

Projective Zariski Topology:

$I: (\text{subsets } \mathbb{P}^n) \xrightarrow{\text{non-unit}} (\text{hom. ideals } \mathbb{F}[x_1, \dots, x_{n+1}]): V$

Then $V I$ is called Zariski closure.

Zariski-closed sets are finite intersections of projective hypersurfaces.

Main difference with affine Zariski topology:

$$I(\emptyset) \neq \mathbb{F}[\vec{x}]$$

Since $\emptyset \subseteq \mathbb{P}^n$ corresponds to the conical set $\{\vec{0}\} \subseteq \mathbb{F}^{n+1}$ we have

$$\begin{aligned} I(\emptyset) &= I(\vec{0}) = M_{\vec{0}} \\ &= x_1 \mathbb{F}[\vec{x}] + \dots + x_{n+1} \mathbb{F}[\vec{x}]. \end{aligned}$$

Called the "irrelevant ideal."

I claim that $M_{\vec{0}}$ is the unique maximal homogeneous ideal.

Proof: Let $M \subsetneq \mathbb{F}[\vec{x}]$ maximal among

homogeneous ideals. Since $I(V(M)) \supseteq M$ is also homogeneous, we must have $I(V(M)) = M$.

Now I claim that $V(M)$ is minimal among projective varieties. To see this, let $V \subseteq V(M)$ be any projective variety. By definition, we have

$$V = V(I(V)).$$

Applying I to containment $V \subseteq V(M)$ gives

$$M = I(V(M)) \subseteq I(V),$$

which again implies $M = I(V)$ because $I(V)$ is homogeneous.

Then applying V gives

$$V(M) = V(I(V)) = V.$$

Hence $V(M)$ is minimal. But \emptyset is the unique minimal variety, hence

$V(M) = \emptyset$. Finally, we conclude

$$M = I(V(M)) = I(\emptyset) = M_{\mathcal{O}}.$$

QED.

When \mathbb{F} is algebraically closed, we get order-reversing bijections:

$$\begin{aligned} (\text{proj. varieties}) &\overset{\sim}{\leftrightarrow} (\text{rad. hom. ideals}) \\ (\text{irr. proj. varieties}) &\overset{\sim}{\leftrightarrow} (\text{prime hom. ideals}). \end{aligned}$$

It follows that every rad. hom. ideal is contained in irrelevant ideal $M_{\mathcal{O}}$.

Ideals whose radical closure is contained in $M_{\mathcal{O}}$ are called "relevant," because they correspond to projective sets.

Minimal Prime Homogeneous Ideals:

Since $\mathbb{F}[\vec{x}]$ is UFD, we already know

minimal non-empty primes = principal primes.

Now I claim

$$\begin{array}{l} \text{minimal non-empty} \\ \text{hom. primes} \end{array} = \begin{array}{l} \text{hom. principal} \\ \text{primes} \end{array}$$

i.e. $F \in \mathbb{F}[\vec{x}]$ with F homogeneous & irreducible, i.e., projective hypersurfaces.

Proof: Let $P = \mathbb{F}[\vec{x}]$ with F (non-zero, non-unit) hom. & irreducible.

So we already know that P is minimal among all prime ideals, hence also among homogeneous prime ideals. ✓

Conversely, let $P \neq 0$ be minimal among homogeneous primes. Choose some non-zero, non-unit $F \in P$. Factor F into primes

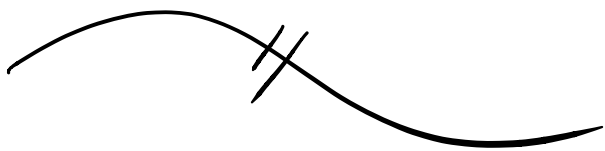
$$F = F_1 F_2 \cdots F_k.$$

Since F is homogeneous, each factor F_i is homogeneous. And since P is

$$P = \sum_{i < j} (x_i p_j - x_j p_i) [F[\vec{x}]]$$

for some point $\vec{p} = (p_1, p_2, \dots, p_{n+1}) \in \mathbb{F}P^n$.

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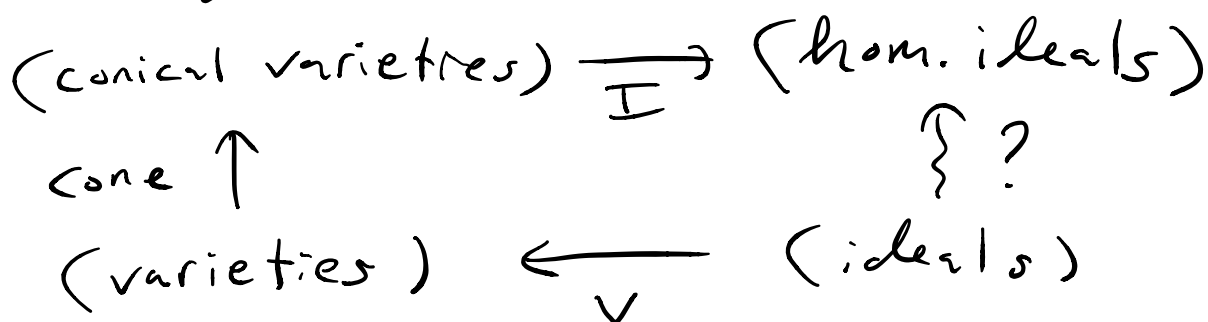


Anything in between is more complicated.

Next: Relationship between affine & projective varieties.



To begin: Coning over an affine variety turns any ideal into a homogeneous ideal:



Define $\text{Cone}(I) := I(\text{Cone}(V(I)))$.

Claim: $\text{Cone}(I) \subseteq I$ is generated by the homogeneous elements of I .

Let $J =$ ideal generated by all homogeneous elements of I .

If $f \in J \subseteq I$, then

$$f = F_1 f_1 + \dots + F_m f_m$$

for some homogeneous $F_1, \dots, F_m \in I$.

Want to show: If $\vec{p} \in V(I)$ then

$$f(\lambda \vec{p}) = 0 \quad \forall \lambda \in \mathbb{F}. \quad \text{Well if}$$

$\vec{p} \in V(I)$ then $F_i(\vec{p}) = 0$ for all i .

$$\text{Hence } F_i(\lambda \vec{p}) = \lambda^{\deg F_i} F_i(\vec{p}) = 0.$$

$$\begin{aligned} \text{Hence } f(\lambda \vec{p}) &= \sum F_i(\lambda \vec{p}) f_i(\lambda \vec{p}) \\ &= 0. \end{aligned} \quad \checkmark$$

We showed $J \subseteq \text{Cone}(I)$.

Conversely, let $f \in \text{Cone}(I)$.

Since $\text{Cone}(I)$ is homogeneous,
have $f^{(k)} \in \text{Cone}(I) \forall k$.

And since $\text{Cone}(I) \subseteq I$ we have

$$f = \sum f^{(k)} \in J. \quad \checkmark$$