Next Topic: Bézout's Theorem.

i.e., There exists a natural definition of "intersection multiplicity"

$$\text{mult}_{\bar{p}}(C, D)$$

for points on two curves $\bar{p} \in C \cap D$ such that, if C, D have <u>no common component</u> then

$$\sum_{\bar{p} \in C \cap D} \text{mult}_{\bar{p}}(C, D) = (\deg C)(\deg D).$$

But First: I forgot to finish our discussion of conics.

Theorem: Let $F(x, y, z) \in \mathbb{C}[x, y, z]$ be homogeneous of degree 2, so $F(\bar{x}) = \bar{x}^T A \bar{x}$ for unique symmetric matrix A. Then TFAE:

① $F(\bar{x})$ is irreducible

② $\det A \neq 0$

③ $V(F)$ is non-singular.

today we will say that a double line is singular

Proof: ① ⟹ ② Suppose $\det A = 0$, hence $\operatorname{rank}(A) = 1$ or 2. We have seen $\exists\, \varphi \in PGL$ such that

$$F^{\varphi}(x,y,z) = x^2$$
$$F^{\varphi}(x,y,z) = x^2 + y^2$$

Each of these is reducible:

$$x^2 + y^2 = (x - iy)(x + iy).$$

Furthermore, reducibility is preserved under PGL. Indeed, if $F^{\varphi} = GH$ for some (necessarily homogeneous) G, H, then applying φ^{-1} gives

$$F = F^{\varphi\varphi^{-1}} = G^{\varphi^{-1}} H^{\varphi^{-1}},$$

where $G^{\varphi^{-1}}, H^{\varphi^{-1}}$ are homogeneous of the same degrees as G, H. ///

(2) $\Rightarrow$ (3): Suppose $V(F)$ is singular. Since singularity is preserved under PGL, we may put F in diagonal form:

$$F^{\varphi} = x^2$$

$$\text{or } x^2 + y^2$$

$$\text{or } x^2 + y^2 + z^2.$$

But I claim that $x^2+y^2+z^2$ is non-singular. Indeed,

$$\nabla(x^2+y^2+z^2) = (2x, 2y, 2z)$$
$$= (0,0,0)$$

implies that $(x,y,z) = (0,0,0)$,

which is not a valid point of $\mathbb{CP}^2$.

just for today

[Note that $V(x^2)$ has singularities at $(0,*,*)$ and $V(x^2+y^2)$ has singularities at $(0,0,*)$.]

Since $V(F^\varphi)$ is singular, this implies that $F^\varphi = x^2$ or x^2+y^2.

In matrix form, $\exists$ invertible matrix B such that

$$
\begin{aligned}
x^2 \text{ or } x^2+y^2 &= F^\varphi(\bar{x}) \\
&= F(B\bar{x}) \\
&= (B\bar{x})^T A (B\bar{x})
\end{aligned}
$$

$$\bar{x}^T \begin{pmatrix} * & & \\ & * & \\ & & 0 \end{pmatrix} \bar{x} = \bar{x}^T B^T A B \bar{x}$$

Since this is true for all $\bar{x}$ we conclude that $B^T A B = \begin{pmatrix} * & & \\ & * & \\ & & 0 \end{pmatrix}$, hence

$$\det(B^T A B) = \det\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$$

$$\det(B)^2 \det(A) = 0$$

$\Rightarrow$ $\det(A) = 0$ because $\det(B) \neq 0$.

③ $\Rightarrow$ ①: We will show more generally that for <u>any</u> homogeneous polynomial,

$V(F)$ is non-singular

$\Rightarrow$ F is irreducible.

Suppose $F = GH$. We'll prove later that $\exists\, \bar{p} \in \mathbb{CP}^2$ such that

$G(\bar{p}) = H(\bar{p}) = 0$, hence also

$F(\bar{p}) = 0$.

Choose affine chart containing $\bar{p}$ & dehomogenize:

$f = gh$, $g(\bar{p}) = h(\bar{p}) = 0$.

We want to show $(\nabla F)_{\bar{p}} = (0,0)$,
i.e., $\bar{p}$ is a singular point of $V(F)$.

Taylor Expansion at $\bar{p}$:

$$F(\bar{p}+\bar{x}) = F^{(0)} + F^{(1)} + F^{(2)} + \cdots$$
$$g(\bar{p}+\bar{x}) = g^{(0)} + g^{(1)} + g^{(2)} + \cdots$$
$$h(\bar{p}+\bar{x}) = h^{(0)} + h^{(1)} + h^{(2)} + \cdots$$

homogeneous polynomials.

where
$$F^{(0)} = F(\bar{p})$$
$$F^{(1)} = (\nabla F)_{\bar{p}}\, \bar{x}$$
$$F^{(2)} = \bar{x}^T (HF)_{\bar{p}}\, \bar{x}$$
etc.

Since $F = gh$ we get

$$F^{(0)} = g^{(0)} h^{(0)}$$
$$F^{(1)} = g^{(0)} h^{(1)} + g^{(1)} h^{(0)}$$
$$F^{(2)} = g^{(2)} h^{(0)} + g^{(1)} h^{(1)} + g^{(0)} h^{(2)}$$
etc.

product rule for gradient vectors.

Since $g^{(0)} = h^{(0)} = 0$ we get

$$f^{(1)} = a\, h^{(1)} + g^{(1)}\, 0 = 0,$$

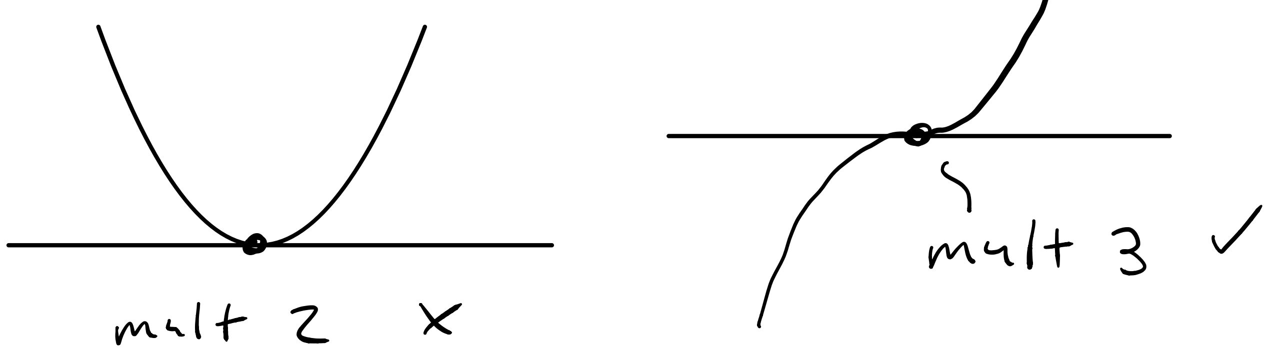
as desired. ///

While we're here, let me observe that this result is useful for the study of inflection points.

Definition: An inflection of a curve C is a nonsingular point $\bar{p}$ with tangent line L such that

$$\mathrm{mult}_{\bar{p}}(C, L) \geq 3.$$

Picture:



Theorem of Inflections:

Let $F(x,y,z) \in \mathbb{C}[x,y,z]$ be homogeneous of degree ≥ 2.

Let $\bar{p}$ be a smooth point of $C = V(F)$

Then $\bar{p}$ is an inflection iff

$$\det(HF)_{\bar{p}} = 0.$$

$$\det \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}_{\bar{p}} = 0$$

<u>Proof</u>: Consider Taylor expansion:

$$F(\bar{p}+\bar{x}) = F(\bar{p}) + (\nabla F)_{\bar{p}}\, \bar{x} + \frac{1}{2} \bar{x}^T (HF)_{\bar{p}}\, \bar{x} + \text{higher terms.}$$

Since $(\nabla F)_{\bar{p}} \neq \bar{0}$, the tangent line has form $L = V((\nabla F)_{\bar{p}}\, \bar{x})$.

Let's also consider the conic

$$Q = V\left(\bar{x}^T (HF)_{\bar{p}}\, \bar{x}\right).$$

Relate Q to $\text{mult}_{\bar{p}}(C, L)$:

To compute multiplicity consider the Taylor expansion:

$$F(\bar{p} + t\bar{q}) = F(\bar{p}) + t\,(\nabla F)_{\bar{p}}\, \bar{q} + \frac{t^2}{2}\, \bar{q}^T (HF)_{\bar{p}}\, \bar{q} + t^3 (\text{stuff}).$$

By definition, multiplicity is ≥ 3

$$\iff \left\{ \begin{array}{r} F(\bar{p}) = 0 \\ (\nabla F)_{\bar{p}}\, \bar{q} = 0 \\ \bar{q}^T (HF)_{\bar{p}}\, \bar{q} = 0 \end{array} \right\}$$

If $\bar{p}$ is an inflection then for all $\bar{q} \in L$ (i.e. $(\nabla F)_{\bar{p}}\,\bar{q} = 0$) we get

$$\bar{q}^T (HF)_{\bar{p}}\, \bar{q} = 0$$

i.e., $L \subseteq Q$. And conversely.

So we want to show that

$$L \subseteq Q \iff \det (HF)_{\bar{p}} = 0.$$

First let $L \subseteq Q$, i.e.,

$$V\big((\nabla F)_{\bar{p}}\, \bar{x} \big) \subseteq V\big(\bar{x}^T (HF)_{\bar{p}}\, \bar{x} \big)$$

From Study's Lemma this implies that $(\nabla F)_{\bar{p}}\,\bar{x} \mid \bar{x}^T (HF)_{\bar{p}}\, \bar{x}$,

so it follows from the above theorem on conics that $\det (HF)_{\bar{p}} = 0$.

Conversely, suppose that $\det (HF)_{\bar{p}} = 0$.

From the above theorem we know that Q is a double line or two (intersecting) lines. We want to show that L is one of these lines. This will follow from two facts:

i) $\bar{p} \in Q$

ii) L is tangent to Q at $\bar{p}$.

To prove these it is convenient to change notation slightly:

$$(x_1, x_2, x_3) := (x, y, z)$$

$$F_i := F_{x_i}.$$

Then since F is homogeneous of degree d, we have Euler's Formula:

$$\sum F_i x_i = F_1 x_1 + F_2 x_2 + F_3 x_3 = dF,$$

and evaluating at $\bar{p} = (p_1, p_2, p_3)$ gives

$$\sum_i F_i(\bar{p})\, p_i = d \cdot F(\bar{p}) = 0 .$$

Furthermore, since the derivative F_i is homogeneous of degree $(d-1)$ we have

$$\sum_j F_{ij}\, x_j = (d-1)\, F_i ,$$

and evaluating at $\bar{p}$ gives

$$\sum_j F_{ij}(\bar{p})\, p_j = (d-1)\, F_i(\bar{p}) .$$

Then putting everything together gives

$$\bar{p}^T (HF)_{\bar{p}}\, \bar{p} = \sum_{i,j} F_{ij}(\bar{p})\, p_i\, p_j$$

$$= \sum_i p_i \sum_j F_{ij}(\bar{p})\, p_j$$

$$= \sum_i p_i\, (d-1)\, F_i(\bar{p})$$

$$= (d-1) \sum F_i(\bar{p}) p_i$$

$$= (d-1) d F(\bar{p})$$

$$= 0,$$

which proves i) ✓

To show ii), I fist claim that the equation of the tangent line to a conic $\bar{x}^T A \bar{x} = 0$ at a point $\bar{p}$ (assume $\bar{p}^T A \bar{p} = 0$) is

$$\bar{p}^T A \bar{x} = 0.$$

Indeed, consider any line $\bar{p} + tq$ through $\bar{p}$, and sustitute into the conic:

$$0 = \bar{x}^T A \bar{x}$$

$$= (\bar{p} + t\bar{q})^T A (\bar{p} + t\bar{q})$$

$$= \bar{p}^T A \bar{p} + 2t\, \bar{p}^T A \bar{q} + t^2 \bar{q}^T A \bar{q}$$

$$= 2t\, \bar{p}^T A \bar{q} + t^2 \bar{q}^T A \bar{q}.$$

Thus $\bar{p} + t\bar{q}$ is tangent to the conic if and only if $\bar{p}^T A \bar{q} = 0$. ✓

Finally it follows from Euler's formula that the tangent line to Q at $\bar{p}$ has the equation

$$0 = \bar{p}^T (HF)_{\bar{p}}\, \bar{x}$$

$$= \sum F_{ij}(\bar{p})\, x_i p_j$$

$$= \sum_i x_i \sum_j F_{ij}(\bar{p})\, p_j$$

$$= \sum x_i\, (d-1)\, F_i(\bar{p})$$

$$= (d-1) \sum F_i(\bar{p})\, x_i$$

$$= (d-1) (\nabla F)_{\bar{p}}\, \bar{x}$$

QED.