Projective equivalence of curves:

$$\mathrm{Aut}(\mathbb{F}\mathbb{P}^n) = GL_{n+1}(\mathbb{F}) / \text{scalars}$$
$$= PGL_{n+1}(\mathbb{F})$$

PGL also acts on the set of homogeneous polynomials of any degree.

$S_d :=$ homogeneous polynomials of degree d.

$$\mathbb{F}[x_1, \dots, x_n] = \bigoplus_{d \geq 0} S_d$$

"graded ring"

Given φ: PGL, claim $\varphi : S_d \to S_d$.

Proof: If F hom of degree d, then

$$F(\lambda \bar{x}) = \lambda^d F(\bar{x}) \text{ for all } \lambda.$$

If $\mathbb{F}$ is infinite then converse is also true. (Last semester.)

Suppose $F \in S_d$, then for any

$\varphi \in PGL$ & $\lambda \in \mathbb{F}$, we have

$$\begin{aligned} F^{\varphi}(\lambda \bar{x}) &= F(\varphi^{-1}(\lambda \bar{x})) \\ &= F(\lambda \varphi^{-1}(\bar{x})) \\ &= \lambda^d F(\varphi^{-1}(\bar{x})) \\ &= \lambda^d F^{\varphi}(\bar{x}). \end{aligned}$$

linearity of φ

Hence $F^{\varphi} \in S_d$. ///

Curves/Hypersurfaces vs. Polynomials:

$$\varphi(V(F)) = V(F^{\varphi})$$

Then $V(F)$ & $V(F^{\varphi})$ are hypersurf. of the same degree, called "projectively equivalent."

What about affine curves/hypersurfaces?

$$\mathrm{Aut}(\mathbb{F}^n) = \ ?$$

[Warning: Holomorphic automorphisms of $\mathbb{C}^n$ are way too numerous for us! We require nice behavior at ∞.]

Think of $\mathbb{FP}^n = \mathbb{F}^n \cup H_0$

$$= \{(1, x_1, \dots, x_n)\} \cup V(x_0)$$

Let $\mathrm{Aut}(\mathbb{F}^n) = \mathrm{Aut}(\mathbb{FP}^n)$ sending $\mathbb{F}^n \to \mathbb{F}^n$ (also $H_0 \to H_0$)

Say $\varphi \in PGL$ preserves $\mathbb{F}^n$ & H_0

$$\varphi\begin{pmatrix} 1 \\ \hline x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ \hline * \\ \vdots \\ * \end{pmatrix} \quad \& \quad \varphi\begin{pmatrix} 0 \\ \hline x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \hline * \\ \vdots \\ * \end{pmatrix}$$

Write φ as block matrix:

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline & & & \\ & & & \end{array}\right)\begin{pmatrix} 1 \\ \hline x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ \hline * \\ \vdots \\ * \end{pmatrix}$$

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline & & & \\ & & & \end{array}\right)\begin{pmatrix} 0 \\ \hline x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \hline * \\ \vdots \\ * \end{pmatrix}$$

$$\varphi = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline \bar{a} & A \end{array}\right)$$

for some column $\bar{a} \in \mathbb{F}^n$ &
invertible $A \in GL_n(\mathbb{F})$.

we obtain:

$$\varphi(\bar{x}) = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline \bar{a} & A \end{array}\right)\left(\begin{array}{c} 1 \\ \hline x_1 \\ \vdots \\ x_n \end{array}\right)$$

$$= \left(\begin{array}{c} 1 \\ \hline \bar{a} + A\bar{x} \end{array}\right)$$

$$= \underbrace{A\bar{x} + \bar{a}}_{\text{affine transformation.}}$$

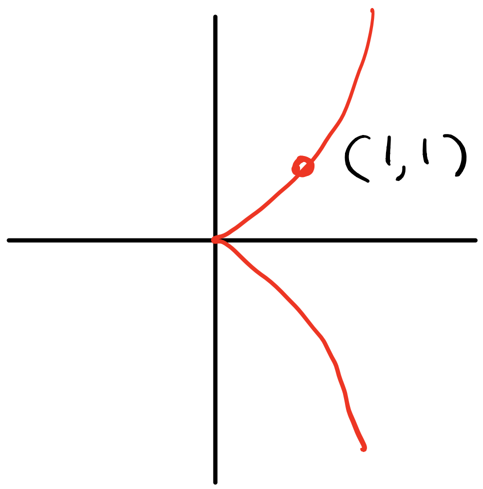
///

Special case: Translation

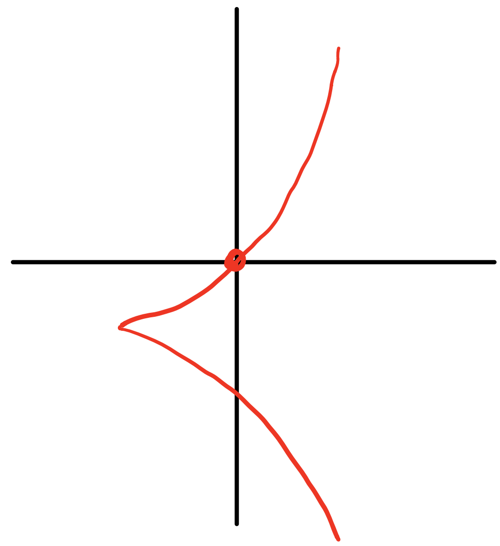
$$\varphi(\bar{x}) = \bar{x} + \bar{a}$$
$$\varphi^{-1}(\bar{x}) = \bar{x} - \bar{a}$$

using this we can send any finite point to the origin.

Example: $f(x,y) = x^3 - y^2$.



$V(f)$



$V(f^{\varphi})$

Let $\varphi(x,y) = (x-1, y-1)$.

$$f^{\varphi}(x,y) = (x+1)^3 - (y+1)^2$$
$$= x^3 + 3x^2 + 3x \cancel{+1} - y^2 - 2y \cancel{-1}$$
$$= 3x - 2y + 3x^2 - y^2 + x^3$$

Remark: Translation is equivalent to Taylor expansion.

Given $F(x_1, \dots, x_n) \in \mathbb{F}[x_1, \dots, x_n]$ and point $\bar{a} = (a_1, \dots, a_n) \in \mathbb{F}^n$ we have a unique expression

$$F(\bar{x}+\bar{a}) = \sum_{I \in \mathbb{N}^n} \frac{(D_{\bar{x}}^{I} F)_{\bar{a}}}{I!} \bar{x}^{I}$$

where $I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$

$$D_x^I = \frac{\partial^{i_1 + \cdots + i_n}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}$$

$$I! = i_1! \, i_2! \cdots i_n!$$

$$\bar{x}^I = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}.$$

Example: Expand $F = x^3 - y^2$ at $(1,1)$.

$F_x = 3x^2$, $F_{xx} = 6x$, $F_{xxx} = 6$

$F_y = -2y$, $F_{yy} = -2$, $F_{yyy} = 0$

$F_{xy} = 0$.

$$F(x+1, y+1) = F(1,1) + F_x(1,1)\,x + F_y(1,1)\,y$$
$$+ \frac{1}{2}\left(F_{xx}(1,1)\,x^2 + 2F_{xy}(1,1)\,xy + F_{yy}(1,1)\,y^2\right)$$
$$+ \frac{1}{6}\left(F_{xxx}(1,1)\,x^3 + \cdots\right)$$

$$= 0 + 3x - 2y + \frac{1}{2}\left(6x^2 + 2\cdot 0xy - 2y^2\right) + \frac{1}{6}\left(6x^3\right)$$

$$= 3x - 2y + 3x^2 - y^2 + x^3 \quad \checkmark$$

Affine vs. Projective Curves / Hypersurfaces:

Given curve $V(f) \subseteq \mathbb{R}^2$

$f(x,y) \in \mathbb{R}[x,y]$ of deg d,

define <u>homogenization</u>

$$f^*(x,y,z) = z^d f\left(\frac{x}{z}, \frac{y}{z}\right).$$

& <u>dehomogenization</u>

$$F_*(x,y) = F(x,y,1).$$

Exercise:

- $(FG)_* = F_* G_*$ & $(fg)^* = f^* g^*$

- $(F+G)_* = F_* + G_*$ & $(f+g)^* = z^r f^* + z^s g^*$

  $r = \deg(g)$ & $s = \deg(f)$.

- $(f^*)_* = f$ & $\underbrace{z^r (F_*)^* = F}_{\substack{\text{Curve } V(F) \subseteq \mathbb{RP}^2 \\ \text{contains the line at } \infty.}}$

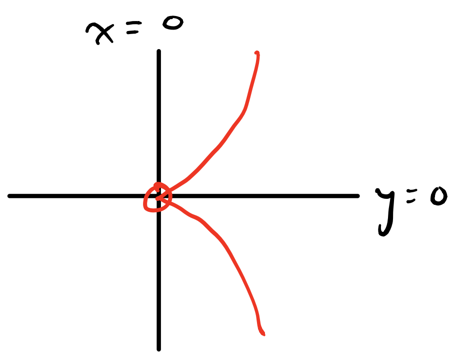
///

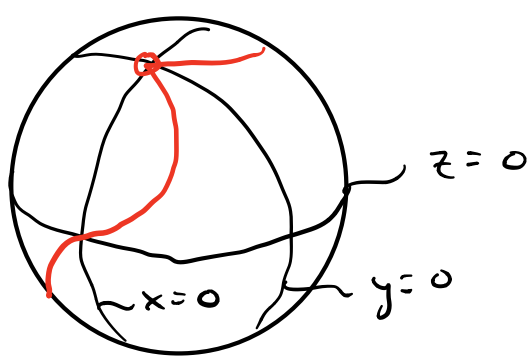
Geometry:

$$\begin{array}{ccc} \mathbb{R}^2 & \subseteq & \mathbb{RP}^2 \\ \cup\!| & & \cup\!| \\ V(f) & \subseteq & \underbrace{\overline{V(f)} := V(f^*)}_{\substack{\text{projective completion} \\ \text{i.e., add the minimum amount} \\ \text{of stuff at infinity.}}} \end{array}$$

Picture: $f(x,y) = x^3 - y^2$.

$f^*(x,y,z) = x^3 - y^2 z$





[see Maple].

Note that projective completion $V(f^*)$ has a unique point at infinity $(z=0)$.

Indeed: $x^3 - y^2 z = 0$

$\downarrow$ $z = 0$

$x^3 = 0$

$x = 0$

$\rightsquigarrow$ $(x,y,z) = (0, \text{anything}, 0)$

$\sim (0,1,0)$

"vertical slope"

To see what happens at this point choose some $\varphi \in PGL$ sending $(0,1,0)$ to the origin $(0,0,1)$.

Many choices: e.g. $\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

So $\varphi\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Then $f(x,y) = x^3 - y^2$

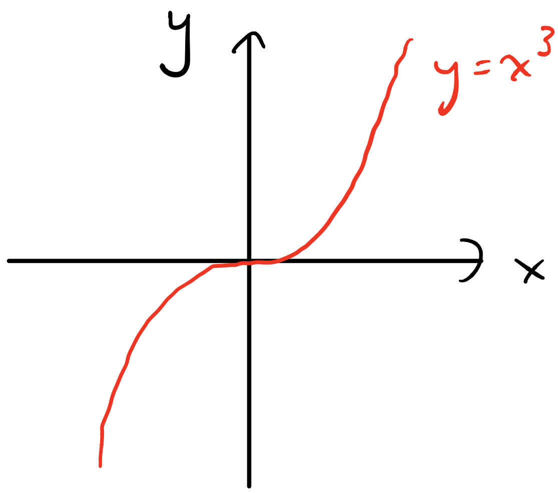
$f^*(x,y,z) = x^3 - y^2 z$

$\downarrow$ flip y & z.

$(\varphi f^*)(x,y,z) = x^3 - z^2 y$

$(\varphi f^*)_*(x,y) = x^3 - y.$

In our new coordinate system:



[Alternatively, we could just dehomogenize at $y=1$ to $f(x,z) = x^3 - z$ which is a curve in the (x,z)-coord plane.]

Summary:

inflection at ∞

cusp at ∞.

Conics: Any degree 2 homogeneous polynomial is called a <u>quadratic form</u>, and has a unique expression

$$F(\bar{x}) = \bar{x}^T A \bar{x}$$

where $A^T = A$ is a symmetrix matrix,

e.g. $x^2 + 2xy - z^2 + 3zx$

$$= (x\ y\ z)\begin{pmatrix} 1 & 1 & 3/2 \\ 1 & 0 & 0 \\ 3/2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The corresponding projective hypersurface $V(F)$ is called a "quadric."

[In the plane, a "conic."]

Theorem: If $\operatorname{char} \mathbb{F} \neq 2$, for any quadratic form $F(\bar{x})$, $\exists\ \varphi \in PGL$

such that

$$F^{q}(\bar{x}) = \underbrace{d_1 x_1^2 + d_2 x_2^2 + \cdots + d_k x_k^2}_{\text{diagonal}}.$$

Over $\mathbb{R}$ we get

$$F^{q}(\bar{x}) = x_1^2 \pm x_2^2 \pm x_3^2 \pm \cdots \pm x_k^2$$

Over $\mathbb{C}$ we get

$$F^{q}(\bar{x}) = x_1^2 + x_2^2 + \cdots + x_k^2.$$

<u>Proof</u>: Standard Theorem of Linear algebra says if $A^T = A$ then $\exists$ invertible B such that

$$B^T A B = D = \begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_r & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

where $d_1, \ldots, d_r \neq 0$, $r = \operatorname{rank}(A)$.

Using this, suppose $F(\bar{x}) = \bar{x}^T A \bar{x}$.

Let $\varphi \in PGL$ defined by $\varphi(\bar{x}) = B^{-1}\bar{x}$, so that

$$
\begin{aligned}
F^{\varphi}(\bar{x}) &= F(\varphi^{-1}(\bar{x})) \\
&= F(B\bar{x}) \\
&= (B\bar{x})^T A (B\bar{x}) \\
&= \bar{x}^T (B^T A B)\, \bar{x} \\
&= \bar{x}^T D \bar{x} \quad \text{as desired.}
\end{aligned}
$$

Over $\mathbb{R}$, let $C = \begin{pmatrix} c_1 & & & & \\ & \ddots & & & \\ & & c_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}$ be

$$
c_i = \begin{cases} 1/\sqrt{d_i} & \text{if } d_i > 0 \\ 1/\sqrt{-d_i} & \text{if } d_i < 0. \end{cases}
$$

Then
$$
C^T B^T A B C = \begin{pmatrix} \pm 1 & & & & \\ & \ddots & & & \\ & & \pm 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}
$$

Over $\mathbb{C}$ we can let

$$
c_i = 1/\sqrt{d_i} \quad \text{if } d_i \neq 0,
$$

so $$C^T B^T A B C = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}.$$

$///$

Corollaries:

- Over $\mathbb{R}$, any conic is projectively equivalent to one of

x^2	double line
$x^2 \pm y^2$	intersecting lines, point
$x^2 + y^2 \pm z^2$	circle, empty

- Over $\mathbb{C}$, any conic is projectively equivalent to one of

x^2	double line
$x^2 + y^2$	two lines
$x^2 + y^2 + z^2$	non-degenerate

[Since $\mathbb{C}$ is algebraically closed, single point & empty set do not occur. They are not "curves."]

Over $\mathbb{C}$, the only projective invariant of a conic $\bar{x}^T A \bar{x} = 0$ is the "rank," i.e., the rank of A.

We say the conic is non-degenerate when the rank is full, i.e., when

$$\det A \neq 0.$$

Claim: Let $F(\bar{x}) = \bar{x}^T A \bar{x}$. Then the following are equivalent:

- $\det A \neq 0$,
- $V(F)$ is irreducible
- $V(F)$ is non-singular.

Proof: Next time.