787 : Topics in Algebra
Algebraic Curves.

Lots of 20th c. math is based on an amazing synthesis from 19th c. math. The following three subjects are basically equivalent:

1. Algebraic Plane Curves defined by a single polynomial in 2 vars:

$$f(x,y) = 0.$$

2. One dimensional (compact) complex manifolds.

3. Field extensions $\mathbb{F}/\mathbb{C}$ of transcendence degree 1.

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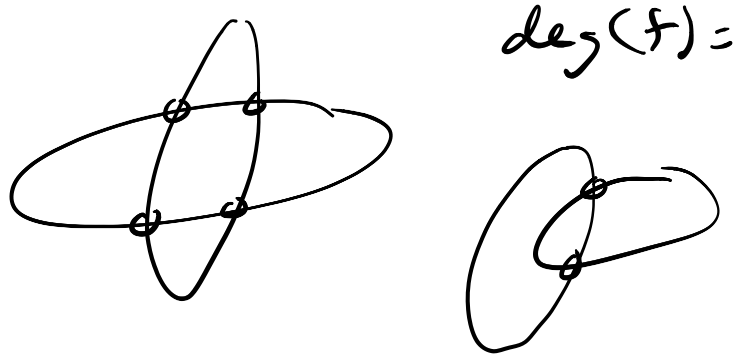
1. Started with Newton/Maclaurin/ Bézout, 1600s & 1700s.

Bézout's Theorem :

Curves $f(x,y)=0$ & $g(x,y)=0$ with $\deg(f)=d$ & $\deg(g)=e$ either:

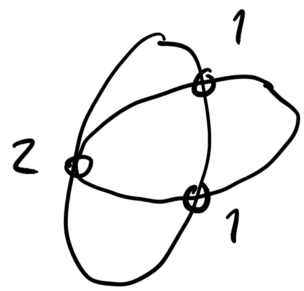
- have a common factor, or
- have $\leq de$ points of intersection.

$\deg(f)=\deg(g)=2$



~ 1812: Poncelet added complex points & points at infinity.

Bézout: Exactly de points of intersection, if counted correctly.



~ 1830s: Plücker / Möbius / Hesse added homogeneous coordinates

Lots of interesting theorems counting inflections, double points, cusps, double tangents, ...

~1860s: Clebsch defines the "genus" of a curve of degree d:

$$g = \frac{(d-1)(d-2)}{2} - \#\text{ double points (counted correctly)}$$

(2) Complex Analysis

~1820s: Abel

$$A(x) = \int^{x} r(t, \sqrt{f(t)})\, dt$$

where $r(x,y)$ is rational
$f(x)$ is polynomial.

More generally:

$$A(x) = \int^{x} r(t, s)\, dt$$

where $r(x,y)$ is rational and t, s are related by $f(t,s) = 0$ where $f(x,y)$ is polynomial.

Abel's Theorem: Given polynomial $f(x,y)$ relating t & s, $\exists$ integer $g \geq 0$ with following property:

For any $a_1, a_2, \ldots \in \mathbb{C}$

$\exists$ $b_1, b_2, \ldots, b_g \in \mathbb{C}$ such that

$$\boxed{\sum_i A(a_i) = \sum_j A(b_j) + \text{something easy}}$$

Very fancy generalization of

$$\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y).$$

~1850s: Riemann

Abel: $A : \mathbb{C} \to \mathbb{C}$ multi-valued.

Riemann: $A : \Sigma \to \mathbb{C}$ single-valued.

where Σ is the Riemann surface corresponding to polynomial $f(x,y)$.

Abel's g = # handles

Abel's periods $b_1, b_2, \ldots, b_g$ correspond to integrating around the handles.

Amazing: Abel's g = Clebsch's g

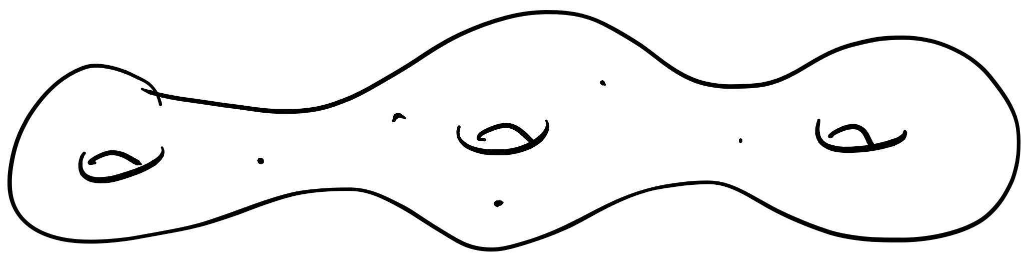
The Riemann Surface:

Equation $f(x,y) = 0$ defines a

$\mathbb{C}$1D subset of $\mathbb{C}^2 \subseteq \mathbb{CP}^2$
$\mathbb{R}$2D subset of $\mathbb{R}^4 \subseteq \mathbb{RP}^4$

It is orientable & <u>compact</u> (if we include points at infinity),

& connected. Hence it is just a multi torus



[Technicality: There are a finite number of singular points (a,b) where $\frac{\partial F}{\partial x}(a,b) = \frac{\partial F}{\partial y}(a,b) = 0$.

These can be complicated but they don't affect the large scale topology.]