Math 662 **Spring 2014**
Homework 4 **Drew Armstrong**

Problem 0. (Drawing Pictures) We have drawn algebraic curves in $\mathbb{R}^2$, $\mathbb{C}^2$ and $\mathbb{R}P^2$. Now we will try to draw an algebraic curve in $\mathbb{C}P^2$. Let $\alpha, \beta, \gamma \in \mathbb{C}$ be distinct complex numbers and consider the polynomial

$$f(x, y) := y^2 - (x - \alpha)(x - \beta)(x - \gamma) \in \mathbb{C}[x, y].$$

We can identify the complex projective line $\mathbb{C}P^1 := \{[x_0 : x_1] : x_0, x_1 \in \mathbb{C} \text{ not both zero}\}$ with the real 2-sphere by stereographic projection. (The points $[1 : x] \in \mathbb{C}P^1$ correspond to finite points $x \in \mathbb{C}$, and the point at infinity $\infty := [0 : 1] \in \mathbb{C}P^1$ is the north pole.) Let $S \subseteq \mathbb{C}P^2$ denote the set of points $(x, y) \in \mathbb{C}P^2$ satisfying $f(x, y) = 0$. (Technically we should homogenize the polynomial $f(x, y)$ to have 3 complex variables, but don't worry about it. Our pictures will not be precise anyway!) Note that for each $x \in \mathbb{C}P^1 \setminus \{\alpha, \beta, \gamma, \infty\}$ the equation $f(x, y) = 0$ has exactly two solutions for $y \in \mathbb{C}P^1$. Thus S can be thought of as a double cover of the sphere $\mathbb{C}P^1$, possibly branched at the four points $\{\alpha, \beta, \gamma, \infty\}$. One can show in fact that there is a single point of S above each $x \in \{\alpha, \beta, \gamma, \infty\}$, instead of two. **Perform cut-and-paste to show that S is topologically equivlalent to a torus.** [Hint: "Cut" from α to β and from γ to ∞. Take the two sheets apart and "paste" them back together. You may assume that Riemann surfaces are orientable (which can be proved using complex analysis), so S is not a Klein bottle.]

Problem 1. (The prime ideals of $\mathbb{Z}[y]$ and $K[x, y]$) Let R be a PID and let $P \leq R[y]$ be a **prime** ideal. You will show that P is one of the following:

- (0),
- (g) for irreducible $g \in R[y]$,
- (p, f) where $p \in R$ is prime, $f \in R[y]$, and the reduction $\bar{f} \in R/(p)[y]$ is irreducible.

Furthermore, the third kind ideals are the **maximal** ideals of $R[y]$.

(a) If P is principal, show that we have $P = (0)$ or $P = (g(y))$ for $g(y) \in R[y]$ irreducible.

(b) If P is not principal, show that P contains f_1, f_2 with no common prime factor in $R[y]$. [Hint: Since P is prime in the UFD $R[y]$ it must contain an irreducible element. (Choose $0 \neq f \in P$ and factor it into irreducibles. One of the factors must be in P.) Let $f_1 \in P$ be irreducible. Since P is not principal there exists $f_2 \in P \setminus (f_1)$. If g is any common divisor of f_1 and f_2 then in particular it is a divisor of f_1. Since f_1 is irreducible this implies that g is a unit or associate to f_1. Show that the second case leads to a contradiction.]

(c) Let $K = \text{Frac}(R)$. Show that the $f_1, f_2 \in P$ from part (b) also have no common factor in $K[y]$. [Hint: If $f_1 = hg_1$ and $f_2 = hg_2$ with $h, g_1, g_2 \in K[y]$ then we can write $h = ah_0$, $g_1 = b_1\gamma_1$, and $g_2 = b_2\gamma_2$ with $a, b_1, b_2 \in K$ and $h_0, \gamma_1, \gamma_2 \in R[y]$ **primitive**. By Gauss' Lemma over a PID we have $f_1 = (ab_1)(h_0\gamma_1)$ with $h_0\gamma_1 \in R[y]$ primitive and it follows that $ab_1 \in R$. Similarly, $ab_2 \in R$. Therefore h_0 divides f_1 and f_2 in $R[y]$. Contradiction.]

(d) If P is not principal, show that $R \cap P = (p)$ for some nonzero prime $p \in R$. [Hint: Consider the $f_1, f_2 \in P$ from part (b). By part (c) we know that f_1, f_2 are coprime in $K[y]$. Since $K[y]$ is a PID there exist $a, b \in K[y]$ with $1 = af_1 + bf_2$. If $c \in R$ is a common denominator of all the coefficients of $a(y)$ and $b(y)$ then $c = (ca)f_1 + (cb)f_2 \in P \cap R$. Hence $P \cap R \neq (0)$.]

(e) Let P be nonprincipal and consider $(p) = R \cap P$ as in part (d). Let $f \mapsto \bar{f}$ be the reduction homomorphism $R[y] \to R/(p)[y]$ and let $\bar{P} \leq R/(p)[y]$ be the image of P under reduction. Show that $\bar{P}$ is a prime ideal of $R/(p)[y]$, and conclude that $P = (p, f)$ for some $f \in R[y]$ such that $\bar{f} \in R/(p)[y]$ is irreducible. [Hint: Note that $R/(p)[y]$ is a PID [Why?], so we have $\bar{P} = (\bar{f})$ for some $f \in R[y]$. Since $\bar{P}$ is prime we know that $\bar{f} \in R/(p)[y]$ is irreducible. Then given any $\varphi \in P$ we have $\bar{\varphi} = \bar{f}\bar{g}$ for some $g \in R[y]$ and we conclude that $\varphi = p \cdot h(y) + f(y)g(y)$ for some $h \in R[y]$.]

(f) Finally, given any irreducible $p \in R$ and $f \in R[y]$ such that $\bar{f} \in R/(p)[y]$ is irreducible, show that $(p, f) < R[y]$ is a **maximal ideal**. Show that any principal prime $(g) \leq R[y]$ is **not maximal**. [Hint: For the first part, show that the composition of homomorphisms $R[y] \to R/(p)[y] \to (R/(p)[y])/(\bar{f})$ is surjective onto a field, with kernel (p, f). For the second part, if g is constant show that $(g) < (g, y) < R[y]$. If $g(y)$ is non-constant let $q \in R$ be any prime that doesn't divide the leading coefficient of $g(y)$. Show that $(g) < (q, g) < R[y]$.]

[Wow, that is some serious algebra.]

Problem 2. **($K[x, y]$ for algebraically closed K)** Now let K be an algebraically closed field and let $\mathfrak{m} < K[x, y]$ be a maximal ideal. By Problem 1 we know that $\mathfrak{m} = (p, f)$, where: $0 \neq p \in K[x]$ is irreducible, $f \in K[x, y]$, and $\bar{f} \in (K[x]/(p))[y]$ irreducible.

(a) Show that $p = x - \alpha$ for some $\alpha \in K$.

(b) Show that $K[x]/(x - \alpha) \approx K$.

(c) Conclude that $f = y - \beta$ for some $\beta \in K$ and hence

$$\mathfrak{m} = (x - \alpha, y - \beta).$$

(d) Find a maximal ideal in $\mathbb{R}[x, y]$ that does **not** look like this. [Hint: Let $p = x^2 + 1$.]

[Congratulations. You just proved the 2-dimensional Nullstellensatz.]

Problem 3. **(Zorn's Lemma)** In a partially ordered set $(\mathcal{P}, \leq)$ we say that $C \subseteq \mathcal{P}$ is a chain if for all $c_1, c_2 \in C$ we have $c_1 \leq c_2$ or $c_2 \leq c_1$. "Zorn's Lemma" is actually an **axiom** which is equivalent to the Axiom of Choice. It says the following:

> *Let $(\mathcal{P}, \leq)$ be* **nonempty**. *If every chain $C \subseteq \mathcal{P}$ has an upper bound (i.e., there exists $u \in \mathcal{P}$ such that $c \leq u$ for all $c \in C$) then $\mathcal{P}$ contains a maximal element (i.e., there exists $m \in P$ such that $p \leq m$ for all $p \in \mathcal{P}$).*

Now let R be a ring, let $I < R$ be a proper ideal, and let $S \subseteq (R, \times, 1)$ be a subsemigroup that is disjoint from I. (Note that we always have $1 \in S$.)

(a) Use Zorn's Lemma to prove that the set of ideals containing I and disjoint from S has a maximal element. [Remark: If $S = \{1\}$, this result implies that every proper ideal is contained in a maximal ideal.]

(b) Prove that this maximal element is a prime ideal. [Hint: Let $P < R$ be a maximal element. If $f, g \notin P$ then the ideals $P + (f)$ and $P + (g)$ are strictly bigger than P, hence they both intersect S. Use this fact to show that $fg \notin P$.]

Problem 4. **(The Radical of an Ideal)** Given a ring R we say that $f \in R$ is nilpotent if there exists n such that $f^n = 0$. We define the nilradical as the set of nilpotent elements:

$$\sqrt{0} := \{f \in R : f^n = 0 \text{ for some } n\}.$$

(a) Prove that $\sqrt{0}$ is an ideal. [Hint: Binomial Theorem.]

(b) Prove that $\sqrt{0}$ is the intersection of all prime ideals of R. [Hint: If $f \in R$ is nilpotent show that it belongs to every prime ideal. Conversely, suppose that $f \in R$ is **not** nilpotent. Since $0 \notin S = \{1, f, f^2, \ldots\}$, Problem 3 implies that there exists a prime ideal **not** containing f.]

(c) More generally, given any ideal $I \leq R$ we define its radical:

$$\sqrt{I} := \{f \in R : f^n \in I \text{ for some } n\}.$$

Prove that $\sqrt{I}$ is the intersection of all prime ideals containing I. [Hint: The "same" proof works.]