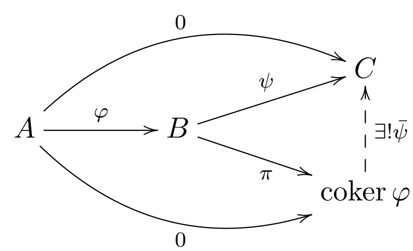
Math 762 **Spring 2016**
Midterm Exam **Drew Armstrong**

Problem 1. Cokernels in Ab**.** Let $\varphi : A \to B$ be a homomorphism of **abelian** groups.

(a) State the universal property of the cokernel $\pi : B \to \operatorname{coker}\varphi$.

We say that $\pi : B \to \operatorname{coker}\varphi$ is the cokernel of $\varphi : A \to B$ if
- $\pi \circ \varphi = 0$, and
- for all homomorphisms $\psi : B \to C$ such that $\psi \circ \varphi = 0$, there exists a unique homomorphism $\bar{\psi} : \operatorname{coker}\varphi \to C$ such that $\bar{\psi} \circ \pi = \psi$.

We can summarize these two conditions with the following diagram:



(b) State the universal property of the quotient homomorphism $\pi : B \to B/\operatorname{im}\varphi$.

We say that $\pi : B \to B/\operatorname{im}\varphi$ is a quotient homomorphism if
- $\operatorname{im}\varphi \subseteq \ker\pi$, and
- for all homomorphisms $\psi : B \to C$ such that $\operatorname{im}\varphi \subseteq \ker\psi$, there exists a unique homomorphism $\bar{\psi} : B/\operatorname{im}\varphi \to C$ such that $\bar{\psi} \circ \pi = \psi$.

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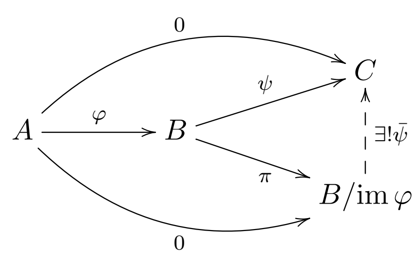


(c) Use your answers from (a) and (b) to show that $\pi : B \to B/\operatorname{im}\varphi$ is the cokernel of φ.

Observe that the two diagrams are the same.

[Remark: This argument also works in the category Grp as long as $\operatorname{im}\varphi \subseteq B$ is a normal subgroup. More generally, the cokernel of $\varphi : A \to B$ is the quotient homomorphism $\pi : B \to B/\langle \operatorname{im}\varphi \rangle^B$, where $\langle \operatorname{im}\varphi \rangle^B$ is the conjugate closure of $\operatorname{im}\varphi$ in B.]

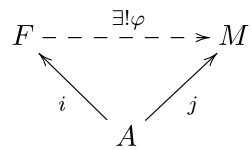
Problem 2. Independent and Spanning Sets. Let R be a ring. By an "R-module" we will mean a **left** R-module.

(a) State the universal property of the free R-module generated by the set A.

We say that F is the free R-module generated by A if
- we have a set function $i : A \to F$, and

- for all R-modules M and set functions $j : A \to M$ there exists a unique R-module homomorphism $\varphi : F \to M$ such that $\varphi \circ i = j$.

We can summarize these two conditions with the following diagram:



(b) Let M be an R-module and let $A \to M$ be an indexed subset of M. State what it means for this subset to be: (1) R-linearly independent, (2) R-spanning, and (3) an R-basis. Use the universal property from part (a) in your answer.

Consider an indexed subset $j : A \to M$ and let $i : A \to F$ be the free module generated by A. Then by part (a) there exists a canonical R-module homomorphism $\varphi : F \to M$ such that $\varphi \circ i = j$. We say that

(1) $A \to M$ is R-linearly independent if φ is injective,

(2) $A \to M$ is R-spanning if φ is surjective,

(3) $A \to M$ is an R-basis if φ is bijective.

(c) Prove that an R-module has a basis if and only if it free.

Proof. If $i : A \to F$ is the free module generated by A then the canonical homomorphism $\varphi : F \to F$ is the identity map. Since the identity map is bijective we conclude that $A \to F$ is a basis. Conversely, let M be an R-module and let $j : A \to M$ be a basis. If $i : A \to F$ is the free module generated by A this means that the canonical morphism $\varphi : F \to M$ satisfying $\varphi \circ i = j$ is a bijection, hence it is an isomorphism of R-modules. □

[Remark: In the next problem I will use the fact that the free R-module F generated by $i : A \to F$ can be identified with the collection of formal sums $\sum_{a \in A} r_a i_a$ in which $r_a = 0$ for all but finitely many $a \in A$.]

Problem 3. Vector Spaces are Free. Let K be a field and let V be a K-module.

(a) Prove that every miminal K-spanning subset of V is a basis.

Proof. Let $j : A \to V$ be a minimal spanning set and assume for contradiction that there exists a linear relation

$$\sum_{a \in A} r_a j_a = 0$$

in which $r_{a'} \neq 0$ for some $a' \in A$. Then since K is a field we can divide by $r_{a'}$ to get

$$j_{a'} = \sum_{a \neq a'} \left(-\frac{r_a}{r_{a'}} \right) j_a.$$

Finally, given any element $u = \sum_{a \in A} s_a j_a \in V$ we can write

$$u = \sum_{a \neq a'} \left(s_a - \frac{s_{a'} r_a}{r_{a'}} \right) j_a,$$

which contradicts the minimality of $j : A \to V$. We conclude that $j : A \to V$ is K-linearly independent, hence it is a basis. □

(b) Prove that every maximal K-independent subset of V is a basis. [It then follows from Zorn's Lemma that every vector space has a basis, but don't prove this.]

Proof. Let $j : A \to V$ be a maximal K-independent set. To show that the set is K-spanning, consider any $u \in V \setminus \operatorname{im} j$. By maximality of $j : A \to V$ there exists a K-linear relation

$$ru + \sum_{a \in A} r_a j_a = 0$$

in which not all coefficients are zero. If $r = 0$ then we obtain a nontrivial relation $\sum_{a \in A} r_a j_a = 0$, which contradicts the independence of A. Hence $r \neq 0$. Then since K is a field we can divide by r to obtain

$$u = \sum_{a \in A} \left(-\frac{r_a}{r} \right) j_a.$$

We conclude that $j : A \to V$ is a K-spanning set, hence it is a basis. □

(c) If the vector space V is **finitely generated**, prove (without using Zorn's Lemma) that V has a basis. [Hint: Let $u_1, u_2, \ldots, u_n \in V$ be a K-spanning set. If this set is **not** K-linearly independent then ...]

Proof. Let $u_1, u_2, \ldots, u_n \in V$ be a K-spanning set. If this set is not K-linearly independent then we know from part (a) that the K-spanning set is **not minimal**. That is, there exists an element u_i such that $\{u_1, \ldots, u_n\} \setminus \{u_i\}$ is still a K-spanning set. If this smaller set is still not K-linearly independent we can repeat the argument until a K-linearly independent K-spanning set is reached. □

Problem 4. Cyclic Modules. Let R be a ring. We say that a (left) R-module M is cyclic if it has an R-spanning set of size one.

(a) If $I \subseteq R$ is a (left) ideal, prove that the abelian quotient group R/I is a (left) R-module (i.e., construct a well-defined (left) linear R-action).

Proof. For all $r \in R$ and $s + I \in R/I$ we will define $r(s + I) := rs + I$. To show that this operation is well-defined, suppose that $s_1 + I = s_2 + I$, i.e., $s_1 - s_2 \in I$. Then since I is a left ideal we have

$$rs_1 - rs_2 = r(s_1 - s_2) \in I,$$

and it follows that $r(s_1 + I) = rs_1 + I = rs_2 + I = r(s_2 + I)$ as desired. Then to show that the (well-defined) operation $\lambda_r(s + I) = rs + I$ defines a ring homomorphism $\lambda : R \to \mathrm{End}_{\mathsf{Ab}}(R/I)$, note that for all $r_1, r_2 \in R$ and $s + I \in R/I$ we have

$$\begin{aligned}(r_1 + r_2)(s + I) &= (r_1 + r_2)s + I \\ &= (r_1 s + r_2 s) + I \\ &= (rs_1 + I) + (rs_2 + I) \\ &= r_1(s + I) + r_2(s + I)\end{aligned}$$

and

$$\begin{aligned}(r_1 r_2)(s + I) &= (r_1 r_2)s + I \\ &= r_1(r_2 s) + I \\ &= r_1(r_2 s + I)\end{aligned}$$

$$= r_1(r_2(s+I));$$

note that for all $r \in R$ and $s_1 + I, s_2 + I \in R/I$ we have

$$\begin{aligned} r((s_1+I)+(s_2+I)) &= r((s_1+s_2)+I) \\ &= r(s_1+s_2)+I \\ &= (rs_1+rs_2)+I \\ &= (rs_1+I)+(rs_2+I) \\ &= r(s_1+I)+r(s_2+I); \end{aligned}$$

and, finally, note that for all $s + I \in R/I$ we have $1(s+I) = 1s + I = s + I$. □

(b) Let $I \subseteq R$ be a (left) ideal of R. Prove that the (left) R-module R/I is cyclic.

Proof. I claim that $\{1 + I\} \subseteq R/I$ is an R-spanning set of size one. Indeed, for all $r + I \in R/I$ we have $r + I = r(1 + I)$. □

(c) If M is a cyclic (left) R-module, prove we have $M \approx R/I$ for some (left) ideal $I \subseteq R$. [Hint: Use the definition of spanning from Problem 2(b).]

Proof. If $\{m\} \subseteq M$ is an R-spanning set then we obtain a canonical surjective R-module homomorphism $\varphi : R \twoheadrightarrow M$ defined by $m \mapsto rm$. [We can think of R as the free R-module generated by one element.] Then by the 1st Isomorphism Theorem for modules we obtain an isomorphism $M = \operatorname{im}\varphi \approx R/\ker\varphi$, where $\ker\varphi$ is a left R-submodule of R (i.e., a left ideal). □