Projective tangent lines.

Lines in $\mathbb{F}\mathbb{P}^2$ have the form

$$ax + by + cz = 0.$$

$$\bar{a}^T \bar{x} = 0.$$

Now let $F(x,y,z)$ be homogeneous, with $F(\bar{p}) = F(p,q,r) = 0$.

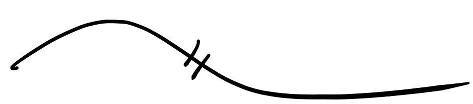
The projective tangent line to the curve $V(F)$ at point $\bar{p}$ has the equation

$$(\nabla F)_{\bar{p}}\, \bar{x} = 0,$$

where

$$(\nabla F)_{\bar{p}} = (F_x(\bar{p}), F_y(\bar{p}), F_z(\bar{p})).$$

Say $\bar{p} \in V(F)$ is a singular point if $(\nabla F)_{\bar{p}} = (0,0,0)$.



Relation to the affine case:

Euler's Homogeneous function theorem says that for hom. poly. $F(x,y,z)$ of degree d, we have

$$xF_x + yF_y + zF_z = dF$$

Proof: Exercise. ///

Using this we will prove that the tangent line at $\bar{p}$ can be computed in any affine chart containing $\bar{p}$.

Proof: Let $\bar{p} = (p, q, 1)$.

Let $F(x,y,z)$ be hom. of degree d,

with $F(p,q,1) = 0$

$$f(x,y) = F(x,y,1)$$

$$z \nmid F$$

so that $F = z^d f\left(\frac{x}{z}, \frac{y}{z}\right)$.

First we observe:

$$\left.\begin{aligned} F_x(p,q,1) &= f_x(p,q) \\ F_y(p,q,1) &= f_y(p,q) \end{aligned}\right\} \text{Exercise.}$$

From Euler's identity:

$$\begin{aligned} (*) \quad & p F_x(p,q,1) + q F_y(p,q,1) + 1 F_z(p,q,1) \\ &= d\, F(p,q,1) \\ &= 0 \end{aligned}$$

Starting with the projective tangent line at $\bar{p} = (p,q,1)$:

$$x F_x(\bar{p}) + y F_y(\bar{p}) + z F_z(\bar{p})$$

$$\overset{(*)}{\parallel}$$

$$(x-p) F_x(\bar{p}) + (y-q) F(\bar{p}) + (z-1) F_z(\bar{p})$$

$$\Big\downarrow \text{ dehomogenize } (z=1)$$

$$(x-p) f_x(p,q) + (y-q) f_y(p,q) + 0.$$

$$\Big\downarrow \text{ rehomogenize}$$

$$(x - zp)F_x(\bar{p}) + (y - zq)F_y(\bar{p})$$

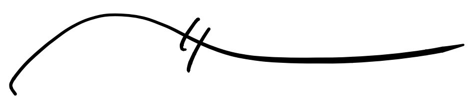
$$\|$$

$$xF_x(\bar{p}) + yF_y(\bar{p}) - z\big(\underbrace{pF_x(\bar{p}) + qF_y(\bar{p})}\big)$$

$$\| \quad (*)$$

$$xF_x(\bar{p}) + yF_y(\bar{p}) + zF_z(\bar{p}) \qquad \checkmark$$

Moral: Tangent line (singularity) is a Zariski-local property.
Later we will see that it's even <u>more</u> local.



To analyze a singularity $\bar{p} \in V(f)$.

$$f(\bar{p} + \bar{x}) = f^{(0)}(\bar{x}) + f^{(1)}(\bar{x}) + f^{(2)}(\bar{x}) + \cdots$$

where $f^{(k)}(\bar{x})$ is homogeneous of degree k, and

$$f^{(0)}(\bar{x}) = f(\bar{p})$$
$$f^{(1)}(\bar{x}) = (\nabla f)_{\bar{p}}\, \bar{x}$$
$$f^{(2)}(\bar{x}) = \bar{x}^T (Hf)_{\bar{p}}\, \bar{x}$$
$$\vdots$$
etc.

We say that $\bar{p} \in V(f)$ is a point of multiplicity m if

$$f(\bar{p}+\bar{x}) = f^{(m)}(\bar{x}) + \cdots + f^{(n)}(\bar{x})$$

i.e. if $f^{(0)}(\bar{x}) = \cdots = f^{(m-1)}(\bar{x}) = 0$ and $f^{(m)}(\bar{x}) \neq 0$.

If $\mathbb{F}$ is algebraically closed then the homogeneous polynomial $f^{(m)}(x,y)$ of degree m splits into linear factors:

$$f^{(m)}(x,y) = \prod_i (a_i x + b_i y)^{e_i}$$

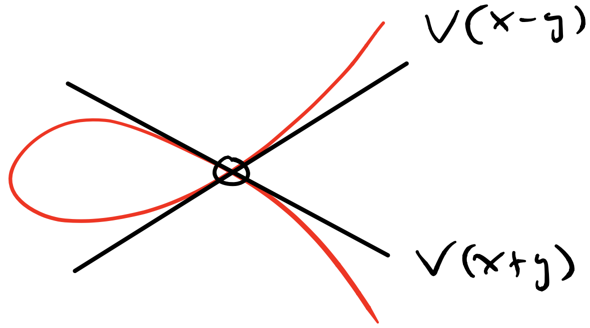
where $e_1 + e_2 + \cdots = m$. [This is just Fund Thm. of Algebra.]

Proof: Exercise. See last semester's notes. ///

Geometric meaning: The linear factors $a_i x + b_i y$ are the tangent lines at the point $\bar{p}$. There are m of them, counted with multiplicity.

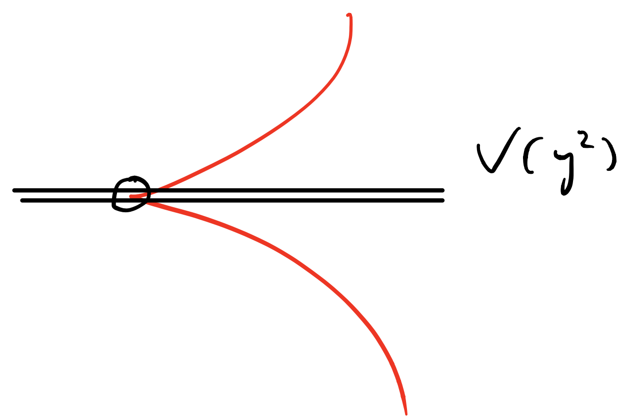
Say that $\bar{p}$ is an "ordinary point of multiplicity m" when it has m distinct tangent lines.

Pictures:



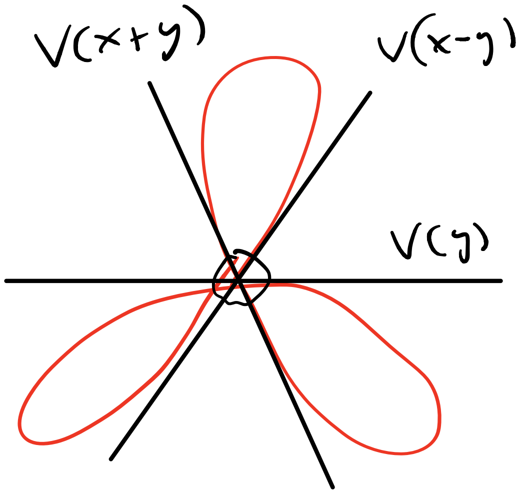
ordinary mult 2

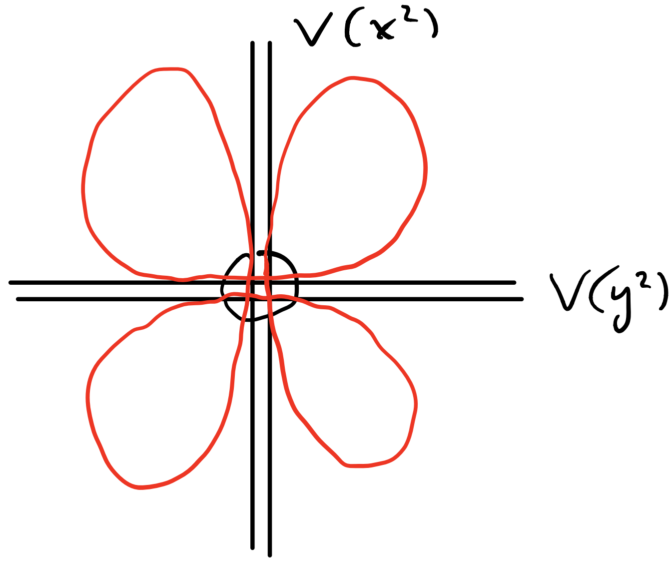
$$\underbrace{(x-y)(x+y)}+x^3=0$$



non-ordinary mult 2.

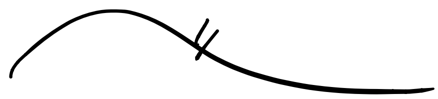
$$\underbrace{y^2}-x^3=0$$





ordinary mult 3 $\qquad$ non-ordinary mult 4

$$\underbrace{3y(x-y)(x+y)} + (x^2+y^2)^2 = 0 \qquad \underbrace{x^2y^2} - (x^2+y^2)^3 = 0$$



Another point of view:

Can parametrize a line through $\bar{p}, \bar{q}$ as $\bar{p} + t\bar{q}$. To compute the intersection of $V(f)$ & $\{\bar{p} + t\bar{q}\}$:

$$f(\bar{p} + t\bar{q}) = f^{(0)}(t\bar{q}) + f^{(1)}(t\bar{q}) + f^{(2)}(t\bar{q}) + \cdots$$

$$= f^{(0)}(\bar{q}) + t\, f^{(1)}(\bar{q}) + t^2 f^{(2)}(\bar{q}) + \cdots$$

Say $V(f)$ & line $\bar{p}+t\bar{q}$ intersect with multiplicity m when

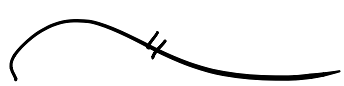
$$f^{(0)}(\bar{q}) = f^{(1)}(\bar{q}) = \cdots = f^{(m-1)}(\bar{q}) = 0$$

$$\& \ f^{(m)}(\bar{q}) \neq 0.$$

i.e. if $t=0$ is a root of $f(\bar{p}+t\bar{q})$ of order m, in which case

$$f(\bar{p}+t\bar{q}) = t^m g(t).$$

If $\deg(f) = n$ then $\deg(g) = n-m$, so that curve & line intersect in $n-m$ other points (typically distinct).



Compare multiplicity of intersection vs. multiplicity of a point:

Suppose $\bar{p} \in V(F)$ has mult. m, so that $f^{(m)} \neq 0$. Then the line $\bar{p} + t\bar{q}$ is one of the m tangents precisely when $f^{(m)}(\bar{q}) = 0$.

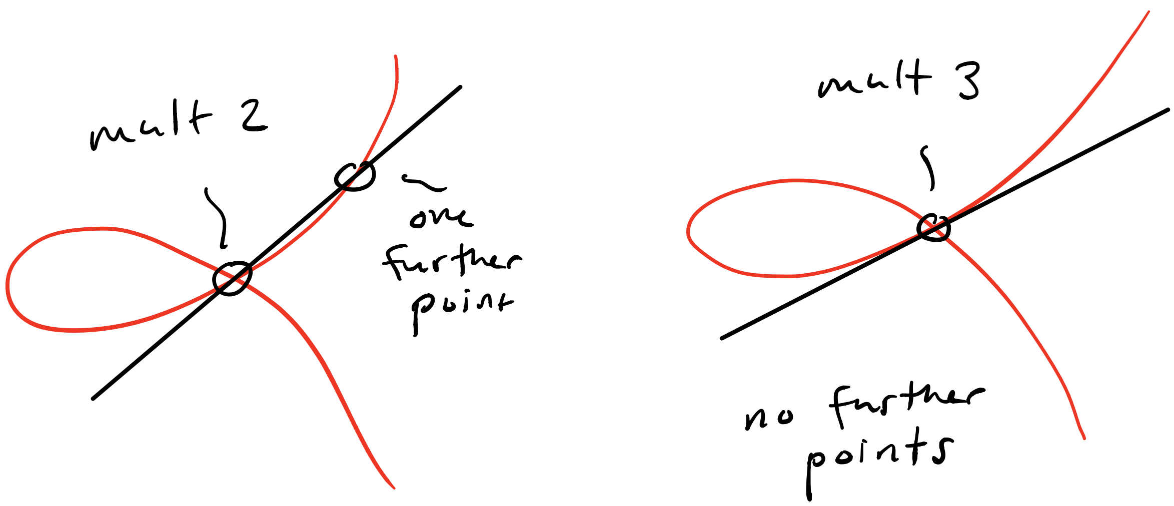
So let $\bar{p} \in C$ have mult m & let $\bar{p} \in L$ be a line. Then

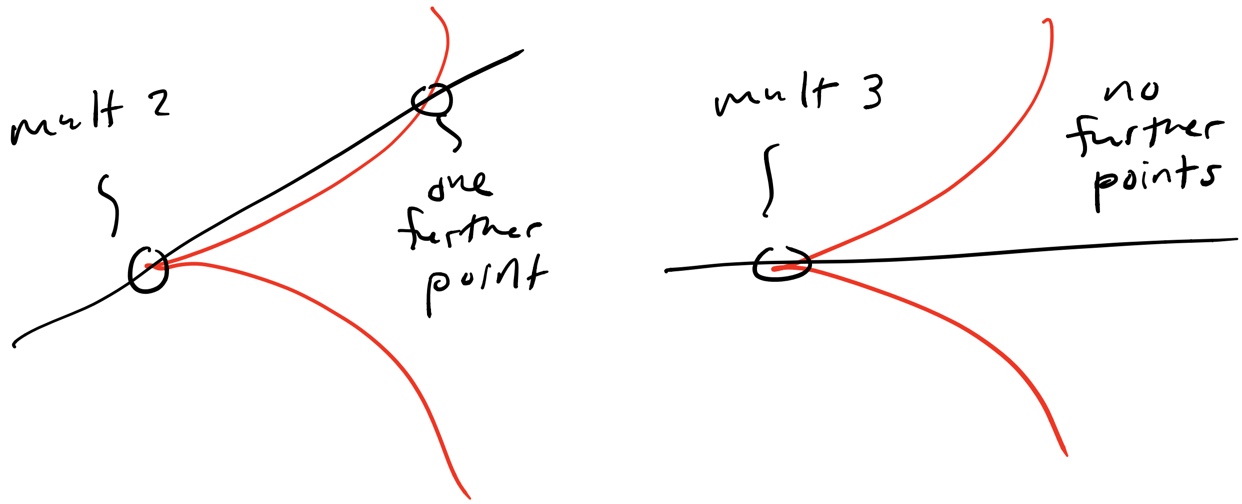
$$\text{mult}_{\bar{p}}(C, L) = \text{mult}_{\bar{p}}(C)$$

for all but finitely many lines L. For the m <u>tangent lines</u> we have

$$\text{mult}_{\bar{p}}(C, L) > \text{mult}_{\bar{p}}(C).$$

Pictures:



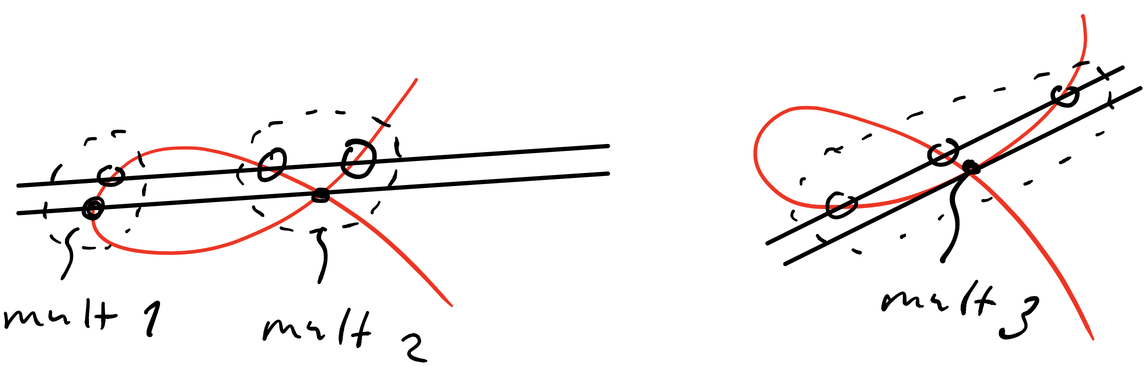


Alternative definition of multiplicity of a point on a curve:

$$\mathrm{mult}_{\bar{p}}(C) = \min\{\mathrm{mult}_{\bar{p}}(C, L) \text{ for all lines through } \bar{p}\}.$$

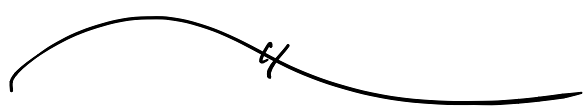
Geometric Meaning (over $\mathbb{C}$):

If $\mathrm{mult}_{\bar{p}}(C, L) = m$ then a small parallel translation of L splits the intersection into m distinct points.



Note: In the left picture the multiplicities of the two points of intersection add to the degree of the curve. This "should" hold in general, since a generic line intersects a curve of degree n in n distinct points.

[Topological arguments are often convincing, but not rigorous.]



Bézout's Theorem is a vast generalization of the above remarks.

Idea: Given curves C, D and a point $\bar{p} \in C \cap D$, there "should" be a natural "intersection multiplicity"

$$\operatorname{mult}_{\bar{p}}(C, D)$$

with the property that

$$\sum_{\bar{p} \in C \cap D} \mathrm{mult}_{\bar{p}}(C, D) = \deg(C) \cdot \deg(D)$$

["should": This is almost an axiom of algebraic geometry. We just have to find the right <u>definition</u>.]

Since algebraic curves can be described in (at least) 3 different ways, it is not surprising that there is no one "best" way to define $\mathrm{mult}_{\bar{p}}(C, D)$.

[We'll see 3 in this class:

- eliminination theory, resultants
- power series, germs of holomorphic functions
- local rings]

There is a homological flavor to intersection theory. Just as with cohomology, there are many equivalent definitions. More important are the general properties that it satisfies.

Theorem (Fulton Alg. Curves, pg. 37).
There exists a <u>unique</u> intersection theory $\mathrm{mult}_{\bar{p}}(C, D)$ satisfying:

- $\mathrm{mult}_{\bar{p}}(C, D) \in \mathbb{N}$ when $\bar{p} \in C \cap D$ and C, D have no common component containing $\bar{p}$.
- $\mathrm{mult}_{\bar{p}}(C, D) = 0 \iff \bar{p} \notin C \cap D$.
- preserved under affine change of coordinates.
- $\mathrm{mult}_{\bar{p}}(C, D) = \mathrm{mult}_{\bar{p}}(D, C)$
- $\mathrm{mult}_{\bar{p}}(C, D) \geq \mathrm{mult}_{\bar{p}}(C) \cdot \mathrm{mult}_{\bar{p}}(D)$
- If $C = V(\prod F_i^{r_i})$ & $D = V(\prod G_j^{s_j})$,

$$\mathrm{mult}_{\bar{p}}(C, D) = \sum r_i s_j \, \mathrm{mult}_{\bar{p}}(V(F_i), V(G_j))$$

- $\mathrm{mult}_{\bar{p}}(V(F), V(G))$

$$= \mathrm{mult}_{\bar{p}}(V(F), V(G + HF))$$

for any polynomial H. //

Whew!