We have seen the ring-theoretic definition of algebraic curves:

$$\text{curve} = \underset{\text{(radical)}}{\text{principal ideal}} \subseteq \mathbb{F}[x,y]$$

i.e. (f) where f has no repeated prime factors. [Correspondence is unique when $\mathbb{F}$ algebraically closed.]

$$\underset{\text{cure}}{\text{irreducible}} = \underset{\text{(prime)}}{\text{principal ideal}} \subseteq \mathbb{F}[x,y]$$

i.e., (f) with f irreducible.

Every curve is a unique union of irreducible curves:

$$C(f_1 \cdots f_m) = C(f_1) \cup \cdots \cup C(f_m).$$

The purely ring-theoretic study of curves is too difficult for us right now (i.e., "too much; too soon").

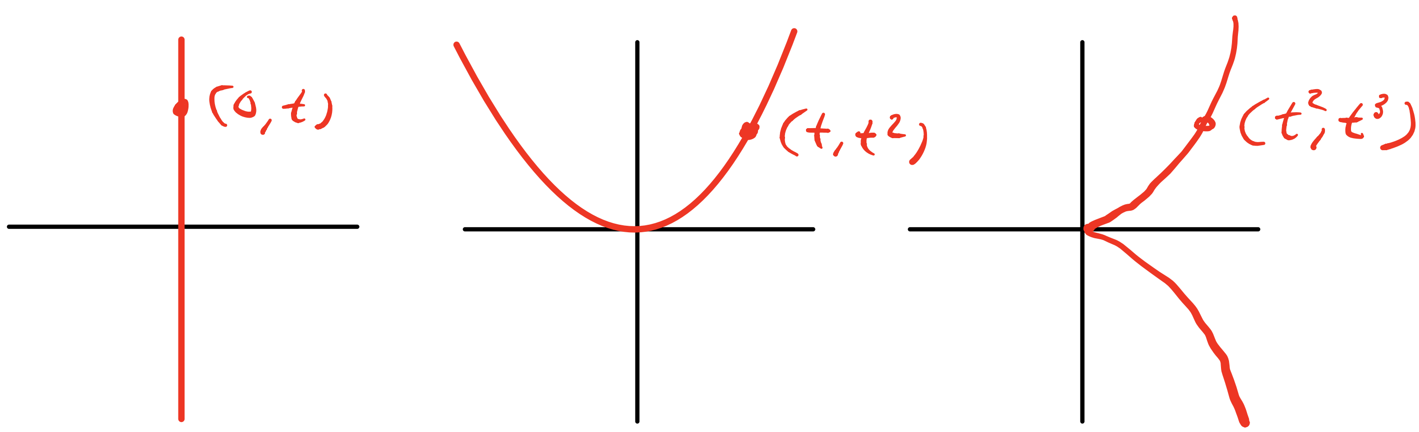
So we will first develop a geometric understanding of curves in $\mathbb{R}^2$, $\mathbb{C}^2$, $\mathbb{RP}^2$, $\mathbb{CP}^2$.

Important Question: What is the correct notion of "isomorphism of curves" (more generally, of varieties)?

Example: we have bijections

$$V(x) \longleftrightarrow V(x^2-y) \longleftrightarrow V(x^3-y^2)$$

$$(0,t) \longleftrightarrow (t,t^2) \longleftrightarrow (t^2,t^3)$$



$x=0$ $\qquad$ $x^2=y$ $\qquad$ $x^3=y^2$

Are these isomorphisms of curves?

Answer: Yes & No.

$$V(x) \overset{?}{\approx} V(x^2-y) \overset{?}{\approx} V(x^3-y^2)$$

bi linearly?	NO	NO
bi polynomially?	YES	NO
bi rationally?	YES	YES.

[In fact, any two curves that can be parametrized by t are birationally isomorphic. But not all curves can be parametrized.

Theorem: parametrizable $\iff$ genus 0.]

Two important steps in geometry:

$$\begin{array}{ccc} \mathbb{C}^2 & \hookrightarrow & \mathbb{CP}^2 \\ \uparrow & & \uparrow \\ \mathbb{R}^2 & \hookrightarrow & \mathbb{RP}^2 \end{array}$$

Poncelet ~1812.

Definition of the "projective plane"?

① Synthetic

② Analytic.

① Synthetic: A projective plane consists of (P, L, I) where

P = set of points

L = set of lines

I = set of incidences (point $\in$ line).

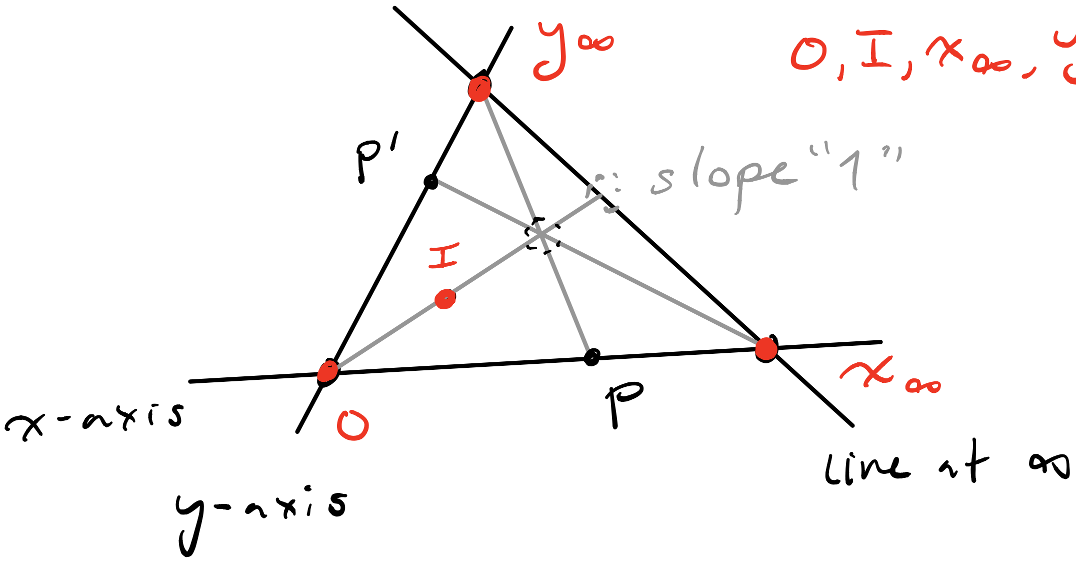
satisfying 3 axioms

- 2 points $\rightsquigarrow$ unique line
- 2 lines $\rightsquigarrow$ unique point
- (non-degeneracy): $\exists$ 4 points, no 3 collinear.

Surprise: Such a plane carries some inherent "coordinates."

Construction due to von Staudt (1870s) & Hilbert (1899).

Fix 4 points, no 3 collinear:



Call them O, I, x_∞, y_∞

Get a bijection between

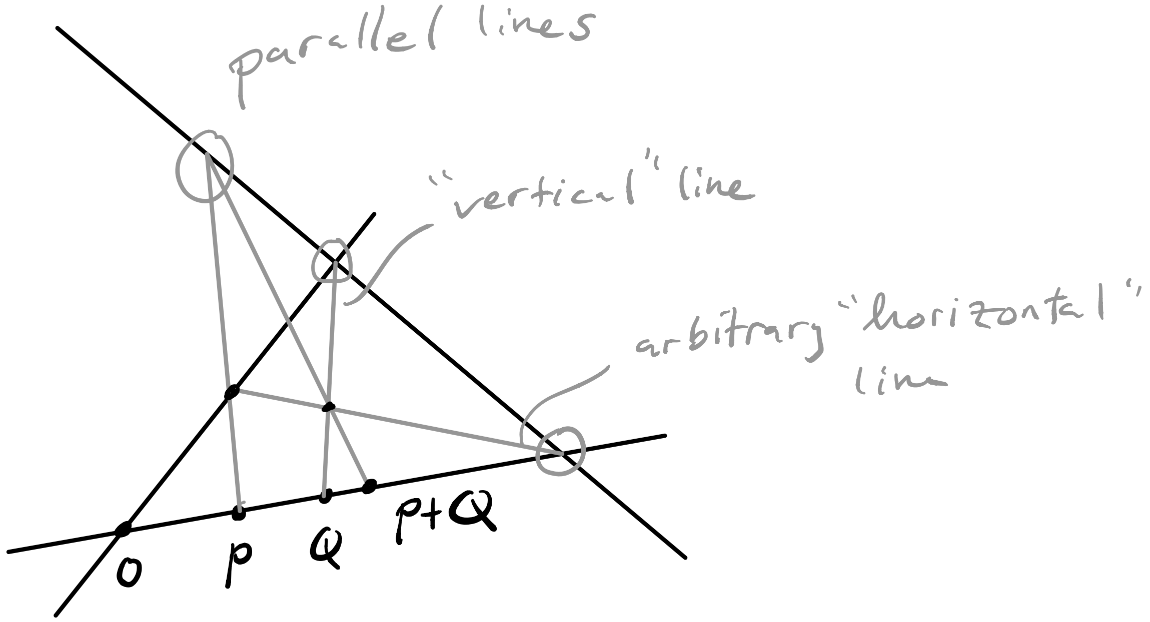
$$\left\{\begin{matrix}\text{points of}\\ x\text{-axis}\end{matrix}\right\} - x_\infty \leftrightarrow \left\{\begin{matrix}\text{points}\\ \text{of } y\text{-axis}\end{matrix}\right\} - y_\infty$$

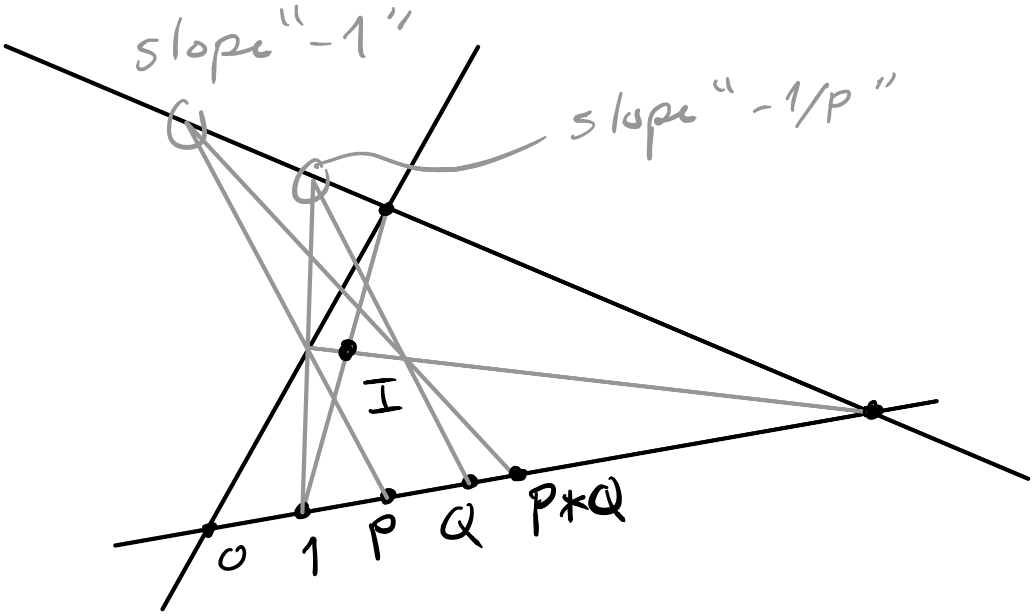
$$p \longleftrightarrow p'$$

Let $R = \{\text{poins on } x\text{-axis}\} - x_\infty$.

Claim: R has the structure of an "almost ring," i.e., we can add & multiply elements:

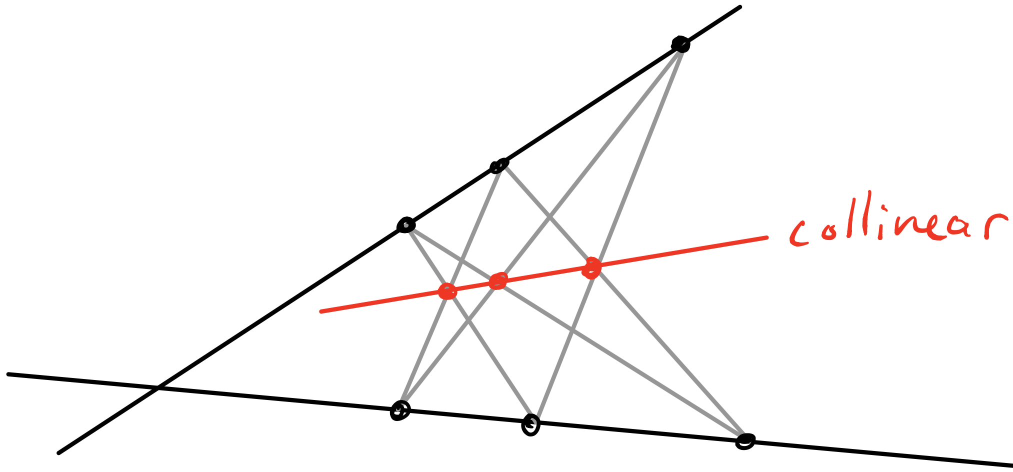
[Convention: lines that meet at ∞ are called "parallel."]





Hibert's Theorem (Foundations of Geometry, 1899):

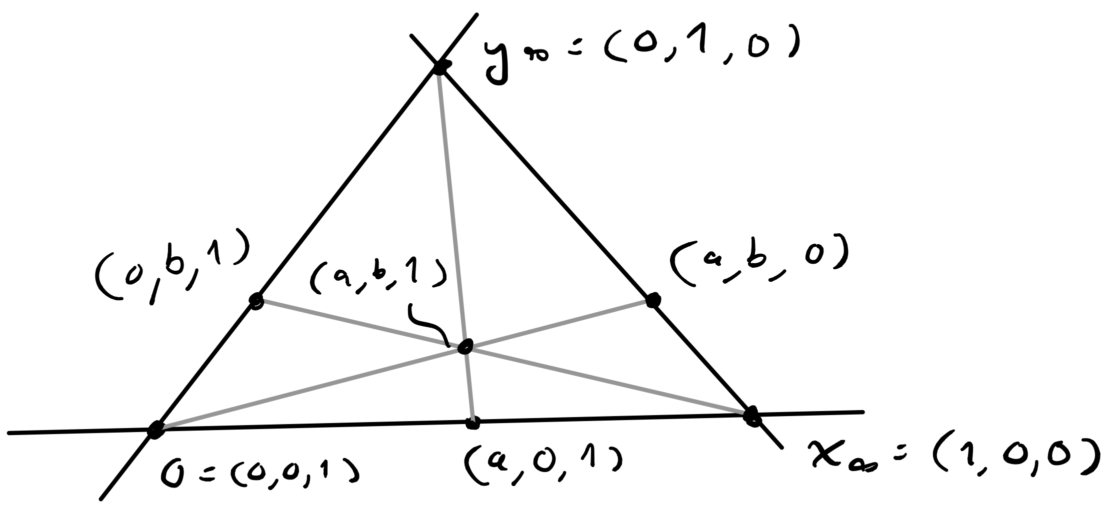
R is a field $\iff$ plane satisfies "Pappus' Theorem"



Given a hexagon with alternating points on two lines, intersection points of opposite sides are collinear.

Proof omitted :) ///

It follows from Hilberts Theorem that we get "homogeneous coordinates"



Coordinates $a, b \in \mathbb{R}$.

$$\Pi = \{(a,b,c) : \underset{\text{not all zero}}{a,b,c \in \mathbb{R}}\} / \text{scalars}$$

$$(a,b,c) \sim (\lambda a, \lambda b, \lambda c) \quad \forall \lambda \neq 0.$$

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② Analytic: Conversely, for any field $\mathbb{F}$ we can define the projective plane over $\mathbb{F}$ by homog. coordinates:

$$\mathbb{FP}^2 = \{\mathbb{F}^3 - \bar{0}\} / \text{scalars}$$

The Fundamental Theorem of the real projective plane relates the synthetic & analytic pictures.

Consider a function $\varphi : \mathbb{RP}^2 \to \mathbb{RP}^2$ sending points to points. Then TFAE:

- $\varphi : \mathbb{RP}^2 \to \mathbb{RP}^2$ is bijective on points & preserves collinearity.

- $\varphi : \mathbb{RP}^2 \to \mathbb{RP}^2$ is induced by an invertible 3×3 matrix acting on homogeneous coordinates.

Proof Sketch:

Let $\varphi : \mathbb{RP}^2 \to \mathbb{RP}^2$ be a collineation, sending $\{O, I, x_\infty, y_\infty\} \to \{P, Q, R, S\}$.

There exists a unique matrix A sending $PQRS$ back to O, I, x_∞, y_∞.

Now $A \circ \varphi : \mathbb{RP}^2 \to \mathbb{RP}^2$ is a collineation fixing points O, I, x_∞, y_∞.

From constructions above, this implies that $A\circ\varphi$ just acts on coordinates by some field automorphism $\alpha : \mathbb{R} \to \mathbb{R}$,

$$A \circ \varphi\,(x, y, z) = (\alpha(x), \alpha(y), \alpha(z)).$$

Finally, observe that $\mathbb{R}$ has no non-trivial automorphism, so

$$A \circ \varphi = id \Rightarrow \varphi = A^{-1} \in GL_3(\mathbb{R}).$$ ///

Remarks:

- Automorphisms of $\mathbb{C}/\mathbb{R}$ are just id & complex conjugation.
- Automorphisms of $\mathbb{C}/\mathbb{Q}$ are much more complicated, but not "geometrically relevant", i.e., do not preserve the usual topology on $\mathbb{C}$.
- Further Justification:

$$\begin{array}{l} \text{biholomorphic automorphisms of } \mathbb{CP}^n \end{array} \begin{array}{l} = \text{bicontinuous} \\ = \text{bipolynomial} \\ = \text{bilinear.} \end{array}$$

$$\underbrace{\mathrm{Aut}}_{\text{in any sense that you choose!}}(\mathbb{CP}^n) = GL_{n+1}(\mathbb{C})/\text{scalars} = PGL_{n+1}(\mathbb{C}).$$

- Automorphisms of <u>affine</u> $\mathbb{C}^n$ are much more complicated (e.g. Jacobian Conjecture!)