb{C}^2 : f(a,b) = 0\}$$

Let's also add "points at infinity" to make it compact.

If $f(x,y)$ has degree d, we define the "homogenization"

$$F(x,y,z) = z^d f\left(\frac{x}{z}, \frac{y}{z}\right).$$

Example: $f(x,y) = (x-1)(y+3)(y^2-x^3-x^2)$

$$\rightsquigarrow F(x,y,z) = (x-z)(y+3z)(y^2z-x^3-x^2z)$$

Since F is homogeneous of degree d we have

$$F(\lambda x, \lambda y, \lambda z) = \lambda^d F(x,y,z).$$

Thus we obtain a "projective curve"

$$C_F(\mathbb{C}) = \{(a,b,c) \in \mathbb{CP}^2 : F(a,b,c) = 0\}$$

$$\left[\mathbb{CP}^2 = (\mathbb{C}^3 \setminus (0,0,0))/\text{scalars}\right]$$

Called the "projective completion"

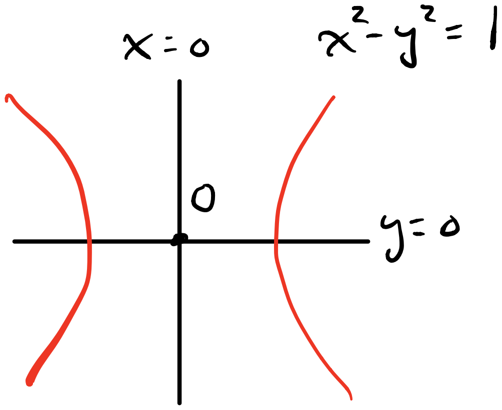
$$\begin{tikzcd} \mathbb{C}^2 \arrow[r] & \mathbb{CP}^2 \\ C_f \arrow[u] \arrow[r] & C_F \arrow[u] \end{tikzcd}$$

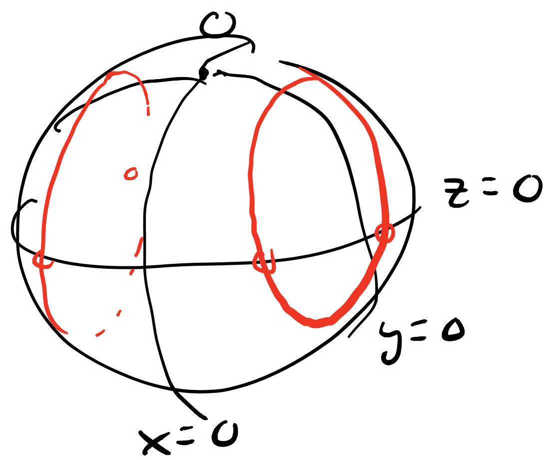
Picture: $\mathbb{RP}^2$ = lines in $\mathbb{R}^3$ through the origin.

Spherical model:

$\mathbb{RP}^2$ = pairs of antipodal points on sphere $\subseteq \mathbb{R}^3$ centred at the origin.

Examples:

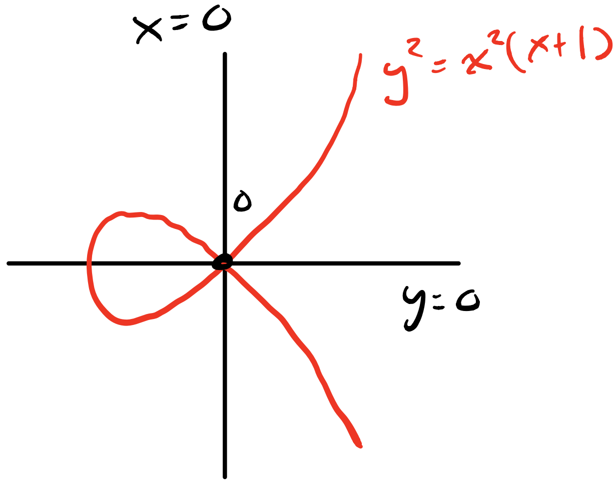
- 

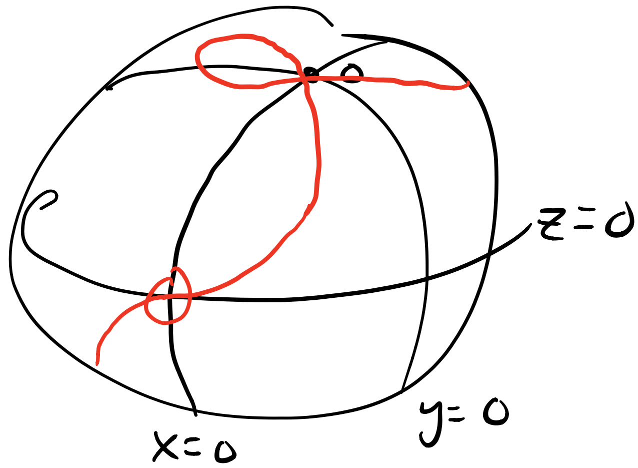


Hyperbola in $\mathbb{R}^2$ becomes an antipodal pair of ellipses in $\mathbb{RP}^2$, passing through the "line at ∞" which is the equator $z=0$.

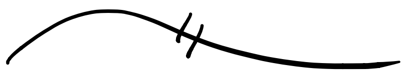
- $F(x,y) = y^2 - x^3 - x^2$

$$F(x,y,z) = y^2z - x^3 - x^2z.$$

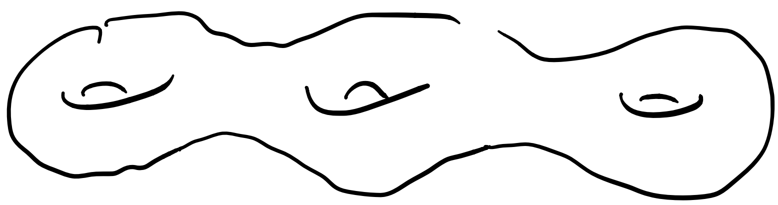




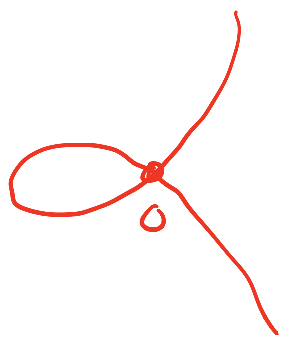
Turns out that the "nodal cubic" has a single point at infinity, which is an inflection point.



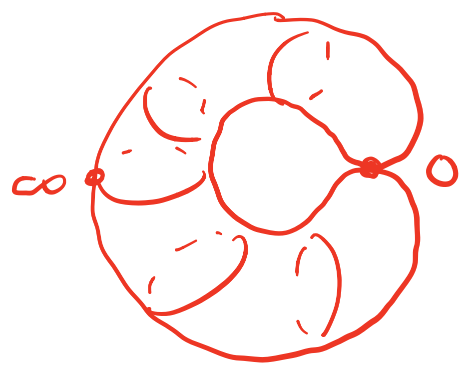
In compact $\mathbb{CP}^2$, any nonsingular curve $C_F(\mathbb{C})$ is a real 2D surface that is compact, orientable & connected, hence is just a multi-torus:



Remark: Singular points are not so bad. Example: nodal cubic



real picture



complex picture
(genus = ? 0 or 1)

Convention: Delete the singular points to determine the genus.

Example: Cuspidal cubic has genus zero. Compare to Clebsch:

$$g = \frac{(d-1)(d-2)}{2} - \#\text{nodes}$$

$$= \frac{(3-1)(3-2)}{2} - 1$$

$$= 1 - 1 = 0 \quad \checkmark$$

(3) Field Extensions.

If $\mathbb{E} \supseteq \mathbb{F}$ is a field extension then

- $\mathbb{E}$ is a vector space over $\mathbb{F}$.
- $\mathbb{E}$ is an algebra over $\mathbb{F}$.

Steinitz (1910): Any two maximal linear independent sets have same size called the dimension (degree):

$$[\mathbb{E} : \mathbb{F}] = \dim_{\mathbb{F}}(\mathbb{E})$$

Def: Element $\alpha \in \mathbb{E} \supseteq \mathbb{F}$ is

- algebraic $/\mathbb{F}$ if we have $f(\alpha) = 0$ for some poly $f(x) = \mathbb{F}[x]$.
- otherwise is transcendental $/\mathbb{F}$.

Elements $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{E}$ are algebraically independent $/\mathbb{F}$ if the evaluation homomorphism

$$\mathbb{F}[x_1, x_2, \ldots, x_n] \longrightarrow \mathbb{F}$$
$$f(x_1, \ldots, x_n) \longmapsto f(\alpha_1, \ldots, \alpha_n)$$

is injective, i.e., has trivial kernel.

Steinitz (1910): Any two maximal algebraic independent sets have same size, called transcendence degree of $\mathbb{E}/\mathbb{F}$:

$$\mathrm{tr.deg}_{\mathbb{F}}(\mathbb{E}).$$

Furthermore: If $\alpha_1, \ldots, \alpha_n$ is a transcendence basis for $\mathbb{E}/\mathbb{F}$, then we have

$$\mathbb{E} = \underbrace{\mathbb{F}(\alpha_1, \ldots, \alpha_n)}_{\substack{\text{rational expressions} \\ \text{in "variables"} \\ \alpha_1, \alpha_2, \ldots, \alpha_n}} [\beta]$$

β: one element algebraic over the $\alpha_1, \ldots, \alpha_n$.

Here $\beta \in \mathbb{E}$ is called a "primitive elt."

Existence of primitive elements for algebraic extensions was first proved by Galois. [Technicality: Assume $\mathbb{E}$ has char 0 or is a finite field.]

Steinitz $\rightsquigarrow$ Noether
1910 1920s.

Noether Normalization:
If $S \supseteq R$ is a finitely generated R-algebra, then there exists a transcendence basis $\alpha_1, \ldots, \alpha_n$ such that

$$R \subseteq R[\alpha_1, \ldots, \alpha_n] \underbrace{\subseteq}_{\text{"integral"}} S$$

Geometric Meaning: Any variety is "birationally equivalent" to a hypersurface in affine space...