Math 762 **Spring 2016**
Homework 4 **Drew Armstrong**

Problem 1. Cokernel of a Direct Sum. Here's something that confused me in class, so I'll have you prove it. (Just like you have to wear a coat when your mother is cold.) Let R be a ring and suppose we have a homomorphism $\varphi : M_1 \oplus M_2 \to N_1 \oplus N_2$ of R-modules. Since $\oplus$ is both the product **and** the coproduct in R-Mod, there exist canonical injections and projections as in the following diagram:

$$
\begin{tikzcd}
M_1 \arrow[rrr, "\varphi_{11}"] \arrow[dr, "\iota_1"] & & & N_1 \\
& M_1 \oplus M_2 \arrow[r, "\varphi"] & N_1 \oplus N_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] & \\
M_2 \arrow[rrr, "\varphi_{22}"] \arrow[ur, "\iota_2"] & & & N_2
\end{tikzcd}
$$

Define the "component homomorphisms" $\varphi_{ij} := \pi_i \circ \varphi \circ \iota_j : M_j \to N_i$ for $i, j \in \{1, 2\}$ and assume that φ_{12} and φ_{21} are both the zero map. In this case prove that we have an isomorphism

$$\frac{N_1 \oplus N_2}{\operatorname{im} \varphi} \approx \frac{N_1}{\operatorname{im} \varphi_{11}} \oplus \frac{N_2}{\operatorname{im} \varphi_{22}}.$$

Problem 2. Chinese Remainder Theorem for PIDs. Let R be a PID and consider any element $a \in R$ with unique prime factorization $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_d^{\alpha_d}$.

(a) Prove that there exist elements $r_1, \ldots, r_d \in R$ such that the following identity holds in the field of fractions:
$$\frac{1}{a} = \frac{r_1}{p_1^{\alpha_1}} + \cdots + \frac{r_d}{p_d^{\alpha_d}}.$$
[Hint: Use induction on d.]

(b) Use part (a) to construct an isomorphism of R-modules
$$\frac{R}{(a)} \approx \frac{R}{(p_1^{\alpha_1})} \oplus \cdots \oplus \frac{R}{(p_d^{\alpha_d})}.$$

Problem 3. Idempotents = Internal Direct Sums. Let R be any ring and let M be a (left) R-module. Let $E = \operatorname{End}_R(M)$ be the (noncommutative) ring of endomorphisms. We say that an endomorphism $e \in E$ is idempotent if $e^2 = e \circ e = e$.

(a) Given $e \in E$, prove that e is idempotent if and only if $\mathrm{id} - e$ is idempotent. We call this an orthogonal pair of idempotents because $e \circ (\mathrm{id} - e) = (\mathrm{id} - e) \circ e = 0$.

(b) If $e \in E$ is idempotent, prove that M decomposes as a direct sum of R-submodules
$$M = \operatorname{im} e \oplus \operatorname{im} (\mathrm{id} - e).$$

(c) Conversely, if $M = M_1 \oplus M_2$ is a direct sum of (left) R-submodules prove that there exist idempotents $e_1, e_2 \in E$ such that $\operatorname{im} e_1 = M_1$, $\operatorname{im} e_2 = M_2$, $e_1 + e_2 = \mathrm{id}$ and $e_1 \circ e_2 = e_2 \circ e_1 = 0$. [Hint: Think of $\oplus$ as the product in R-Mod.]

Problem 4. Generalized Eigenspaces of a Matrix. Let K be a field and consider a matrix $A \in \operatorname{Mat}_n(K)$. Let $\varphi_A : K[x] \to \operatorname{Mat}_n(K)$ be the canonical homomorphism from the free algebra $K[x]$ $(= K\langle x \rangle)$. Since $K[x]$ is a PID we know that the kernel has the form $\ker \varphi_A = (m_A(x))$ for some unique monic polynomial $m_A(x)$ called the minimal polynomial of

A. Let $m_A(x) = f_1(x)^{m_1} \cdots f_d(x)^{m_d}$ be the unique factorization into irreducible polynomials (note that $m_A(x)$ is not necessarily irreducible because $\mathrm{Mat}_n(K)$ is not an integral domain). Then by Problem 2 there exist polynomials $g_i(x) \in K[x]$ such that

$$\frac{1}{m_A(x)} = \sum_i \frac{g_i(x)}{f_i(x)^{m_i}}.$$

For each i we define the polynomial $p_i(x) := m_A(x)g_i(x)/f_i(x)^{m_i} = g_i(x)\prod_{i\neq j} f_j(x)^{m_j}$ and the matrix $P_i := p_i(A) = \varphi_A(p_i(x)) \in \mathrm{Mat}_n(K)$.

(a) Prove that we have $\sum_i P_i = I$.
(b) Prove that for $i \neq j$ we have $P_iP_j = 0$.
(c) Prove that for all i we have $P_i^2 = P_i$. [Hint: Use (a) and (b).]
(d) Conclude from Problem 3 that we have a direct sum decomposition of K-subspaces

$$K^n = \bigoplus_{i=1}^{d} \operatorname{im} P_i.$$

Problem 5. Jordan-Chevalley Decomposition. Now let K be an **algebraically closed** field and consider a matrix $A \in \mathrm{Mat}_n(K)$. In this problem we will prove that there exist unique matrices $S, N \in \mathrm{Mat}_n(K)$ such that:

- S is diagonalizable and N is nilpotent,
- $A = S + N$,
- $SN = NS$.

(a) Since K is algebraically closed we can factor the minimal polynomial as $m_A(x) = \prod_i(x-\lambda_i)^{m_i}$ for some $\lambda_i \in K$ and $m_i \in \mathbb{N}$. Let P_i be the projections from Problem 4 corresponding to the factors $f_i(x)^{m_i} = (x-\lambda_i)^{m_i}$. Prove that $(A-\lambda_i I)^{m_i}P_i = 0$.
(b) Existence: Prove that the matrix $S := \sum_i \lambda_i P_i$ is diagonalizable and that the matrix $N := A - S$ is nilpotent. Then since $S = \sum_i \lambda_i p_i(A)$ for some polynomials $p_i(x)$ it automatically follows that $SN = NS$. [Hint: Define the polynomial $\phi(x) := \prod_i(x-\lambda_i)$ and show that $\phi(S) = 0$. Use this to define projection matrices $Q_1, \ldots, Q_d$ as in Problem 4 and obtain a direct sum decomposition $K^n = \oplus_i \operatorname{im} Q_i$. Then show that the image of Q_i consists of λ_i-eigenvectors for S. To show that N is nilpotent, note that for all polynomials $h(x) \in K[x]$ we have $h(N) = \sum_i h(A-\lambda_i I)P_i$. Then use part (a).]
(c) Uniqueness: I need to come up with a good hint for this.