Thurs Oct 18

Recall the definition of group action:

Say group G "acts on" set X if

① $1(x) = x \quad \forall x \in X$

(identity element acts like identity map)

② $g(h(x)) = (gh)(x) \quad \forall g, h \in G, x \in X$

(composition acts like group operation).

DEF: Let $GL_n(\mathbb{F})$ act on $Mat_n(\mathbb{F})$ by

$$P(A) := PAP^t \quad \text{"congruence"}$$

It's an action because

$$\begin{aligned} P(Q(A)) &= P(QAQ^t) \\ &= P(QAQ^t)P^t \\ &= (PQ)A(PQ)^t = (PQ)(A). \end{aligned}$$

The action restricts to symmetric matrices:

$$\begin{aligned} (P(A))^t &= (PAP^t)^t \\ &= (P^t)^t A^t P^t \\ &= PAP^t = P(A). \end{aligned}$$

Orbits = equivalence classes of bilinear forms.

Q: Can we describe/classify the orbits?

Theorem: If $\mathbb{F}$ is algebraically closed (or even just quadratically closed, i.e. every element has a square root) and char $\mathbb{F} \neq 2$ then $\forall\ A \in Mat_n(\mathbb{F})$ with $A^t = A$ $\exists$ $P \in GL_n(\mathbb{F})$ such that

$$PAP^t = \begin{array}{c} \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right) \begin{matrix} \}p \\ \}z \end{matrix} \\ \begin{matrix} \underbrace{\quad}_{p} & \underbrace{\quad}_{z} \end{matrix} \end{array}$$

Thus:

orbits of symmetric forms $\longleftrightarrow$ $(p,z) \in \mathbb{Z}^2_{\geq 0}$ with $p+z=n$.

Proof: By the Structure Theorem $\exists$ $P \in GL_n(\mathbb{F})$ with

$$PAP^t = \begin{array}{c} \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right) \begin{matrix} \}p \\ \}z \end{matrix} \\ \begin{matrix} \underbrace{\quad}_{p} & \underbrace{\quad}_{z} \end{matrix} \end{array}$$

Where $D = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_p \end{pmatrix}$, $a_i \neq 0 \ \forall i$.

Now let $S = \left(\begin{array}{ccc|c} 1/\sqrt{a_1} & & & \\ & \ddots & & 0 \\ & & 1/\sqrt{a_p} & \\ \hline & 0 & & I \end{array}\right)$, which EXISTS.

Then

$$(SP)A(SP)^t = S(PAP^t)S^t = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)$$

These are inequivalent because dim(rad) is intrinsic to the form. ∎

Corollary: Over $\mathbb{C}$ there is only one symmetric non-degenerate form, i.e. the "dot product" I.

Q: What about $\mathbb{R}$?

Theorem: Given $A \in \mathrm{Mat}_n(\mathbb{R})$ with $A^t = A$, $\exists\, P \in GL_n(\mathbb{R})$ such that

$$PAP^t = \begin{array}{c} \left(\begin{array}{c:c:c} I & & \\ \hline & -I & \\ \hline & & 0 \end{array}\right) \begin{array}{l} \}\, p \\ \}\, m \\ \}\, z \end{array} \\ \begin{array}{ccc} \underbrace{\quad}_{p} & \underbrace{\quad}_{m} & \underbrace{\quad}_{z} \end{array} \end{array}$$

Thus:

orbits of symmetric forms $\longleftrightarrow$ $(p, m, z) \in \mathbb{Z}^3_{\geq 0}$ with $p + m + z = 0$.

Proof: By Structure Theorem $\exists\, P \in GL_n(\mathbb{R})$ with

$$PAP^t = \left(\begin{array}{ccc|ccc|ccc} a_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & a_p & & & & & & \\ \hline & & & a_{p+1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & a_{p+m} & & & \\ \hline & & & & & & a_{p+m+1} & & \\ & & & & & & & \ddots & \\ & & & & & & & & a_n \end{array}\right)$$

where $a_1, \ldots, a_p > 0$
$a_{p+1}, \ldots, a_{p+m} < 0$
$a_{p+m+1}, \ldots, a_n = 0$.

Now let $s_i := \begin{cases} 1/\sqrt{a_i} & \text{if } a_i > 0 \\ 1/\sqrt{-a_i} & \text{if } a_i < 0 \\ 1 & \text{if } a_i = 0. \end{cases}$

and $S = \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{pmatrix}$. Then.

$$(SP)A(SP)^t = S(PAP^t)S^t = \left(\begin{array}{c|c|c} I & & \\ \hline & -I & \\ \hline & & 0 \end{array}\right).$$

The fact that these are <u>inequivalent</u> for different (p, m, z) is called "Sylvester's Law of Inertia" (1852). (proof omitted) ∎

Notation: (p, m, z) is called the "inertia" or the "signature" of the form.

New Language: Quadratic Forms.

DEF: A quadratic form $Q(x) = Q(x_1, \dots, x_n)$ $\in \mathbb{F}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree 2, i.e.

$$Q(x) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j \quad , \quad a_{ij} \in \mathbb{F}.$$

We can also think of it as a function

$$Q : \mathbb{F}^n \longrightarrow \mathbb{F}$$
$$(x_1, \dots, x_n) \longmapsto \sum_{i \le j} a_{ij} x_i x_j$$

The function satisfies

① $\forall \alpha \in \mathbb{F}$, $Q(\alpha x) = \alpha^2 Q(x)$.

② If $\operatorname{char} \mathbb{F} \ne 2$, define

$$B(x, y) := \frac{1}{2} \left[Q(x+y) - Q(x) - Q(y) \right]$$

Note that $B(x, y) = B(y, x)$.

Claim: $B: \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ is bilinear

Proof: Note that

$$Q(x) = \sum_{i \le j} a_{ij} x_i x_j = x^t A x$$

where $A = (A_{ij})$ and

$$A_{ij} = \begin{cases} a_{ij} & \text{if } i = j \\ a_{ij}/2 & \text{if } i < j \\ a_{ji}/2 & \text{if } i > j \end{cases}$$

The result follows. $\blacksquare$

Example: $Q(x,y) = x^2 + 4xy + 3y^2$ can be written as

$$Q(x,y) = (x, y) \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Finally, note that

$$B(x,x) = \frac{1}{2}\left[Q(x+x) - Q(x) - Q(x) \right]$$

$$= \frac{1}{2}\left[Q(2x) - 2Q(x) \right]$$

$$= \frac{1}{2}\left[4Q(x) - 2Q(x) \right] = Q(x).$$

Summary: If $\operatorname{char} \mathbb{F} \neq 2$, $\exists$ bijection

$$\{\text{quadratic forms}\} \longleftrightarrow \{\text{symmetric bilinear forms}\}$$

$$Q(x) \longmapsto B(x,y) := \frac{1}{2}\left[Q(x+y) - Q(x) - Q(y)\right]$$

$$Q(x) := B(x,x) \longleftarrow\!\!\shortmid \quad B(x,y)$$

Application: Conic Sections

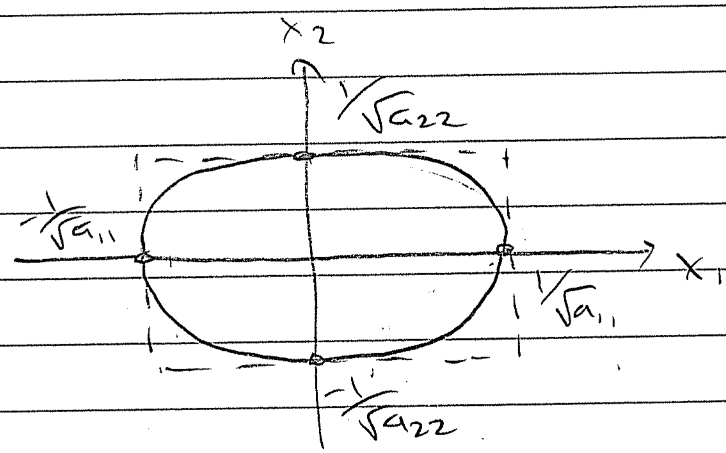
Q: Given $a_{11}, a_{12}, a_2, b_1, b_2, c \in \mathbb{R}$, what does the zero locus

$$f(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + b_1x_1 + b_2x_2 + c = 0$$

look like in $\mathbb{R}^2$?

We know 3 cases

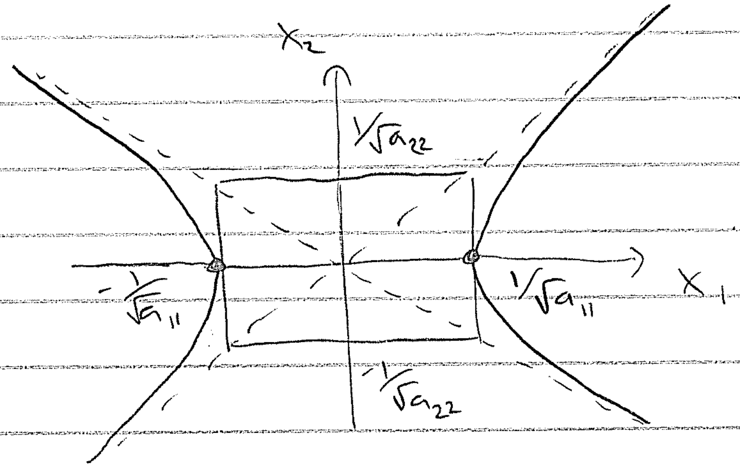
<u>Ellipse</u>



$f(x_1, x_2)$
$= a_{11}x_1^2 + a_{22}x_2^2 - 1$
$(a_{11}, a_{22} > 0)$

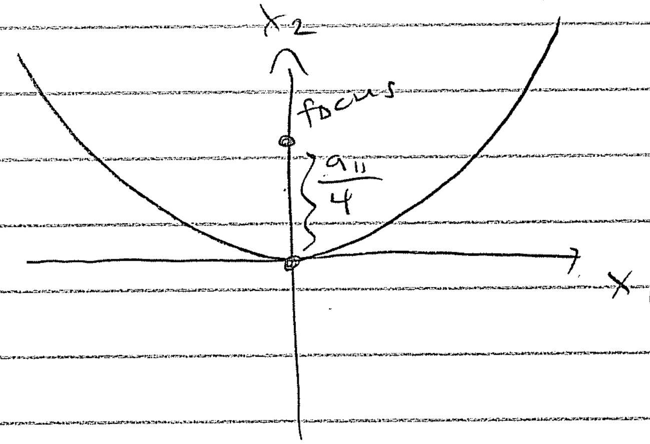
Hyperbola

$f(x_1, x_2)$
$= a_{11}x_1^2 - a_{22}x_2^2 - 1$
$(a_{11}, a_{22} > 0)$



Parabola

$f(x_1, x_2) = a_{11}x_1^2 - x_2$
$(a_{11} > 0)$



In general, we can write

$$f(x) = x^t A x + B x + c$$

$$= \underbrace{(x_1, x_2)^t \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{quadratic form}} + (b_1, b_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + c$$

Theorem: If $\det A \neq 0$ then $f(x) = 0$ is "affinely equivalent" to an ellipse or hyperbola.

Proof: By Structure Theorem $\exists P \in GL_n(\mathbb{F})$ such that $P^t A P$ is diagonal.

Changing variables $Px' = x$ gives

$$f(x') = x'^t A' x' + B' x' + c$$

where $A' = P^t A P$ and $B' = BP$

Drop the primes to get

$$f(x) = a_{11} x_1^2 + a_{22} x_2^2 + b_1 x_1 + b x_2 + c$$

Now "complete the squares" $x_i = \left(x_i' - \frac{b_i}{2a_{ii}}\right)$.

(This is a "translation" of the plane.)

Drop primes again to get

$$f(x) = a_{11} x_1^2 + a_{22} x_2^2 + c.$$

If $c = 0$ we get a single point, so let $c \neq 0$ and scale to get

$$f(x) = a_{11} x_1^2 + a_{22} x_2^2 - 1$$

If $a_{11}, a_{22} < 0$ the locus is $\emptyset$

So assume $a_{11} > 0$.

Then

$a_{22} > 0 \implies$ ellipse

$a_{22} < 0 \implies$ hyperbola $\blacksquare$

The cases $\det A = 0$ are "degenerate"

(Things are cleaner in projective space and over an alg. closed field...)

Remark: The classification works in any dimension

$$f(x) = x^t A x + Bx + c.$$

In $\mathbb{R}^3$: for $\det A \neq 0$,

sign. of A	shape
$(1,1,1)$	ellipsoid
$(1,1,-1)$	one sheet hyperboloid
$(1,-1,-1)$	two sheet hyperboloid.

Sylvester.

Tues Oct 23

Appendix from last class.

Q: Why is a degree 2 poly $f(x,y) \in \mathbb{R}[x,y]$ called a "conic section"?

Let $f(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$.

Then we can write $f(x,y) = 0$ as

$$(x,y)\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (d,e)\begin{pmatrix} x \\ y \end{pmatrix} + f = 0.$$

as we did last time. But this is much cleaner if we "homogenize":

$$F(x,y,z) := z^2 f\left(\frac{x}{z}, \frac{y}{z}\right)$$

$$= ax^2 + bxy + cy^2 + dxz + eyz + fz^2.$$

"homogeneous of degree 2"

i.e.

$$F(x,y,z) = (x,y,z)\begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= x^t M x.$$

We have $f(x,y) = 0 \iff$

$$F(x,y,z) = 0 \quad \text{and} \quad z = 1$$

$$\Updownarrow$$

$$x^t M x = 0 \quad \text{and} \quad x^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1.$$

By structure theorem $\exists\, P \in GL_n(\mathbb{R})$ with $P^t M P = D$ diagonal. Change coordinates to $x = Px'$. Then

$$x^t M x = 0 \quad \text{and} \quad x^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$$

$$\Updownarrow$$

$$x'^t D x' = 0 \quad \text{and} \quad x'^t \left(P^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = 1.$$

We know $D = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} \pm 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & 0 \end{pmatrix}$

or $\begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix}$

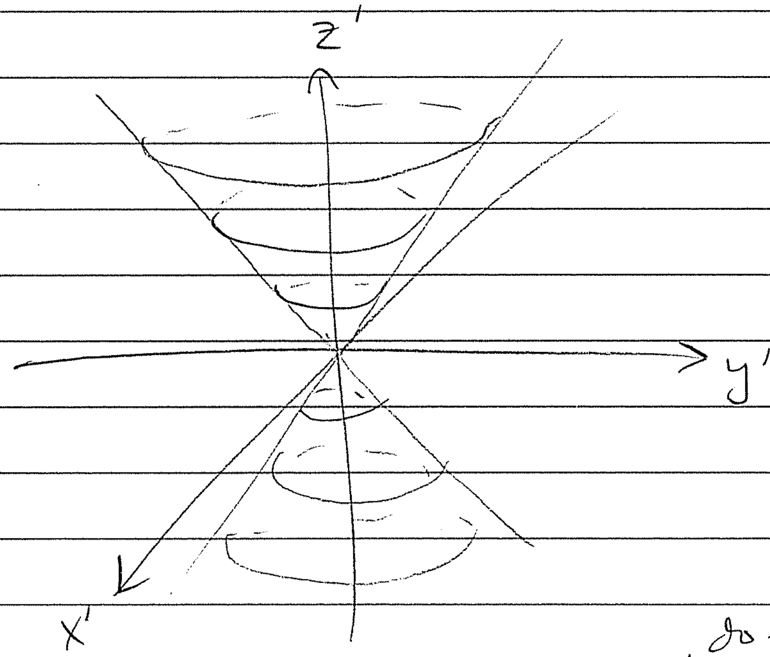
Note $x'^t D x' = 0 \iff x'^t (-D) x' = 0.$

Turns out there is <u>only</u> <u>one</u> non<u>trivial</u> <u>case</u>:

$$D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

Then $x'^t D x' = 0 \iff x'^2 + y'^2 - z'^2 = 0$

This is a "cone":



dot product

So $\quad x'^t D x' = 0 \quad$ and $\quad x'^t \left(P^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = 1$

Intersection of a cone and a plane.
Change coords back: $f(x,y) = 0$ is intersection of

$$P(\text{cone}) \quad \text{and plane} \quad x^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1.$$