<u>Soon</u> we will prove

Study's lemma: Over alg closed field we have bijections

$$\{\text{squarefree polynomials}\} \longleftrightarrow \{\text{affine plane curves}\}$$

$$\{\text{squarefree hom. polynomials}\} \longleftrightarrow \{\text{projective plane curves}\}.$$

i.e. every curve can be written as $V(f)$ for some <u>unique</u> squarefree polynomial (no repeated prime factors).

$$f = p_1 p_2 \cdots p_k$$

$$V(f) = V(p_1) \cup \cdots \cup V(p_k).$$

Projective completion:

$$\begin{array}{ccc} \mathbb{F}^2 & \subseteq & \mathbb{FP}^2 \\ \cup\!\shortmid & & \cup\!\shortmid \\ V(f) & \subseteq & \overline{V(f)} := V(f^*) \end{array}$$

$\overline{V(f)} - V(f)$ consists of finitely many points at infinity.

General properties of homogenization

$$f \mapsto f^*$$

$$\underset{\deg d}{f(x,y)} \mapsto z^d f\left(\frac{x}{z}, \frac{y}{z}\right)$$

show that irreducibility and components are preserved under completion.

Furthermore, we say that curves

$$C, D \subseteq \mathbb{F}^2$$

are projectively equivalent when

$$\varphi(\overline{C}) = \overline{D}$$

for some $\varphi \in PGL_3(\mathbb{F})$.

[Remark: We like proj. geometry better because of its "finiteness/compactness". But affine geometry is easier to work with algebraically.]

Facts :

- All curves of degree 1 (ie. with minpoly of degree 1) are projectively equivalent.

<u>Proof</u> : Let $F(\bar{x}) = \bar{a}^T \bar{x}$ be any hom. polynomial of degree 1. $(\bar{a} \neq \bar{0})$

e.g. $F(x,y,z) = (a \ b \ c) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$= ax + by + cz.$$

Let A be any invertible matrix such that $\bar{a}^T A = (0 \ 0 \ 1)$
($\bar{a}^T$ is the 3rd column of A^{-1})

Define $\varphi(\bar{x}) = A\bar{x}$, so that

$$F^{\varphi}(\bar{x}) = \bar{a}^T (A\bar{x})$$
$$= (\bar{a}^T A)\bar{x}$$
$$= (0 \ 0 \ 1) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z .$$

Conclusion: Any line in $\mathbb{FP}^2$ is projectively equivalent to the standard line at infinity $z=0$.

- Any curve of degree 2 can be written as $V(F)$, $F(\bar{x}) = \bar{x}^T A \bar{x}$ for some unique $A^T = A$, not necessarily invertible. $(A \neq 0)$.

From last time, such curve is projectively equivalent to one of:

- x^2 — double line
- $x^2 + y^2$ — two intersecting lines
- $x^2 + y^2 + z^2$ — non-degerate.

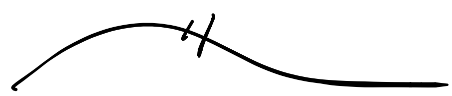
[Remark: We'll see that any non-degenerate conic is isomorphic to the Riemann sphere $\mathbb{CP}^1$.]

So, over $\mathbb{C}$ (any alg closed field), the only projective invariant of the conic $C : \bar{x}^T A \bar{x} = 0$ is its rank.

$$\operatorname{rank}(C) := \operatorname{rank}(A).$$

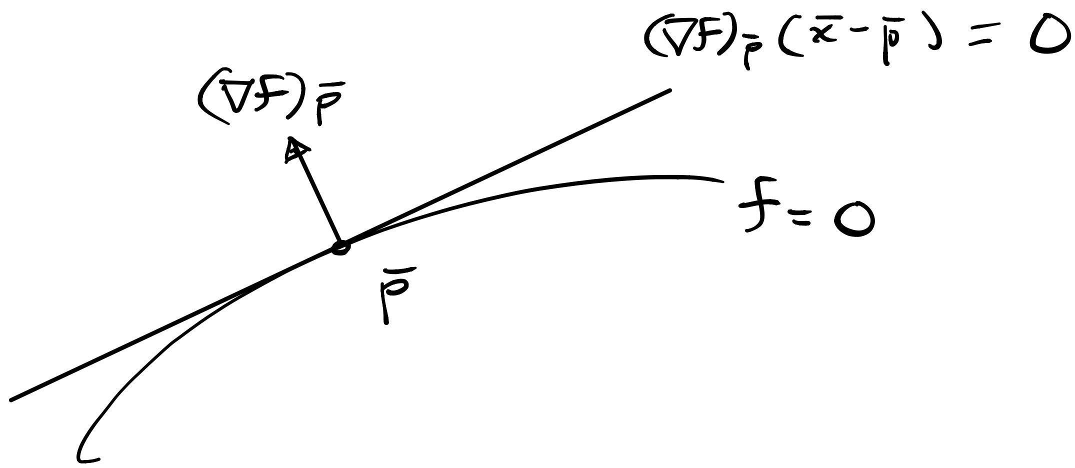
Claim: The following are equivalent.

- $\det(A) \neq 0$
- C is non-singular
- $\bar{x}^T A \bar{x}$ is irreducible.



We need to define singularity.

Recall from Calculus:



Where the gradient vector is

$$\nabla f = (f_{x_1}, f_{x_2}, \ldots, f_{x_n})$$

$$(\nabla f)_{\bar{p}} = (f_{x_1}(\bar{p}), \ldots, f_{x_n}(\bar{p}))$$

Equation of the tangent hyperplane is

$$\underbrace{(\nabla f)_{\bar{p}}}_{\substack{\text{I guess this} \\ \text{is a \underline{row} vector?}}} (\bar{x} - \bar{p}) = 0.$$

In terms of Taylor series:

$$f(\bar{x}) = f(\bar{p}) + (\nabla f)_{\bar{p}} (\bar{x} - \bar{p})$$
$$+ \frac{1}{2} (\bar{x} - \bar{p})^T (Hf)_{\bar{p}} (\bar{x} - \bar{p})$$
$$+ \text{higher terms},$$

where Hf is the <u>Hessian</u> matrix

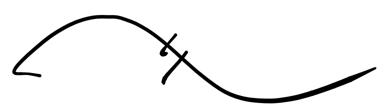
$$Hf = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots \\ f_{x_2x_1} & \ddots & \\ \vdots & & \end{pmatrix}$$

which is symmetric because polynomials are nice functions.

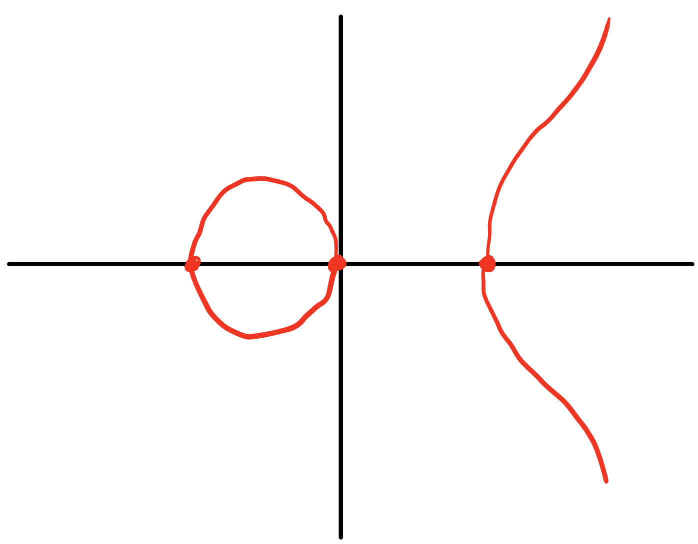
Definition: Given $\bar{p} \in V(f)$ i.e. with $f(\bar{p}) = 0$, we say $\bar{p}$ is a singular point of $V(f)$ when

$$(\nabla f)_{\bar{p}} = \bar{0}.$$

Otherwise $\bar{p}$ is a non-singular point.



Example:
$$\begin{aligned} f(x,y) &= x(x+1)(x-1) - y^2 \\ &= x(x^2-1) - y^2 \\ &= x^3 - x - y^2. \end{aligned}$$



Suppose $\bar{p} = (p, q) \in V(f)$ so

$$y^2 = x^3 - x$$

$$q^2 = p^3 - p$$

The gradient at $\bar{p}$ is

$$\nabla f = (3x^2 - 1, -2y)$$

$$(\nabla f)_{\bar{p}} = (3p^2 - 1, -2q).$$

If $(\nabla f)_{\bar{p}} = (0,0)$ then $q = 0$

$$\rightsquigarrow \quad 0 = q^2 = p^3 - p = p(p^2 - 1)$$

$$\rightsquigarrow \quad p = 0$$

$$\rightsquigarrow \quad 3p^2 - 1 = -1 \neq 0$$

contradiction.

There are <u>no singular points</u>.

More generally, let

$$f(x,y) = g(x) - y^2$$

for some polynomial with <u>no repeated roots</u>.

Proof: Let $\bar{p} = (p, q) \in V(f)$ so that $g(p) - q^2 = 0$

$$g(p) = q^2.$$

The gradient:

$$\nabla f = (g'(x), -2y)$$

$$(\nabla f)_{\bar{p}} = (g'(p), -2q) \stackrel{?}{=} 0$$

$\rightsquigarrow$ $-2q = 0$ & $g'(p) = 0$

$q = 0$

But then $g(p) = q^2 = 0$. Since $g(p) = g'(p) = 0$ this contradicts the fact that g has no multiple roots. ///

Jargon: $y^2 = g(x)$ with no multiple roots is called "hyper-elliptic" ("elliptic" when $\deg(g) = 3$.)

[Remark: There is a theorem that any non-singular cubic is projectively equivalent to an elliptic curve.]

Example: $f(x,y) = x^2(x+1) - y^2$.

$$x^3 + x^2 - y^2$$

$$\nabla f = (3x^2 + 2x, -2y) \stackrel{?}{=} (0,0)$$

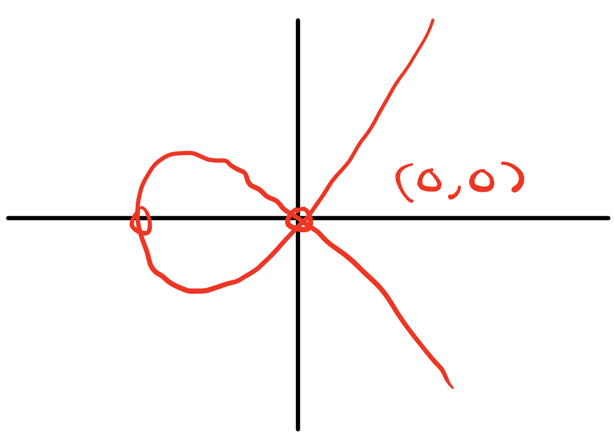
$\rightsquigarrow$ $-2y = 0$ & $3x^2 + 2x = 0$

$y = 0$ $\quad$ $x(3x+2) = 0$

$x = 0$

$\longrightarrow$ $(x,y) = (0,0)$

[$(-2/3, 0)$ not on curve].



unique singularity at the origin.

More specifically ?

Taylor expansion near $\bar{0}$:

$$f(\bar{0}+\bar{x}) = f(\bar{0}) + (\nabla f)_{\bar{0}}\,\bar{x} + \bar{x}^T (Hf)_{\bar{0}}\,\bar{x} + \cdots$$

$$= 0 + 0 + \bar{x}^T (Hf)_{\bar{0}}\,\bar{x} + \cdots$$

Approximately quadratic near $\bar{0}$.

$$x^2(x+1) - y^2$$

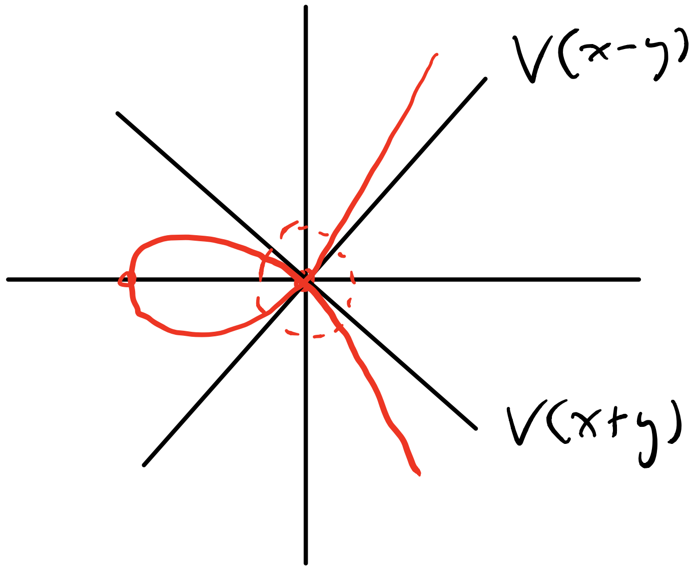
$$= \underset{\substack{\uparrow \\ (0,0) \text{ on} \\ \text{the curve}}}{0} + \underset{\substack{\uparrow \\ \text{singular} \\ \text{point}}}{0} + \underset{\substack{\uparrow \\ \text{locally} \\ \text{quadratic}}}{(x^2-y^2)} + \underset{\substack{\uparrow \\ \text{too much} \\ \text{information}}}{x^3}$$

Since smallest nonzero form is

$$x^2 - y^2 = (x-y)(x+y),$$

curve is <u>locally</u> a union of two lines

$$V(x^2-y^2) = V(x-y) \cup V(x+y).$$

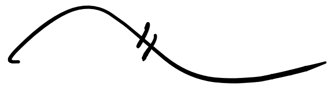


Say that $V(f)$ has two distinct tangents at $(0,0)$.

[Remark: $f(x,y) = x^2(x+1) - y^2$ is irreducible in the ring $\mathbb{F}[x,y]$ ($\mathbb{F}$ alg. closed), but it becomes reducible in the ring of formal power series

$$\mathbb{F}[[x,y]] = \{ \sum a_{ij} x^i y^j \}$$

where infinitely many coefficients may be nonzero. There will be two irreducible factors corresponding to the two "branches."]

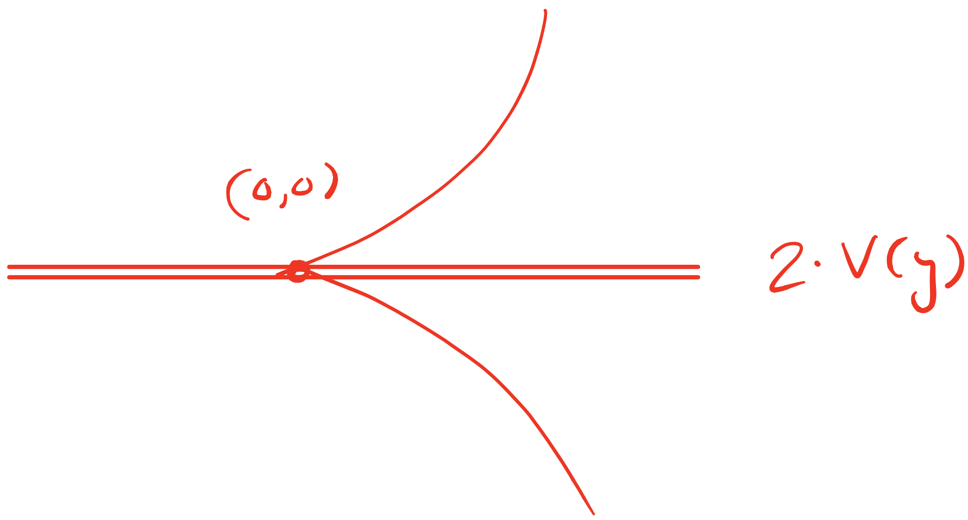


Example: $f(x,y) = x^3 - y^2$.

$$\nabla f = (3x^2, -2y) \overset{?}{=} (0,0)$$

$$\longrightarrow (x,y) = (0,0)$$

unique singularity at origin.



Taylor expansion at (0,0):

$$f = 0 + 0 + (-y^2) + x^3$$

(0,0) on the curve

singular

locally a "double line" $2 \cdot V(y)$

too much information

What about singularities at ∞?

Lines in $\mathbb{F}\mathbb{P}^2$ have the form

$$\bar{a}^T \bar{x} = 0$$
$$ax + by + cz = 0.$$

Let F be homogeneous polynomial. If $F(\bar{p}) = 0$ then the projective tangent line at $\bar{p}$ has equation

$$(\nabla F)_{\bar{p}} \, \bar{x} = 0.$$

Q: Why not $(\nabla F)_{\bar{p}} (\bar{x} - \bar{p}) = 0$?

Well, that's not a homogeneous polynomial, so it doesn't necessarily define any projective set.

<u>However</u>, it turns out that the two equations are <u>equivalent</u>,

so everything is okay.

Proof uses "Euler's Identity" for homogeneous functions

$$(\nabla F)_{\bar{p}} \, \bar{p} = 0.$$

Next time ...