Projective Zariski Topology:

$$I : (\text{subsets } \mathbb{F}\mathbb{P}^n) \rightleftarrows (\text{hom. ideals } \mathbb{F}[x_1, \dots, x_{n+1}]) : V$$

non-unit

Then VI is called Zariski closure.

Zariski-closed sets are finite intersections of projective hypersurfaces.

Main difference with affine Zariski topology:

$$I(\emptyset) \neq \mathbb{F}[\vec{x}]$$

Since $\emptyset \subseteq \mathbb{F}\mathbb{P}^n$ corresponds to the conical set $\{\vec{0}\} \subseteq \mathbb{F}^{n+1}$ we have

$$I(\emptyset) = I(\vec{0}) = M_{\vec{0}}$$
$$= x_1 \mathbb{F}[\vec{x}] + \cdots + x_{n+1} \mathbb{F}[\vec{x}].$$

Called the "irrelevant ideal."

I claim that $M_{\vec{0}}$ is the unique maximal homogeneous ideal.

Proof: Let $M \subsetneq \mathbb{F}[\vec{x}]$ maximal among homogeneous ideals. Since $I(V(M)) \supseteq M$ is also homogeneous, we must have $I(V(M)) = M$.

Now I claim that $V(M)$ is minimal among projective varieties. To see this, let $V \subseteq V(M)$ be any projective variety. By <u>definition</u>, we have

$$V = V(I(V)).$$

Applying I to containment $V \subseteq V(M)$ gives

$$M = I(V(M)) \subseteq I(V),$$

which again implies $M = I(V)$ because $I(V)$ is homogeneous.

Then applying V gives

$$V(M) = V(I(V)) = V.$$

Hence $V(M)$ is minimal. But $\emptyset$ is the unique minimal variety, hence

$V(M) = \emptyset$. Finally, we conclude

$$M = I(V(M)) = I(\emptyset) = M_{\vec{0}}.$$

QED.

When $\mathbb{F}$ is algebraically closed, we get order-reversing bijections:

$$(\text{proj. varieties}) \overset{\sim}{\longleftrightarrow} (\text{rad. hom. ideals})$$
$$(\text{irr. proj. varieties}) \overset{\sim}{\longleftrightarrow} (\text{prime hom. ideals}).$$

It follows that every rad. hom. ideal is contained in irrelevant ideal $M_{\vec{0}}$. Ideals whose radical closure is contained in $M_{\vec{0}}$ are called "relevant," because they correspond to projective sets.

Minimal Prime Homogeneous Ideals: Since $\mathbb{F}[\vec{x}]$ is UFD, we already know

$$\begin{matrix}\text{minimal} \\ \text{non-empty primes}\end{matrix} = \text{principal primes}.$$

Now I claim

$$\begin{array}{c}\text{minimal non-empty} \\ \text{hom. primes}\end{array} = \begin{array}{c}\text{hom. principal} \\ \text{primes}\end{array}$$

i.e. $F\mathbb{F}[\vec{x}]$ with F homogeneous & irreducible, i.e., projective hypersurfaces.

<u>Proof</u>: Let $P = F\mathbb{F}[\vec{x}]$ with F (non-zero, non-unit) hom. & irreducible. So we already know that P is minimal among <u>all</u> prime ideals, hence also among homogeneous prime ideals. ✓

Conversely, let $P \supsetneq 0$ be minimal among homogeneous primes. Choose some non-zero, non-unit $F \in P$. Factor F into primes

$$F = F_1 F_2 \cdots F_k.$$

Since F is homogeneous, each factor F_i is homogeneous. And since P is prime we know that $F_i \in P$ for some i, hence $F_i\,\mathbb{F}[\vec{x}] \subseteq P$.

Finally, since F_i is homogeneous & irreducible, the ideal $F_i\,\mathbb{F}[\vec{x}]$ is homogeneous and prime, hence by minimality of P we have

$$F_i\,\mathbb{F}[\vec{x}] = P$$

as desired. ///

Maximal Relevant Homogeneous Primes:

Recall, we have an order-reversing bijection between irreducible projective varieties & prime homogeneous ideals.

Since the <u>points</u> of $\mathbb{F}\mathbb{P}^n$ are the minimal (irreducible) varieties, the maximal (relevant) prime homogeneous ideals are just the ideals of the points, i.e.,

$$P = \sum_{i<j} (x_i p_j - x_j p_i)\,\mathbb{F}[\vec{x}]$$

for some point $\vec{p} = (p_1 : p_2 : \cdots : p_{n+1}) \in \mathbb{FP}^n$.

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Anything in between is more complicated.

Next: Relationship between affine & projective varieties.

To begin: Coning over an affine variety turns any ideal into a homogeneous ideal:

$$\begin{tikzcd}
(\text{conical varieties}) \arrow[r, "I"'] & (\text{hom. ideals}) \\
(\text{varieties}) \arrow[u, "\text{cone}"] & (\text{ideals}) \arrow[l, "V"] \arrow[u, squiggly, "?"']
\end{tikzcd}$$

Define $\text{Cone}(I) := I(\text{Cone}(V(I)))$.

Claim: $\text{Cone}(I) \subseteq I$ is generated by the homogeneous elements of I.

Let J = ideal generated by all homogeneous elements of I.

If $f \in J \subseteq I$, then

$$f = F_1 f_1 + \cdots + F_m f_m$$

for some homogeneous $F_1, \ldots, F_m \in I$.

Want to show: If $\vec{p} \in V(I)$ then $f(\lambda \vec{p}) = 0 \quad \forall \lambda \in \mathbb{F}$. Well if $\vec{p} \in V(I)$ then $F_i(\vec{p}) = 0$ for all i.

Hence $F_i(\lambda \vec{p}) = \lambda^{\deg F_i} F_i(\vec{p}) = 0$.

Hence $f(\lambda \vec{p}) = \sum F_i(\lambda \vec{p}) f_i(\lambda \vec{p})$

$$= 0.$$ ✓

We showed $J \subseteq \text{Cone}(I)$.

Conversely, let $f \in \text{Cone}(I)$.

Since Cone(I) is homogeneous,
have $f^{(k)} \in \text{Cone}(I)\ \forall k$.
And since $\text{Cone}(I) \subseteq I$ we have

$$f = \sum f^{(k)} \in J.$$