Thurs Apr 4

Back to Reflection Groups.

Goal: <u>Classify</u> FGGR in all dimensions.
So far we have them up to rank 3

Recall: we have a bijection

$$\text{FGGR} \longleftrightarrow \text{FCSM}.$$

Now we introduce a new concept that is "dual" to closed mirror systems in exactly the same way that f.g. cones are "dual" to polyhedral cones.

Given a FCSM Σ, let

$$\Phi(\Sigma) := \left\{ \pm \text{ unit normal vectors to the hyperplanes} \right\}.$$

Recall, given any $\alpha \in \mathbb{R}^n$, the reflection in the hyperplane $\alpha^\perp$ has formula

$$\boxed{t_\alpha(\beta) = \beta - 2\left(\frac{\beta \cdot \alpha}{\alpha \cdot \alpha}\right)\alpha}$$

for all $\beta \in \mathbb{R}^n$

Then the set $\Phi(\Sigma)$ is "closed" in the following way:

$\forall \alpha, \beta \in \Phi(\Sigma)$ we have $t_\alpha(\beta) \in \Phi(\Sigma)$

Proof: Since t_α is orthogonal, note that $t_\alpha(\beta)$ is a unit vector $\perp$ to the hyperplane $t_\alpha(\beta^\perp)$.

But since $\alpha^\perp, \beta^\perp \in \Sigma$ and since t_α is the same thing as $t_{\alpha^\perp}$, the "closed" property of Σ implies that

$$t_\alpha(\beta^\perp) = t_{\alpha^\perp}(\beta^\perp) \in \Sigma$$

So $t_\alpha(\beta)$ is a unit normal vector for $t_\alpha(\beta) \in \Sigma$, i.e., $t_\alpha(\beta) \in \Phi(\Sigma)$

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Conversely, we define the concept of a "root system" (please excuse the strange name).

Definition: A finite root system is a finite set Φ of vectors satisfying

(RS1) $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ for all $\alpha \in \Phi$.

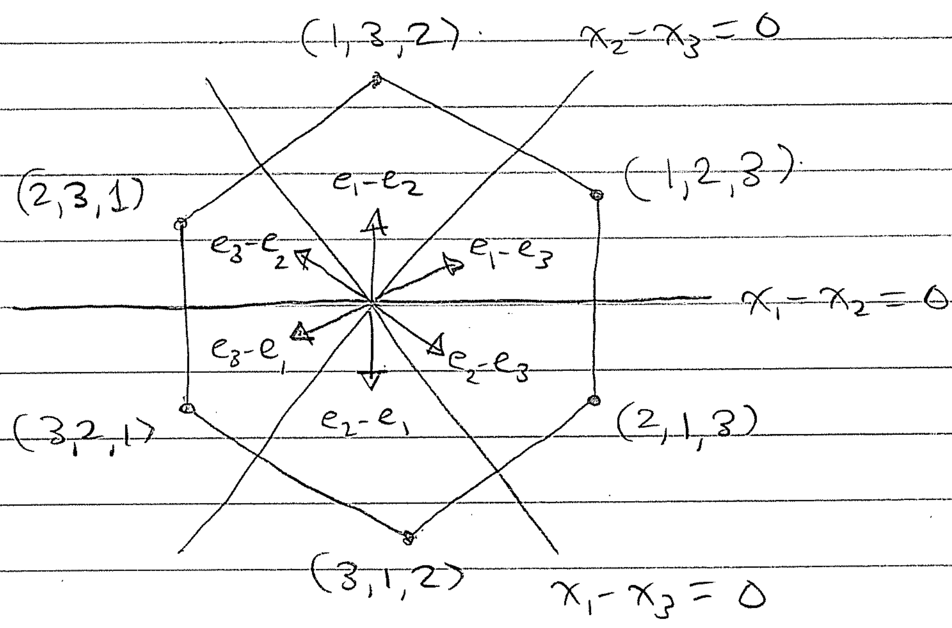
(RS2) $t_\alpha(\beta) \in \Phi$ for all $\alpha, \beta \in \Phi$

Thus we have bijections

$$\text{FGGR} \longleftrightarrow \text{FCSM} \longleftrightarrow \text{FRS}.$$

and we will sometimes blur the distinctions

Example: type $A_2 = G_2(3) = \mathfrak{S}_3 = D_6$



In general, the root system of type A_{n-1} is

$$\Phi = \{ e_i - e_j \ : \ 1 \leq i, j \leq n \}$$

It lives in the $(n-1)$-dimensional space

$$\mathbb{R}^n_0 = (e_1 + e_2 + \cdots + e_n)^\perp$$

The corresponding mirror system is

$$\Sigma = \{ x_i - x_j = 0 \ : \ 1 \leq i, j \leq n \}$$

and the corresponding group is the symmetric group $\mathfrak{S}_n$ of permutations

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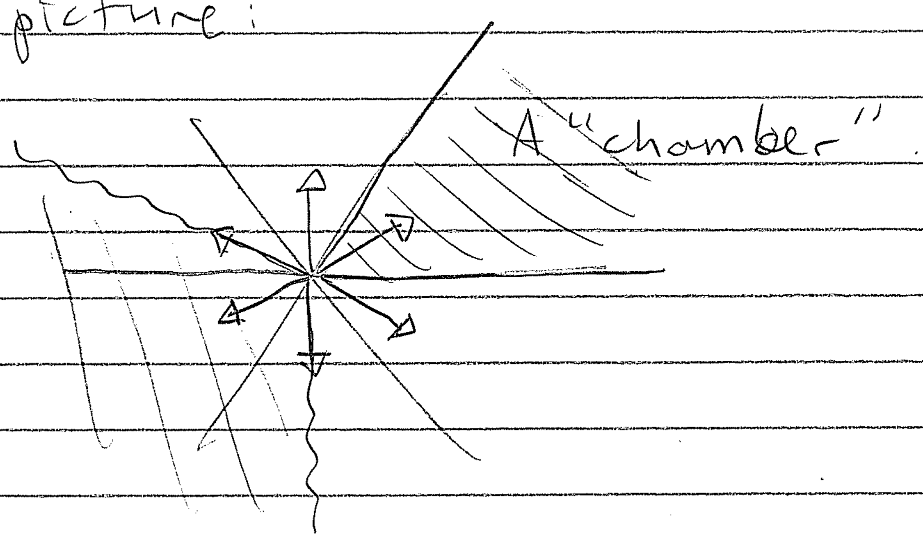
The relationship between root systems and mirror systems is suggested by this picture:



Idea: Each "chamber" of the arrangement is dual to a cone generated by some special set of roots.

Let's prove this.

Definition: Let Φ be a root system, and consider any $x \in \mathbb{R}^n$ such that $x \cdot \alpha \neq 0$ for all $\alpha \in \Phi$. (This is possible since Φ is finite.)

Define the positive and negative roots relative to x:

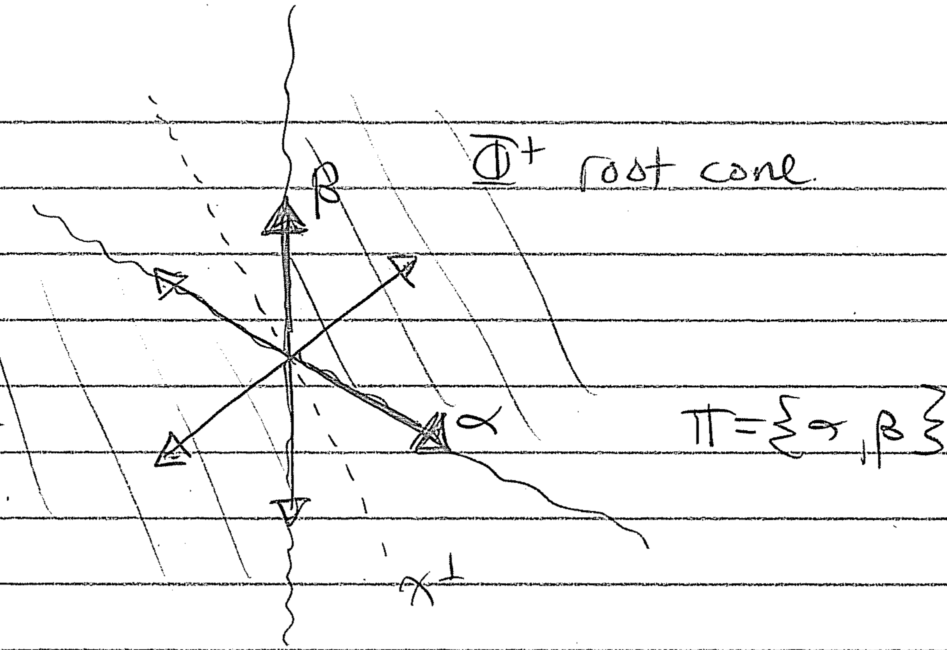
$$\Phi^+ = \{\alpha \in \Phi : x \cdot \alpha > 0\}$$
$$\Phi^- = \{\alpha \in \Phi : x \cdot \alpha < 0\}$$

Then we have $\Phi^- = -(\Phi^+)$ and

$$\Phi = \Phi^+ \sqcup \Phi^-$$

The root cone is the cone $\mathbb{R}^+ \Phi^+$ generated by Φ^+

Picture:



We saw before that every f.g. cone has a unique simple basis.

Let Π be the simple basis of $\mathbb{R}^+\Phi^+$ and call these the simple roots (again, relative to x).

Assume that $V = \mathbb{R}\Phi$. Then

Theorem: The simple roots are a linear basis of V. Hence the root cone is simplicial. We say

$$|\Pi| = \dim(V)$$

is the "rank" of the root system.

To prove this we need a Lemma:

$\forall \alpha, \beta \in \Pi$ we have $\alpha \cdot \beta \leq 0$

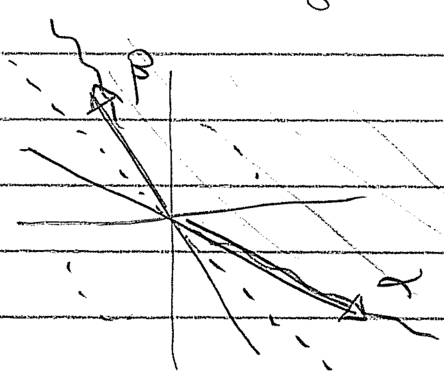
Proof: Consider the plane $P = \mathbb{R}\langle \alpha, \beta \rangle \subseteq \mathbb{R}^n$ and the intersection

$$\Phi_0 = \Phi \cap P$$

Evidently Φ_0 is itself a root system in P. Projecting α to P we get

$$\Phi_0^+ = \Phi^+ \cap P.$$

We see that Φ_0 has rank 2 with simple roots $\Pi_0 = \{\alpha, \beta\}$ Thus Φ_0 consists of $2m$ equiangular vectors



And it is now evident that $\alpha \cdot \beta \leq 0$ (since $m \geq 2$)

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[Remark: This method of proof is very important. Many properties of root systems are "local" in the sense that they only depend on rank 2 subsystems.]

Proof of Theorem:

Clearly Π is a spanning set for $V = \mathbb{R}\Phi$. We must show that Π is independent.

Assume for contradiction that we have a nontrivial relation

$$\sum_{\alpha \in \Pi} a_\alpha \alpha = 0$$

Separate the coefficients into + and − to write

$$\sum_{\beta} b_\beta \beta = \sum_{\gamma} c_\gamma \gamma =: \sigma$$

with $b_\beta, c_\gamma > 0$. Since all β are in the root cone we have $\sigma \neq 0$.

On the other hand we have

$$
\begin{aligned}
0 &\leq \delta \cdot \delta \\
&= \left(\sum_\beta b_\beta \beta\right) \cdot \left(\sum_\gamma c_\gamma \gamma\right) \\
&= \sum_\beta \sum_\gamma b_\beta c_\gamma (\beta \cdot \gamma) \leq 0.
\end{aligned}
$$

since $\beta \cdot \gamma \leq 0 \;\forall\; \beta, \gamma \in \Pi$.
Hence $\delta \cdot \delta = \|\delta\|^2 = 0$. Contradiction

Corollary: The number $|\Pi|$ is independent of the vector α used to define "positive"; We call this number the <u>rank</u> of Φ.

Question:

How does the <u>structure</u> of

$$\Pi \subseteq \Phi^+ \subseteq \Phi$$

depend on the choice of α?

Answer: We will show that the group

$$G(\Phi) := \langle t_\alpha : \alpha \in \Phi \rangle$$
$$= \langle t_\alpha : \alpha \in \Phi^+ \rangle$$

acts regularly on simple systems.
Thus all simple systems are congruent
and we have a bijection

choices of simple systems $\longleftrightarrow$ elements of $G(\Phi)$.

Furthermore, the "simple reflections"

$$S = \{ t_\alpha : \alpha \in \Pi \}$$

generate the group in a beautiful way.

Example: Type A_{n-1}

Choose a permutation $\rho \in \mathfrak{S}_n$.

This defines a notion of "positive" roots

$$\Phi^+ = \{ e_{p(i)} - e_{p(j)} : 1 \le i < j \le n \}$$

and a simple system

$$\Pi = \{ e_{p(i)} - e_{p(i+1)} : 1 \le i \le n-1 \}.$$

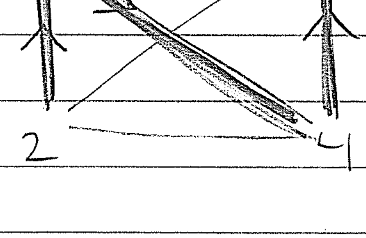
The "simple reflections" in the hyperplanes $x_{p(i)} - x_{p(i+1)} = 0$ correspond to the generating set of

"adjacent transpositions" $(p(i), p(i+1))$.

[Remark: The minimal generating sets of transpositions are the spanning trees of the complete graph K_n.

The simple generating sets are the (oriented) chains

E.g.



$p = 2143$]