We are heading in the direction of "birational equivalence" of curves. This is the most "big picture" category in which to study curves.

Last time we showed that any smooth curve in $\mathbb{CP}^n$ $(n \geq 4)$ can be embedded biholomorphically in $\mathbb{CP}^3$.

Proof Sketch: Consider the "secant variety" of the curve $C \subseteq \mathbb{CP}^n$:

$$\mathrm{Sec}(C) = \left\{ \begin{array}{c} \text{points on secant and} \\ \text{tangent lines to } C \end{array} \right\} \subseteq \mathbb{CP}^n$$

If $n \geq 4$ then $\mathbb{CP}^n \setminus \mathrm{Sec}(C)$ is non-empty, so we can choose a point $\bar{p} \in \mathbb{CP}^n \setminus \mathrm{Sec}(C)$. Then the projection

$$\pi_{\bar{p}} : \mathbb{CP}^n \setminus \bar{p} \to \mathbb{CP}^{n-1}$$
$$C \longrightarrow C'$$

is biholomorphic onto its image:

$$C \cong \pi_{\bar{p}}(C) = C' \subseteq \mathbb{CP}^{n-1}.$$

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What can go wrong?

If $n = 3$ then it might be the case that $\mathrm{Sec}(C) = \mathbb{CP}^3$, so there is nowhere to project from.

Example: Consider the twisted cubic:

$$C = \{(s^3, s^2t, st^2, t^3)\} \subseteq \mathbb{CP}^3$$

First let's project from the point

$\bar{p} = (0,0,1,0) \in \mathbb{CP}^3 \setminus C$

onto the plane $\mathbb{CP}^2 = \{(x,y,0,z)\}$.

The line connecting a general point $\bar{a} = (a,b,c,d)$ & $\bar{p} = (0,0,1,0)$ has the form $\bar{a} + \lambda \bar{p} = (a, b, c+\lambda, d)$.

This line intersects $\mathbb{CP}^2 = \{(x,y,0,z)\}$ in the point $(a,b,0,d)$:

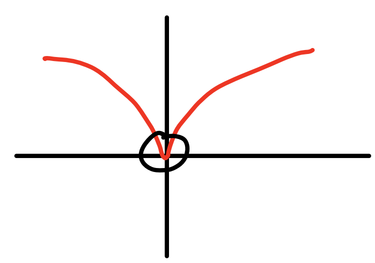
$$\pi_{\bar{p}} : \mathbb{CP}^3 \setminus \bar{p} \longrightarrow \mathbb{CP}^2$$
$$(a,b,c,d) \longmapsto (a,b,0,d).$$

Restricting this projection to the curve C gives

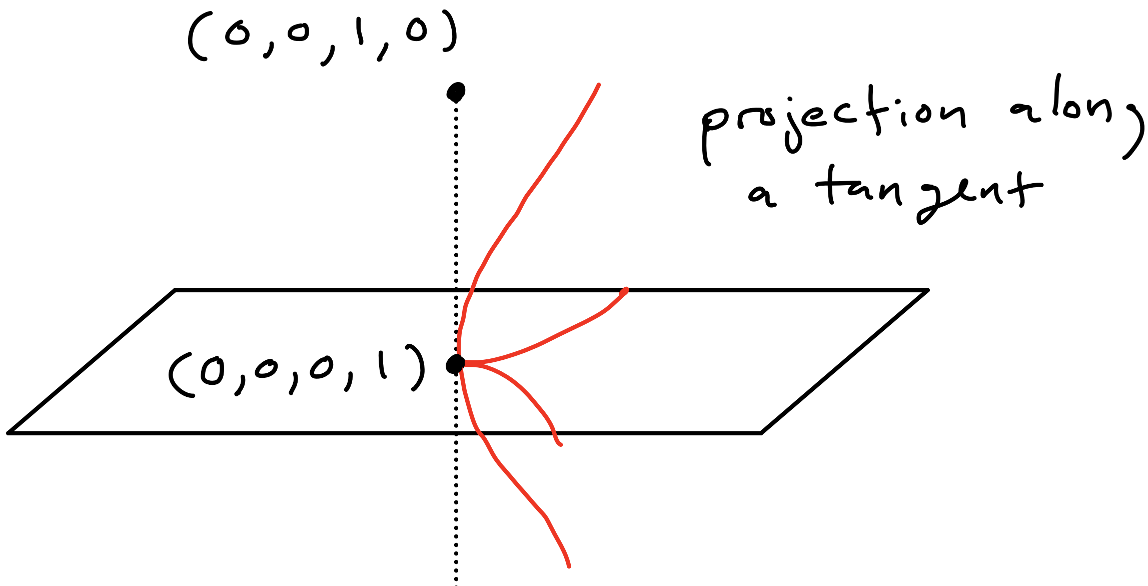
$$\pi_{\bar{p}}(C) = \{(x,y,z) = (s^3, s^2t, t^3)\}$$
$$= V(y^3 - x^2 z)$$

$$\left[\ y^3 = (s^2t)^3 = s^6t^3 = (s^3)^2(t^3) = x^2 z\ \right]$$

This is a cuspidal cubic in the x,y chart:



This happened because $\bar{p} = (0,0,1,0)$ is on a tangent line to C. Namely, the line $(0,0,\lambda,\lambda+1)$ is tangent to C at $(0,0,0,1)$



All the obvious projections have cusps, so let's try a less obvious projection.

Project from $\bar{q} = (1,0,0,1) \in \mathbb{CP}^3 \setminus C$ onto $\mathbb{CP}^2 = \{(0,x,y,z)\}$

The line $\bar{a} + \lambda \bar{q} = (a+\lambda, b, c, d+\lambda)$ meets the plane $\mathbb{CP}^2 = \{(0,x,y,z)\}$ when $a+\lambda = 0$, i.e., $\lambda = -a$.

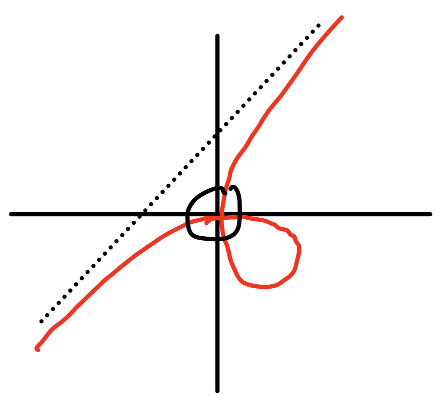
So the projection is given by

$$\begin{aligned} \pi_{\bar q} : \mathbb{CP}^3 \setminus \bar q &\longrightarrow \mathbb{CP}^2 \\ (a,b,c,d) &\longmapsto (0,b,c,d-a). \end{aligned}$$

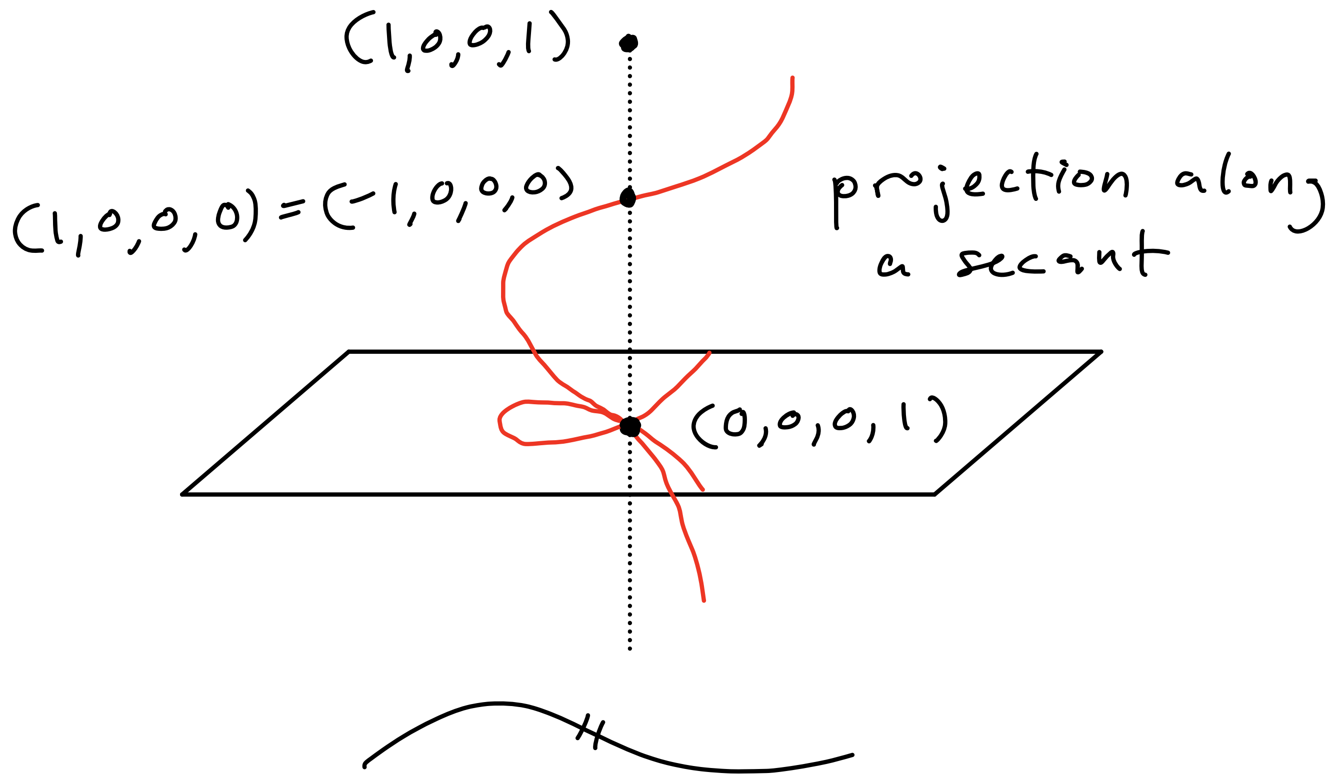
Restricting this to the curve gives

$$\begin{aligned} \pi_{\bar q}(C) &= \{(0,x,y,z) = (0, s^2t, st^2, t^3-s^3)\} \\ &= V(x^3-y^3+xyz), \end{aligned}$$

which is a nodal cubic curve in the x,y plane:



OOPS! The point $\overline{z} = (1,0,0,1)$ was on a secant (non-tangent) to the curve. Indeed, this line is $(\lambda, 0, 0, \lambda+1)$, which intersects C in exactly two points, when $\lambda = 0$ and -1:



I claim that <u>any</u> projection of the twisted cubic $C \subseteq \mathbb{CP}^3$ onto $\mathbb{CP}^2$ is equivalent to one of the above two examples, i.e., has a single cusp or a single node.

In other words, I claim that

$$\mathrm{Sec}(C) = \mathbb{CP}^3,$$

and that any point $\bar{p} \in \mathbb{CP}^3 \setminus C$ lies on either a secant or a tangent (but not both). To prove this we will use a very clever idea from [Harris, Alg. Geom., pg 118].

Idea: There exists an action of PGL_2 on $\mathbb{CP}^3$ with three orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$, satisfying:

$$\mathcal{O}_1 = C$$
$$\mathcal{O}_1 \cup \mathcal{O}_2 = \text{union of tangent lines}$$
$$\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 = \mathrm{Sec}(C)$$

Furthermore, we can view this

strange action of PGL_2 as a subroup of the usual action $PGL_4 \curvearrowright \mathbb{CP}^3$.

Proof: First we observe that

$$C = \{(s^3, s^2t, st^2, t^3)\} \subseteq \mathbb{CP}^3$$

is projectively equivalent to

$$C' = \{(s^3, 3s^2t, 3st^2, t^3)\} \subseteq \mathbb{CP}^3.$$

[In fact, we can choose as coords. <u>any</u> basis of 3-forms in s,t.]

Next we identify $\mathbb{CP}^3$ with the set of 3-forms in some variables u,v:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \equiv au^3 + bu^2v + cuv^2 + dv^3$$

We define the (left) action of PGL_2 (u,v) as usual:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha u + \beta v \\ \gamma u + \delta v \end{pmatrix}$$

And we extend this action to $\mathbb{CP}^3$ as follows:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \bullet \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$:= a(\alpha u + \beta v)^3 + b(\alpha u + \beta v)^2(\gamma u + \delta v)$$
$$+ c(\alpha u + \beta v)(\gamma u + \delta v)^2 + d(\gamma u + \delta v)^3$$

$\vdots$ some computation

$$\equiv \begin{pmatrix} \alpha^3 & \alpha^2\gamma & \alpha\gamma^2 & \gamma^3 \\ 3\alpha^2\beta & \alpha^2\delta + 2\alpha\beta\gamma & \beta\gamma^2 + 2\alpha\gamma\delta & 3\gamma^2\delta \\ 3\alpha\beta^2 & \beta^2\gamma + 2\alpha\beta\delta & \alpha\delta^2 + 2\beta\gamma\delta & 3\gamma\delta^2 \\ \beta^3 & \beta^2\delta & \beta\delta^2 & \delta^3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

One can check that this is a <u>right</u> action on $\mathbb{CP}^3$ because it reverses the order of multiplication.

After that clever definition, the rest is easier. We observe that this action preserves the factorization of 3-forms in (u,v). Thus we have 3 orbits:

Use that $PGL_2 \curvearrowright \mathbb{CP}^1$ is transitive on triples of points.

$$\mathcal{O}_1 = \{ L(u,v)^3 \}$$

$$\mathcal{O}_2 = \{ L_1(u,v)^2 L_2(u,v) \}$$

$$\mathcal{O}_3 = \{ L_1(u,v) L_2(u,v) L_3(u,v) \}$$

It is easy to see that $\mathcal{O}_1$ is equal to the twisted cubic C'. Indeed,

$$(s^3, 3s^2t, 3st^2, t^3)$$

$$\equiv s^3u^3 + 3s^2tu^2v + 3st^2uv^2 + t^3v^3$$

$$= (su+tv)^3.$$

To see that $\mathcal{O}_1 \cup \mathcal{O}_2$ is the union of tangent lines, we note that the

point $\bar{p} = (0,0,1,0) \in \mathbb{CP}^3 \setminus C$
from the example becomes

$$\bar{p}' = (0,0,3,0) \in \mathbb{CP}^3 \setminus C',$$

and that $\bar{p}' \equiv 3u^2v \in \mathcal{O}_2$.

Finally, we note that the point $\bar{q} = (1,0,0,1) \in \mathbb{CP}^3 \setminus C$ from the example becomes $\bar{q}' = (1,0,0,1) \in \mathbb{CP}^3 \setminus C'$ and that

$$\bar{q}' \equiv u^3 + v^3 = \prod_{k=0}^{2} \left(u + e^{2\pi i k/3} v\right) \in \mathcal{O}_3.$$

QED.