More discussion of the genus.

We have seen that a smooth curve in $\mathbb{CP}^2$ of degree d has genus

$$g = \frac{(d-1)(d-2)}{2}.$$

This means that there do not exist smooth plane curves of genera

$$2, 4, 5, 7, 8, 9, 11, \ldots$$

Hower, I claim that there do exist "smooth algebraic curves" of <u>all</u> genera, embedded in $\mathbb{CP}^3$.

[Moreover, we will see that <u>any</u> smooth algebraic curve in $\mathbb{CP}^n$ $(n \geq 4)$ can be embedded in $\mathbb{CP}^3$.]

To begin, we consider the <u>set</u>

$$\mathbb{CP}^1 \times \mathbb{CP}^1 = \{(x_1, x_2 ; y_1, y_2)\}$$

modulo the relations

$$(\lambda x_1, \lambda x_2; \mu y_1, \mu y_2) = (x_1, x_2; y_1, y_2)$$

for all $\lambda, \mu \in \mathbb{C}$.

Question: Is $\mathbb{CP}^1 \times \mathbb{CP}^1$ a variety?

It's not even clear what this should mean. Certainly the product of the Zariski topologies is "wrong" because the only closed sets are finite unions of vertical/horizontal lines.

We will define the topology on $\mathbb{CP}^1 \times \mathbb{CP}^1$ via the "Segre embedding":

$$\sigma : \mathbb{CP}^1 \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^3$$

$$(x_1, x_2; y_1, y_2) \longmapsto (z_{11}, z_{12}, z_{21}, z_{22})$$

where $z_{ij} = x_i y_j$. Note that σ is well defined. Indeed, if

$$(\bar{x}'; \bar{y}') \sim (\bar{x}; \bar{y}) \text{ then}$$

$$z_{11}' = x_1' y_1' = \lambda\mu\, x_1 y_1 = \lambda\mu\, z_{11}$$
$$z_{12}' = x_1' y_2' = \lambda\mu\, x_1 y_2 = \lambda\mu\, z_{12}$$
$$z_{21}' = x_2' y_1' = \lambda\mu\, x_2 y_1 = \lambda\mu\, z_{21}$$
$$z_{22}' = x_2' y_2' = \lambda\mu\, x_2 y_2 = \lambda\mu\, z_{22},$$

so that $(\bar{z}') \sim (\bar{z})$.

And it is <u>always defined</u>. To see this suppose for contradiction that

$$(x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2) = (0,0,0,0)$$

since $(x_1, x_2), (y_1, y_2) \in \mathbb{CP}^1$ we must have $x_1 \neq 0$ or $x_2 \neq 0$, and also $y_1 \neq 0$ or $y_2 \neq 0$. But

$$x_1 \neq 0 \longrightarrow y_1 = y_2 = 0 \quad \lightning$$
$$x_2 \neq 0 \longrightarrow y_1 = y_2 = 0 \quad \lightning$$

Thus σ is a function of sets.

I claim that σ is bijective onto its image, which is the smooth quadric surface

$$Q = V(z_{11} z_{22} - z_{12} z_{21}) \subseteq \mathbb{CP}^3.$$

Indeed, the image of σ is contained in Q. For the other direction, consider a general point

$$(a, b, c, d) \in Q,$$

so that $ad = bc$. Then in any standard affine chart of $\mathbb{CP}^3$ we can define the inverse:

$$\sigma^{-1}(a, b, c, 1) = (b, 1, c, 1)$$

$$\left[\sigma(b, 1, c, 1) = (bc, b, c, 1) = (a, b, c, 1) \right]$$

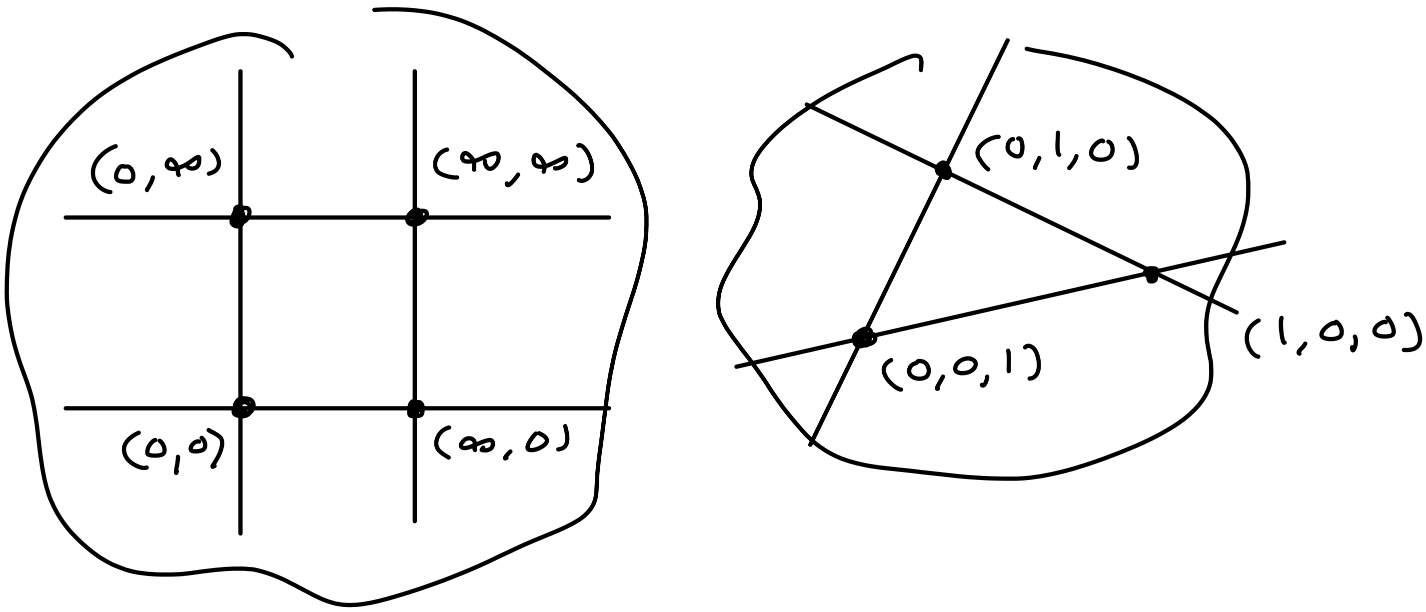
etc.

Thus we have a <u>bijection</u>

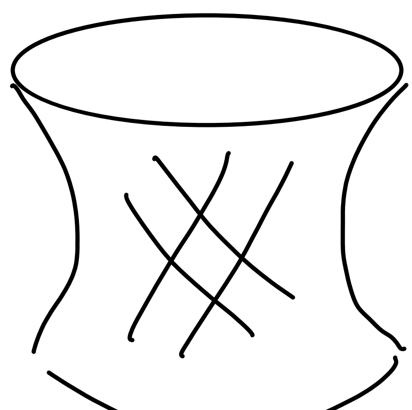
$$\mathbb{CP}^1 \times \mathbb{CP}^1 \longleftrightarrow Q \subseteq \mathbb{CP}^3$$

We *define* the Zariski topology on $\mathbb{CP}^1 \times \mathbb{CP}^1$ via this bijection.

Picture of $\mathbb{CP}^1 \times \mathbb{CP}^1$ vs. $\mathbb{CP}^2$:



They are not isomorphic because, for example, $\mathbb{CP}^1 \times \mathbb{CP}^1$ contains "parallel lines". Here is a slightly more accurate picture of $\mathbb{CP}^1 \times \mathbb{CP}^1$:

$\mathbb{CP}^1 \times \mathbb{CP}^1 := Q =$ 

[Warning: The true picture of Q is a real 4D manifold in $\mathbb{R}^6$!]

Given this structure, we can talk about "curves" in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

These are defined by "bihomogeneous" polynomials $F(\bar{x};\bar{y}) \in \mathbb{C}[x_1, x_2, y_1, y_2]$ satisfying

$$F(\lambda \bar{x}, \bar{y}) = \lambda^m F(\bar{x}, \bar{y})$$
$$F(\bar{x}, \mu \bar{y}) = \mu^n F(\bar{x}, \bar{y}).$$

In this case we say F is homogeneous of bidegree (m,n). [Note that this also implies homogeneous of degree $m+n$.]

Theorem: Consider a generic (irreducible & non-singular) bihomogeneous polynomial

$$F(\bar{x}, \bar{y}) \in \mathbb{C}[x_1, x_2, y_1, y_2]$$

of bidegree (m, n). Then the smooth curve $V(F) \subseteq \mathbb{CP}^1 \times \mathbb{CP}^1$ has genus $(m-1)(n-1)$. [And thus every genus is achieved.]

Proof (Riemann-Hurwitz):
We consider the vertical projection

$$\begin{aligned} \pi : \mathbb{CP}^1 \times \mathbb{CP}^1 &\to \mathbb{CP}^1 \\ (\bar{x}, \bar{y}) &\mapsto \bar{x} \end{aligned}$$

restricted to the curve $V(F)$.

For this purpose we write

$$F(\bar{x},\bar{y}) = \Phi_0(\bar{x})\, y_1^n y_2^0 + \cdots + \Phi_n(\bar{x})\, y_1^0 y_2^n,$$

where each $\Phi_i(\bar{x}) \in \mathbb{C}[x_1,x_2]$ is homogeneous of degree m. The covering $V(F) \to \mathbb{CP}^1$ is generally n-to-1. To find the branch points we will use the following Lemma.

Lemma: Let $f(s,t)$ be homogeneous. Then f has a repeated root of the form $(bs-at)$ if and only if

$$\mathrm{Disc}(f) := \mathrm{Res}_{s,t}(f_s, f_t) = 0.$$

Proof: Exercise. ///

Based on this Lemma, we define

$$G(\bar{x}) = \mathrm{Res}_{\bar{y}}(F_{y_1}, F_{y_2}) \in \mathbb{C}[x_1,x_2].$$

By tracing definitions we see that $G(\bar{x})$ is determinant of an $(n-1)\times(n-1)$ matrix whose entries are homogeneous polynomials in $\mathbb{C}[x_1,x_2]$ of degree m. Hence $G(\bar{x})$ is homogeneous of of degree $2m(n-1)$.

We may assume that F is general, so that G has $2m(n-1)$ distinct roots, which are the branch points:

$$\Delta(\pi) = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{2m(n-1)}\} \subseteq \mathbb{CP}^1$$

For $\bar{c} \in \mathbb{CP}^1 \setminus \Delta$ we have $\#\pi^{-1}(\bar{c}) = n$.

We may assume (tweak coefficients if necessary) that $\#\pi^{-1}(\bar{b}_i) = n-1$ for all i. Finally, by Riemann-Hurwitz we have $(r = 2m(n-1))$

$$\chi(V(F)) = n\left(\chi(\mathbb{CP}^1) - r\right) + \sum_{i=1}^{r} \#\pi^{-1}(\overline{b_i})$$

$$= n\left(2 - 2m(n-1)\right) + 2m(n-1)(n-1)$$

$$= 2 - 2(m-1)(n-1)$$

Hence $\quad 2 - 2g = 2 - 2(m-1)(n-1)$

$$g = (m-1)(n-1).$$

$///$

To find a smooth curve of genus 2 we consider a general polynomial in $\mathbb{C}[x_1, x_2, y_1, y_2]$ of degree $(3,2)$.

For example:

$$F = \frac{1}{2}x_1^3 y_1^2 + x_2^3 y_1 y_2 + \frac{1}{2}x_1^3 y_2^2.$$

We can show that the curve $V(F) \subseteq \mathbb{CP}^1 \times \mathbb{CP}^1$ is non-singular by restricting to the 4 affine charts $x_1 = y_1 = 1$, $x_1 = y_2 = 1$, $x_2 = y_1 = 1$, $x_2 = y_2 = 1$.

[Proof: Exercise.]

To find the branch points, consider

$$G(\bar{x}) = \mathrm{Res}_{\bar{y}}(F_{y_1}, F_{y_2})$$

$$= \det \begin{pmatrix} x_1^3 & x_2^3 \\ x_2^3 & x_1^3 \end{pmatrix}$$

$$= x_1^6 - x_2^6 = \prod_{k=0}^{5} (x_1 - \omega^k x_2)$$

where $\omega = \exp(2\pi i/6)$. Thus we have 6 distinct branch points

$$\Delta = \{ (\omega^k, 1) : k = 0, \ldots, 6 \} \subseteq \mathbb{CP}^1$$

We can think of $V(F)$ as a two-sheeted covering of $\mathbb{C}$ branched over the 6^{th} roots of unity, where each branch point has one preimage.

Thus $V(F)$ has genus 2.

How does it sit inside $\mathbb{CP}^3$ via the Segre embedding?

Idea: We bump up the polynomial F of degree $(3,2)$ to a polynomial of degree $(3,3)$, and there are 2 ways to do this: $y_1 F$ & $y_2 F$. Observe that

$$V(F) = V(y_1 F, y_2 F)$$

$y_1 F$ & $y_2 F$ define surfaces in $\mathbb{CP}^3$

Thus the image of our curve in $\mathbb{CP}^3$ is defined by 3 homogeneous polynomials

$$z_{11}z_{22} - z_{12}z_{21}, \; y_1 F, \; y_2 F$$

Q