Tue Nov 13

Another example of a "torsor":

Let $(V, \mathbb{F})$ be a vector space and let AV be the <u>set</u> V (of "points") with structure forgotten.

Then $(V, +, 0) \curvearrowright AV$ regularly by "translation", hence AV is a V-torsor, called "affine space".

Think:

$$AV = \text{manifold} = \text{"points"}$$
$$V = \text{tangent space} = \text{"vectors"}$$

Now consider quad. form $Q: V \to \mathbb{F}$ with $Q(x) = 0 \Rightarrow x = 0$, so we can think of $Q(x) = $ "$\|x\|^2$" as a length/norm. Let

$\mathrm{Isom}(AV, Q)$

$:= \{$ functions $\varphi: V \to V$ such that $Q(\varphi(x) - \varphi(y)) = Q(x-y) \ \forall x, y \in V \}$

"$\mathrm{dist}_Q(\varphi(x), \varphi(y)) = \mathrm{dist}_Q(x,y)$"

Translation subgroup :

For $\alpha \in V$ define $t_\alpha : AV \to AV$

$$x \longmapsto x + \alpha$$

Then $(V, +, 0) \approx \{t_\alpha : \alpha \in V\} \leq \mathrm{Isom}(AV)$

$$\alpha + \beta \longleftrightarrow t_\alpha \circ t_\beta$$

Abuse notation : $V \leq \mathrm{Isom}(AV, Q)$.

Now let $\varphi \in \mathrm{Isom}(AV)$ and define $\alpha := \varphi(0)$. Then we have

$$(t_{-\alpha} \circ \varphi)(0) = t_{-\alpha}(\varphi(0))$$
$$= t_{-\alpha}(\alpha) = \alpha - \alpha = 0$$

Hence $t_{-\alpha} \circ \varphi =: \mu \in \mathrm{Isom}_0(AV)$

Then $\varphi = t_\alpha \circ \mu$

$\Rightarrow \mathrm{Isom}(AV) = V\,\mathrm{Isom}_0(AV)$

and $V \cap \mathrm{Isom}_0(AV) = \{id\}$.

$\Rightarrow \mathrm{Isom}(AV) = \mathrm{Isom}_0(AV) \times V$

structure ?

Now here's a surprise:

$$\mathrm{Isom}_0(AV, Q) \approx O(V, Q).$$

i.e. if φ preserves Q and $\varphi(0) = 0$
then φ is linear $\left(\varphi(x+cy) = \varphi(x) + c\varphi(y)\right)$.

Proof following M. Artin's "Algebra" pg. 156

Again let

$$B(x,y) := \frac{1}{2}\left[Q(x+y) - Q(x) - Q(y)\right]$$

Lemma: Given $x, y \in V$. If
$B(x,x) = B(x,y) = B(y,y)$ then $x = y$.

Proof: $Q(x-y) = B(x-y, x-y)$

$= B(x,x) - 2B(x,y) + B(y,y) = 0$

Then $Q(x-y) = 0 \Longrightarrow x - y = 0$

we assumed
Q is "anisotropic"

∎

Proof of Theorem:

Suppose $\varphi(0) = 0$ and
$Q(\varphi(x) - \varphi(y)) = Q(x-y) \quad \forall\, x, y \in V$.

Then also have $B(\varphi(x), \varphi(y)) = B(x,y) \quad \forall\, x, y \in V$
since
① $Q(\varphi(x) - \varphi(y)) = Q(\varphi(x)) - 2B(\varphi(x), \varphi(y)) + Q(\varphi(y))$
② $Q(x-y) = Q(x) - 2B(x,y) + Q(y)$.

Since char $\mathbb{F} \neq 2$, ① − ② + ($\varphi(0) = 0$) gives
$B(\varphi(x), \varphi(y)) = B(x,y)$ ✓

Now let $x, y \in V$, $c \in \mathbb{F}$. Want to show
(i) $\varphi(x+y) = \varphi(x) + \varphi(y)$ and (ii) $\varphi(cx) = c\varphi(x)$.

Notation: let w' stand for $\varphi(w)$.

(i) Let $z = x+y$. Want $z' = x' + y'$.
By Lemma, enough to show

$$B(z', z') = B(z', x'+y') = B(x'+y', x'+y').$$

i.e. $B(z',z') = B(z',x') + B(z',y')$
$$= B(x',x') + 2B(x',y') + B(y',y')$$

But is because $B(u',w') = B(u,w)$ $\forall u,w$. ✓

ii) Want to show $(cx)' = cx'$.
By Lemma, enough to show

$$B((cx)',(cx)') = B(cx',(cx)') = B(cx',cx').$$

But

- $B((cx)',(cx)') = B(cx,cx) = c^2 B(x,x)$.
- $B(cx',(cx)') = c\,B(x',(cx)')$
$$= c\,B(x,cx) = c^2 B(x,x)$$
- $B(cx',cx') = c^2 B(x',x') = c^2 B(x,x)$.

$\blacksquare$

We showed

$$\mathrm{Isom}_0(AV,Q) = O(V,Q)$$

I think that's rather cool!

Now we know

$$\text{Isom}(AV, Q) = O(V,Q) \times V$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ structure ?

Let $\mu \in O(V,Q)$ and $t_\alpha \in V$. Then $\forall x \in V$,

$$\begin{aligned}(\mu \circ t_\alpha)(x) &= \mu(t_\alpha(x)) \\ &= \mu(x+\alpha) \\ &= \mu(x) + \mu(\alpha) \\ &= t_{\mu(\alpha)}(\mu(x)) \\ &= (t_{\mu(\alpha)} \circ \mu)(x).\end{aligned}$$

μ is linear!

$$\Longrightarrow \boxed{\mu \circ t_\alpha \circ \mu^{-1} = t_{\mu(\alpha)}}$$

Finally, consider any $\varphi \circ t_\beta \in \text{Isom}(AV)$ and any $t_\alpha \in V$.

Conjugating gives

$$\begin{aligned}(\varphi \circ t_\beta) \circ t_\alpha \circ (\varphi \circ t_\beta)^{-1} &= \varphi \circ t_\beta \circ t_\alpha \circ t_\beta^{-1} \circ \varphi^{-1} \\ &= \varphi \circ t_\alpha \circ \varphi^{-1} \\ &= t_{\varphi(\alpha)} \quad \in V.\end{aligned}$$

Hence we have

☆ Theorem ☆ : Given quad. form $Q : V \to \mathbb{F}$ with $Q(x) = 0 \Rightarrow x = 0$ and char $\mathbb{F} \neq 2$, we have

$$\mathrm{Isom}(AV, Q) = V \rtimes O(V, Q)$$

natural action.

i.e. "isometry" = (vector, orth. trans.)

with multiplication

$$(\alpha_1, \varphi_1) \circ (\alpha_2, \varphi_2) = t_{\alpha_1} \circ \varphi_1 \circ t_{\alpha_2} \circ \varphi_2$$

$$= t_{\alpha_1} \circ t_{\varphi_1(\alpha_2)} \circ \varphi_1 \circ \varphi_2$$

$$= t_{\alpha_1 + \varphi_1(\alpha_2)} \circ (\varphi_1 \circ \varphi_2)$$

$$= (\alpha_1 + \varphi_1(\alpha_2), \varphi_1 \circ \varphi_2)$$

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This can be expressed nicely in coordinates as matrix multiplication.

Fix a basis for V and consider
$A \in O(V, Q) \leq \mathrm{Mat}_n(\mathbb{F})$ and $\alpha \in V = \mathbb{F}^n$

Identify $(\alpha, A) = \left(\begin{array}{c|c} A & \alpha \\ \hline 0 \cdots 0 & 1 \end{array}\right)$

Note that

$$\left(\begin{array}{c|c} A_1 & \alpha_1 \\ \hline 0 \cdots 0 & 1 \end{array}\right)\left(\begin{array}{c|c} A_2 & \alpha_2 \\ \hline 0 \cdots 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{c|c} A_1 A_2 & \alpha_1 + A_1 \alpha_2 \\ \hline 0 \cdots 0 & 1 \end{array}\right)$$

Weird idea:

$\mathrm{Isom}(AV, Q)$ acts linearly on $V \oplus \mathbb{F}$, preserving the form

$$\left(\begin{array}{c|c} Q & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 \end{array}\right) \text{ on } V \oplus \mathbb{F}$$

Hmm . . .

Homework Problem for Later.

Given degenerate $Q: V \to \mathbb{F}$, split the radical

$$V = V^{\perp} \oplus W.$$

Q: What is the relation between

$O(V, Q)$ and $O(W, Q)$?

Note that $O(V^{\perp}, Q)$ is just $GL(V^{\perp})$.

Prove that

$$O(V,Q) = \left\{ \left(\begin{array}{c|c} A & M \\ \hline & G \end{array} \right) \right\} \leq GL(V)$$

where $A \in O(W, Q)$
$G \in GL(V^{\perp})$
$M \in \mathrm{Hom}(V^{\perp}, W)$.

Prove: $O(V,Q) = \left[O(W,Q) \times GL(V^{\perp}) \right] \ltimes \mathrm{Hom}(V^{\perp}, W)_{+}$
(additive group)