Algebraic Curves **Spring 2021**
Homework 1 **Drew Armstrong**

Turn in any one problem by Thursday, Feb 18, on the Google classroom.. You may be able to find solutions in last semester's course notes, or elsewhere on my webpage.

Problem 1. For any ideals $I_1, I_2 \subseteq \mathbb{F}[x_1, \ldots, x_n]$, prove that $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$. This is the only non-formal ingredient of the Zariski topology. [That is, every other property of the Zariski topology follows from formal properties of the maps V, I.]

Problem 2. A closed set S is called **reducible** if can be expressed as $S = S_1 \cup S_2$ where $S_1, S_2 \subsetneq S$ are proper closed subsets. Prove that a variety $V = V(I)$ is irreducible if and only if the ideal I is prime.

Problem 3. Let $(L, \wedge, \vee)$ be a lattice[1] satisfying two extra properties:

- DCC: given $x_0 \geq x_1 \geq \cdots$ we must have $x_n = x_{n+1} = \cdots$ for some n.
- Semi-Distributive: for all x, y, z we have $x \wedge (y \vee z) = (x \vee y) \wedge (x \vee z)$.

Prove that every element $x \in L$ has a "unique factorization" $x = p_1 \vee \cdots \vee p_k$ where each p_i is irreducible (cannot be written as $p = q_1 \vee q_2$ with $q_1, q_2 \lneq p$) and $p_i \not\leq p_j$ for all $i \neq j$.

Problem 4. Let $\mathbb{F}$ be an algebraically closed field. Use the Nullstellensatz and Problem 3 to prove that every radical ideal $I \subseteq \mathbb{F}[x_1, \ldots, x_n]$ has a "unique factorization" $I = P_1 \cap \cdots \cap P_k$ where each P_i is prime and $P_i \not\subseteq P_j$ for all $i \neq j$.

Problem 5. More generally, let R be any commutative ring and let $I \subseteq R$ be any ideal. Prove that $\sqrt{I}$ is the intersection of all prime ideals that contain I. [Hint: For the hard direction, suppose that $f \notin \sqrt{I}$ and consider the set $S = \{1, f, f^2, \ldots\}$. You may assume (Zorn's Lemma) that the set $\{\text{ideals } J \subseteq R\text{: } I \subseteq J \text{ and } S \cap J = \emptyset\}$ contains some maximal element P. Show that this P is a prime ideal.]

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[1]For intuition you can think of L as a collection of sets with $\wedge = \cap$ and $\vee = \cup$.