Thurs Mar 28

Today: Gram Matrices

Recall: Given a real matrix $B^t = B$ (i.e. a real quadratic form), TFAE

- B is positive semidefinite
- $x^t B x \geq 0$ for all $x \neq 0$.
- all eigenvalues of B are ≥ 0.
- $B = A^t A$ for some real A.

here.

Furthermore, B is positive definite when the columns of A are independent. In this case, B is just the Gram matrix of the dot product in some basis.

Say $A = (\alpha_1 \cdots \alpha_m)$. Then

$$B = \begin{pmatrix} \alpha_1^t \alpha_1 & \alpha_1^t \alpha_2 & \\ \alpha_2^t \alpha_1 & \cdots & \\ & & \alpha_m^t \alpha_m \end{pmatrix}$$

Write $x = \sum x_i \alpha_i$
and $y = \sum y_i \alpha_i$ in α-coordinates

Then the dot product is.

$$x^t y = \left(\sum x_i \alpha_i^t\right)\left(\sum y_j \alpha_j\right)$$

$$= \sum_{i,j} x_i y_j \, \alpha_i^t \alpha_j$$

$$= (x_1 \cdots x_m) \begin{pmatrix} \alpha_1^t\alpha_1 & \cdots & \alpha_1^t\alpha_m \\ \vdots & \ddots & \vdots \\ \alpha_m^t\alpha_1 & \cdots & \alpha_m^t\alpha_m \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$= [x]_\alpha^t A^t A [y]_\alpha$$

For any real matrix A we have.

$$\det(A^t A) \geq 0.$$

What does this mean?

Small Cases: $A = \alpha$: column vector

$$\Longrightarrow \det(A^t A) = \alpha^t \alpha = \|\alpha\|^2 \geq 0$$

NOT surprising.

$A = (\alpha_1 \ \alpha_2)$.

$$\Rightarrow \det(A^t A) = \det \begin{pmatrix} \|\alpha_1\|^2 & \alpha_1 \circ \alpha_2 \\ \alpha_2 \circ \alpha_1 & \|\alpha_2\|^2 \end{pmatrix}$$

$$= \|\alpha_1\|^2 \|\alpha_2\|^2 - (\alpha_1 \circ \alpha_2)^2$$

$$\geq 0.$$

$$\Rightarrow \|\alpha_1\|^2 \|\alpha_2\|^2 \geq (\alpha_1 \circ \alpha_2)^2$$

$$\Rightarrow \|\alpha_1\| \cdot \|\alpha_2\| \geq |\alpha_1 \circ \alpha_2|.$$

Cauchy-Schwarz Inequality

How about $A = (\alpha_1 \ \alpha_2 \ \alpha_3)$?

$$\det(A^t A) = \det \begin{pmatrix} \|\alpha_1\|^2 & \alpha_1 \circ \alpha_2 & \alpha_1 \circ \alpha_3 \\ \alpha_2 \circ \alpha_1 & \|\alpha_2\|^2 & \alpha_2 \circ \alpha_3 \\ \alpha_3 \circ \alpha_1 & \alpha_3 \circ \alpha_2 & \|\alpha_3\|^2 \end{pmatrix}$$

$$= \|\alpha_1\|^2 \|\alpha_2\|^2 \|\alpha_3\|^2 + 2(\alpha_1 \circ \alpha_2)(\alpha_2 \circ \alpha_3)(\alpha_1 \circ \alpha_3)$$

$$- \|\alpha_1\|^2 (\alpha_2 \circ \alpha_3)^2 - \|\alpha_2\|^2 (\alpha_1 \circ \alpha_3)^2 - \|\alpha_3\|^2 (\alpha_1 \circ \alpha_2)$$

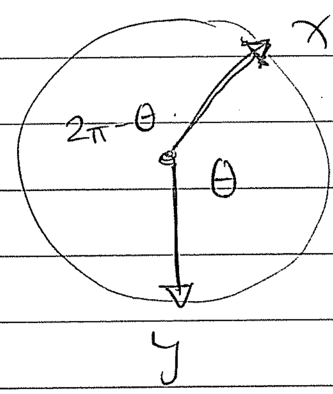
≥ 0 What does this mean?

Given two points x, y on a unit sphere of any dimension, define their *intrinsic distance* in the sphere

$$d(x,y) = \arccos(x \cdot y).$$

By definition of arccos we have $0 \leq d(x,y) \leq \pi$.

Picture



In this case, $d(x,y) = \theta$.

But we need to check that $d(x,y)$ is really a "distance", i.e., we need to prove the triangle inequality

$$d(x,y) \leq d(x,z) + d(z,y).$$

for all x, y, z in the unit sphere.

Let's prove it!

Proof: $d(x,y) \leq d(x,z) + d(z,y)$

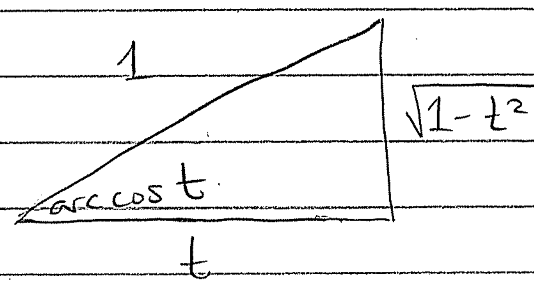
$\iff \arccos(x \circ y) \leq \arccos(x \circ z) + \arccos(z \circ y)$.

$\iff x \circ y \geq \cos(\arccos(x \circ z) + \arccos(z \circ y))$.

Because cos decreases on $[0, \pi]$.

$$x \circ y \geq \cos(A + B).$$
$$= \cos A \cos B - \sin A \sin B$$

But $\sin(\arccos t) = \sqrt{1-t^2}$ because.



Hence

$$x \circ y \geq (x \circ z)(z \circ y) - \sqrt{1-(x \circ z)^2}\sqrt{1-(z \circ y)^2}$$

$$\iff \sqrt{1-(x \circ z)^2}\sqrt{1-(z \circ y)^2} \geq \underbrace{(x \circ z)(z \circ y) - x \circ y}_{\text{If this is } < 0 \text{ there is nothing to prove.}}$$

So we can square both sides

$$(1-(x\circ z)^2)(1-(z\circ y)^2) \geq ((x\circ z)(z\circ y) - x\circ y)^2.$$

$$\iff 1+2(x\circ z)(z\circ y)(x\circ y) - (x\circ z)^2 - (z\circ y)^2 - (x\circ y)^2 \geq 0.$$

$$\iff \det\begin{pmatrix} 1 & x\circ y & x\circ z \\ y\circ x & 1 & y\circ z \\ z\circ x & z\circ y & 1 \end{pmatrix} \geq 0$$

And we know this is true $\blacksquare$

Say this at the beginning.

Next, observe that for any real A we have.

$$\ker(A^t A) = \ker(A).$$

Proof: If $Ax = 0$ Then $A^t A x = A^t 0 = 0$.

Conversely, suppose $A^t A x = 0$. Then

$$\|Ax\|^2 = (Ax)^t (Ax) = x^t (A^t A x) = x^t 0 = 0$$

$$\implies Ax = 0$$

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