Real Points at Infinity.

$\mathbb{RP}^2 = (\mathbb{R}^3 \setminus \text{origin}) / \text{nonzero scalars}.$

Picture:

plane $(x, y, 1)$ of finite points

line $(x : y : 0)$ of points at infinity.

We need a better way to vizualize.

Note:

$$\mathbb{RP}^2 \longleftrightarrow \text{lines through } \vec{0} \in \mathbb{R}^3.$$

Each line intersects the unit sphere $S^2 \leq \mathbb{R}^2$ at two antipodal points.

Thus

$$\mathbb{RP}^2 \longleftrightarrow S^2 / (\text{antipodal map}).$$

point of $\mathbb{RP}^2 \longleftrightarrow$ antipodal pair of points of S^2

line of $\mathbb{RP}^2 \longleftrightarrow$ great circle of S^2.

Furthermore, this sets up a bijection between points & lines of $\mathbb{RP}^2$:

point of $\mathbb{RP}^2$ $\longleftrightarrow$ antipodal points (north & south poles)

$\longleftrightarrow$ great circle (equator)

$\longleftrightarrow$ line of $\mathbb{RP}^2$.

Algebraically:

Send each point $(a:b:c) \in \mathbb{RP}^2$ to the line $ax+by+cz=0$ in $\mathbb{RP}^2$.

Call this line $[a:b:c] \subseteq \mathbb{RP}^2$.

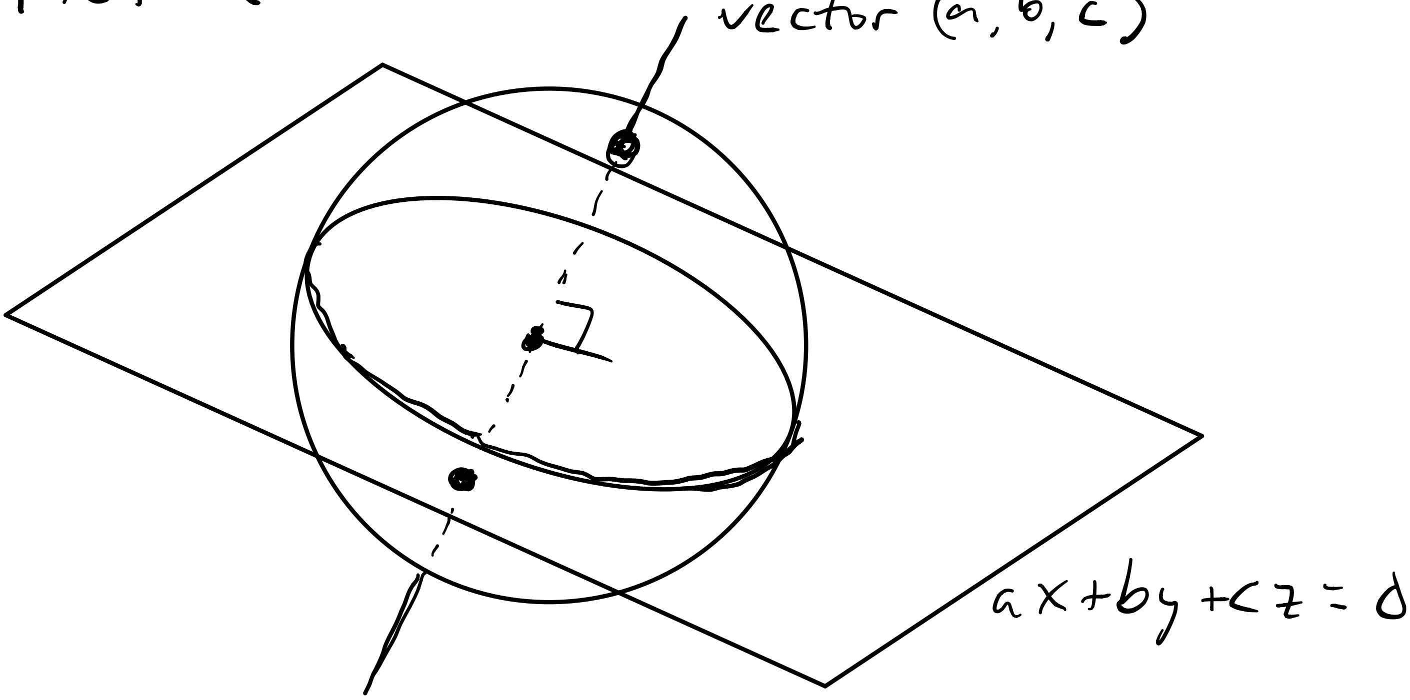
Observe that

$$(a:b:c) \in [a':b':c']$$
$$\Updownarrow$$
$$(a':b':c') \in [a:b:c]$$

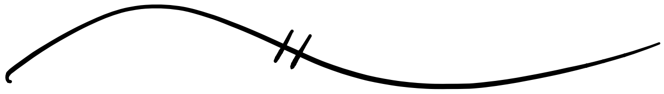
"Point-Line Duality"

In higher dimensions $\mathbb{RP}^n$ we have a duality between points
$(a_1 : a_2 : \cdots : a_n)$ and
hyperplanes $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$.

Picture:



Any great circle could be the line at ∞. Any pair of antipodal points could be the origin.



Gives a better way to visualize projective curves.

Homogeneous $F[x,y,z] \in \mathbb{R}[x,y,z]$

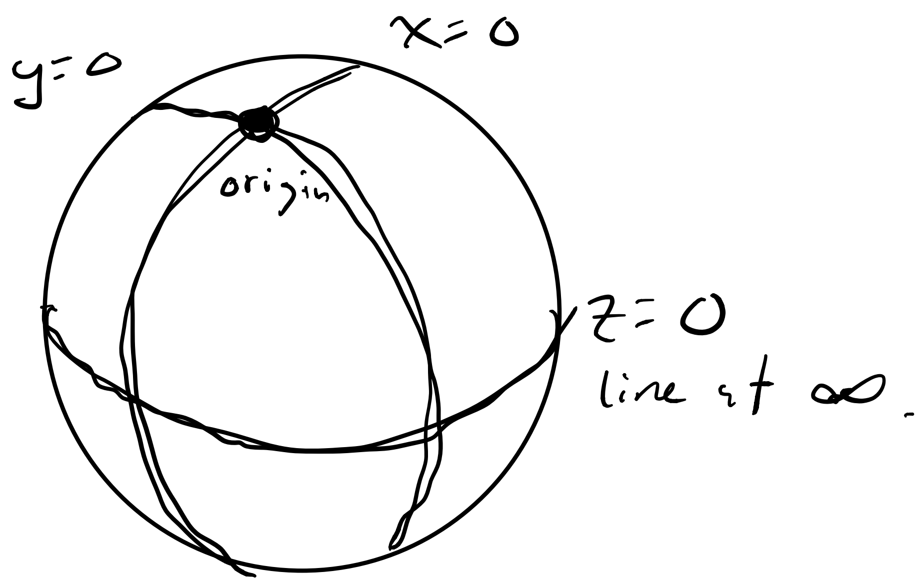
Vizualize curve $C_F \subseteq \mathbb{RP}^2$ as intersection of

surface $F(x,y,z) = 0$ in $\mathbb{R}^3$
(cone)

&

unit sphere $S^2 \subseteq \mathbb{R}$.

From this point of view the lines $x=0$, $y=0$, $z=0$ are mutually perpendicular "axes" on S^2.



Examples:

LOOK AT MAPLE.

What did we see?

Hyperbola: $f(x,y) = x^2 - y^2 - \frac{1}{10}$

$F(x,y,z) = x^2 - y^2 - \frac{z^2}{10}$.

Send $x=0$ to infinity:

$F(1,y,z) = 1 - y^2 - \frac{z^2}{10} = 0$

$y^2 + \frac{z^2}{10} = 1$ ellipse.

Send $y=0$ to infinity:

$F(x,1,z) = x^2 - 1 - \frac{z^2}{10} = 0$

$x^2 - \frac{z^2}{10} = 1$ hyperbola.

Parabola: $f(x,y) = x^2 - 5y$.

$F(x,y,z) = x^2 - 5yz$

Send $y=0$ to ∞:

$$F(x,1,z) = x^2 - 5z = 0$$

parabola.

Send $x=0$ to ∞:

$$F(1,y,z) = 1 - 5yz = 0$$

$$yz = 1/5$$

hyperbola.

It seems that hyperbolas, ellipses, parabolas are all the same if we change the line at infinity.

Definition:
Projective transformation $\mathbb{RP}^2 \to \mathbb{RP}^2$ is the restriction of invertible linear map $\mathbb{R}^3 \to \mathbb{R}^3$. Multiplication by scalars does not change the map.

PGL_3 = projective $\mathbb{RP}^2 \to \mathbb{RP}^2$.

GL_3 = invertible linear $\mathbb{R}^3 \to \mathbb{R}^3$.

Surjective group homomorphism

$$GL_3 \twoheadrightarrow PGL_3$$

with kernel $\{\lambda I : \lambda \neq 0\}$. Hence

$$PGL_3 \approx GL_3 / \text{scalars}.$$

Def: Let $\Phi \in PGL_3$,

$(u, v, w) = \Phi(x, y, z)$,

$F(x, y, z) \in \mathbb{R}[x, y, z]$

homogeneous of degree d.

Then (exercise) $G(x, y, z) := F(u, v, w)$

is homogeneous of degree d,

and we say

$$C_F, C_G \subseteq \mathbb{RP}^2$$

are "projectively equivalent." ///

Given $f(x,y), g(x,y) \in \mathbb{R}[x,y]$ we say $C_f, C_g \subseteq \mathbb{R}^2$ are proj. eqivalent when the (unique) proj. completions $C_F, C_G \subseteq \mathbb{R}\mathbb{P}^2$ are proj. equivalent.

$\left[C_f \rightsquigarrow C_F, \quad F(x,y,z) = z^d f\left(\frac{x}{z}, \frac{y}{z}\right). \right]$

Example: Rotate the sphere model of $\mathbb{R}\mathbb{P}^2$ via $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$.

Hence ellipses, hyperbolas, parabolas are proj. equivalent.

More generally.

Theorem: Given $f(x,y) \in \mathbb{R}[x,y]$ of degree 2, affine curve C_f is proj. eqiv. to one of the following:

1) $x^2 = 0$ (double line)

2) $x^2 \pm y^2 = 0$ (intersecting lines or single point)

3) $x^2 + y^2 \pm 1 = 0$ (circle or $\emptyset$)

We'll prove this next time.